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Optimal Design for Accelerated Life Tests Under Progressively Type-II Hybrid Censoring

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ABSTRACT

This paper presents the constant-stress and step-stress accelerated life tests model with two stress levels under the progressively Type-II hybrid censoring. The optimal test design plans for accelerated life tests are studied. It is assumed that the lifetime of the items follow the exponential distribution. The explicit conditional density functions of order statistics under progressively Type-II hybrid censoring scheme are given to obtain the expected Fisher information matrix. The optimal test design plan with the minimum asymptotic variance of the mean life under the use stress level is determined. The test units allocated to each stress in constant-stress accelerated life test and the changing time to severer accelerated stress in step-stress accelerated life test are obtained. Finally, a numerical example is presented by the Monte Carlo simulation to illustrate the optimal test design plan. It is shown that the step-stress accelerated life test is a better choice.

Keywords: Accelerated Life test, Optimal design plan, Progressively Type-II Hybrid Censoring, Asymptotic variance

1. Introduction

For high reliable products, accelerated life test (ALT) are usually used to assess their reliability. In the ALT, test units are put to accelerated stress conditions such as temperature, pressure and voltage to accelerate failure, shorten the total testing time and reduce the test cost. For simple operation and test equipment, the experimenters often adopt constant-stress accelerated life test (CSALT) and step-stress accelerated life test (SSALT). Grouped units are tested under different accelerated conditions in CSALT. In SSALT, the test stress will changed to higher stress level when some failures take place or when the test lasts for some time. Until prefixed failures and censored time, the SSALT is terminated. Many key references on statistical inference are referred to CSALT and SSALT (Nelson [1], Meeker and Escobar [2], Gouno et al. [3]; Balakrishnan and Xie [4]).

In fact, according to test purpose and properties of test units, experimenters often need to determine test plans, for example, the test time, the failure number, the stress levels and the accelerated models, before CSALT and SSALT. These problems in designing optimal ALT have received much concern since the ALT is widely applied in reliability engineering and other practical areas. Miller and Nelson [5] studied the optimum test plan for simple SSALT with exponential lifetime distribution under the complete failure data and proposed the cumulative exposure (CE) model. Yang [6] dealt with optimal design of 4-level CSALT with Type-II censoring data. Zhang [7] gave the comparison of optimum plans of simple CSALT and SSALT under Type-I censoring and indicated that the optimum SSALT was better than optimum SSALT. Other related studies see References [8-10].

The above literature referred to in CSALT and SSALT are based on complete failure data, Type-I and Type-II censoring, hybrid censoring and progressive Type-I and Type-II censoring scheme, which are classical censoring schemes for testing units. With rapid development in technology, classical censoring schemes gradually display the drawbacks in ALT, for example, the experimenters cannot flexibly terminate the ALT under the progressive Type-I and Type-II censoring. Therefore, Childs et al. [11] proposed the progressively hybrid censoring scheme (PHCS), including Type-I PHCS and Type-II PHCS, which integrates hybrid censoring and progressive Type-II censoring. Also, Childs et al. [11] gave the important results for PHCS.

Since PHCS was put forward, parametric inference has been studied by some authors. Ma and shi [12] considered the parameter inference for Lomax distribution based on progressively Type-II hybrid censored data. Li et al. [13] gave the MLE and approximate confidence intervals based on CE model for the simple SSALT under progressively Type-I hybrid censoring scheme. Zhou et al. [14] considered MLE and bootstrap confidence intervals for the constant stress accelerated life model with increasing stress levels from Geometric process. The point and interval MLEs of Weibull parameters and acceleration factor were discussed under Type-I PHCS for step-stress partially accelerated test by Ismail [15]. Zhao et al. [16] constructed simple CSALT with Burr Type-XII lifetime distribution and obtained the MLE and approximate confidence intervals under Type-I PHCS. Wu et al. [17] studied the MLE, asymptotic confidence intervals, Bayesian estimates and approximate credible intervals of Weibull parameters in constant-stress accelerated competing risks model. However, there are few literature associated to the optimal accelerated life test design under Type-II PHCS.

In this paper, we study the optimum design for simple constant-stress and step-stress accelerated life tests based on the progressively Type-II hybrid censoring data. The rest of the paper is organized as follows: the statistical model and assumptions in CSALT and SSALT under Type-II PHCS are described in Section 2. The likelihood function and the marginal density function of order statistics under Type-II PHCS are obtained. The expected Fisher information matrix and the asymptotic variance of the mean life in constant-stress and step-stress accelerated life models are presented in Section 3. Numerical results are shown in Section 4 to illustrate the rationality of the optimum accelerated life test plan. The conclusions are contained in Section 5.

2. Some assumptions and model description

2.1. Some assumptions

In order to establish the simple constant-stress and step-stress accelerated life tests model under the Type-II PHCS, we first give following assumptions.

A1 There are two stress levels S_1 and S_2 in simple ALT, and $S_0 < S_1 < S_2$, where S_0 is the use stress level.

A2 For any stress level, the lifetime distribution of a test unit is distributed as the exponential distribution.

A3 A cumulative exposure model holds: the remaining life of tested units depends only on the current cumulative failure probability and the current stress, regardless of how the probability is accumulated.

A4 At stress level S_i (i=1,2), the mean life of a test unit is a log-linear function of stress given by

$$\log(\theta_i) = a + b\varphi(S_i) \tag{1}$$

where a and b are unknown parameters, $\varphi(S)$ is a known decreasing function of stress S. When stress S is the temperature condition, $\varphi(S) = S^{-1}$ is Arrhenius model and when stress S is the voltage condition, $\varphi(S) = -\log(S)$ is inverse power model.

2.2. Test procedure under Type-II PHCS

Assume that n units are tested in simple ALT which is ended before the censored time T, the description below shows the test procedures of simple CSALT and SSALT.

(1) Test procedure of CSALT

Put $n_i = n\pi_i$ units into the accelerated stress S_i for $i = 1,2$. The r_i is the expected failure number and $\{R_{i,j}, j = 1,2,...,r_i\}$ are expected removal scheme with $n_i = \sum_{j=1}^{r_i}(1+R_{i,j})$. At the failure time $x_{i,j}$, $R_{i,j}$ units are removed from the remaining survived units. In fact, The test under S_i is finally terminated at $\min(x_{i,r_i}, T)$ in Type-I PHCS and the final failure number d_i has random values $\{d_i = 1,2,\ldots,r_i\}$. The final censored units R^*_{i,d_i+1} at time T can be expressed as

$$R^*_{i,d_i+1} = \sum_{j=d_i+1}^{r_i}(1+R_{i,j}), \quad if\ d_i = 1,2,...,r_i-1, R^*_{i,d_i+1} = 0, if\ d_i = r_i .$$

Therefore, we have the data $\vec{x}_i = \{0 < x_{i,1} < x_{i,2} < ... < x_{i,d_i} < T\}, i = 1,2.$

(2) Test procedure of SSALT

Firstly put n units into the accelerated stress S_1. r is the expected failure number before T and $\{C_j, j = 1,2,...,r\}$ are expected removal scheme with $n = \sum_{j=1}^{r}(1+C_j)$. At the failure time y_j, C_j units are removed. Until d_1 failure units occur before the time τ, the test with $n_2 = n - \sum_{j=1}^{d_1}(1+R_j)$ units switches to the higher accelerated stress S_2. Under the accelerated stress S_2, $C_{d_1+l}, l = 1,2,...,r-d_1$ units are removed at the failure time y_{d_1+l} until d_2 failure units before T. The SSALT is ended at $\min(y_r, T)$ in Type-I PHCS. The final failure number $d = d_1 + d_2$ have random values $\{d = 1,2,\ldots,r\}$. The final censored units C^*_{d+1} at time T can be given by

$$C^*_{d+1} = \sum_{j=d+1}^{r}(1+C_j), \quad if\ d = 1,2,...,r-1, C^*_{d+1} = 0, if\ d = r.$$

Then, we have the data $\vec{y} = \{0 < y_1 < y_2 < ... < y_{d_1-1} < y_{d_1} < \tau < y_{d_1+1} < ... < y_{d_1+d_2} < T\}$ in SSALT. Note that $d_1 > 0$ and $d_2 > 0$.

2.3. Model description

Under the assumption A1, the probability density function (pdf) and distribution function of a test unit for simple constant-stress accelerated life model are given by, respectively

$$f_i(x;\theta_i) = (1/\theta_i)\exp\{-x/\theta_i\}, \quad F_i(x;\theta_i) = 1-\exp\{-x/\theta_i\}, x>, \theta_i > 0, i = 1,2. \tag{2}$$

As $S_1 < S_2$, $\theta_1 > \theta_2$. From the Assumptions A1 and A2, the cumulative distribution function (cdf) of a test unit in SSALT can be expressed as

$$G(x) = \begin{cases} G_1(x) = F_1(x;\theta_1), 0 < x < \tau, \\ G_2(x) = 1 - S_2(x-\tau;\theta_2)S_1(\tau;\theta_1), x \ge \tau, \end{cases} \tag{3}$$

where $S_i(x;\theta_i) = 1 - F_i(x;\theta_i), i = 1,2.$ Substituting equation (2) into equation (3), we have the pdf and cdf of a test unit in SSALT

$$g(x) = \begin{cases} g_1(x) = (1/\theta_1)\exp\{-x/\theta_1\}, 0 < x < \tau, \\ g_2(x) = (1/\theta_2)\exp\{-(x-\tau)/\theta_2 - \tau/\theta_1\}, x \ge \tau. \end{cases} \tag{4}$$

$$G(x) = \begin{cases} G_1(x) = 1-\exp\{-x/\theta_1\}, 0 < x < \tau, \\ G_2(x) = 1-\exp\{-(x-\tau)/\theta_2 - \tau/\theta_1\}, \mathrm{x} \ge \tau. \end{cases} \tag{5}$$

3. Likelihood function and conditional density function

3.1. Likelihood function

As there exist unknown parameters a and b, we first present the maximum likelihood estimates $\hat{a}$ and $\hat{b}$ so that we can determine the optimum test plan. Based on above assumption A3, accelerated models and failure data, the likelihood function $L_C(a,b)$ in CSALT is given by

$$L_C(a,b) \propto \exp\left\{-d_1(a+b\varphi_1) - d_2(a+b\varphi_2) - \sum_{i=1}^{2}\exp\{-a-b\varphi_i\}T_i\right\}, \tag{6}$$

where $\varphi_i = \varphi(S_i), T_i = \sum_{j=1}^{d_i}(1+R_{i,j})x_{i,j} + R^*_{i,d_i+1}T, i = 1,2$ and the likelihood function $L_S(a,b)$ in SSALT is proportional to

$$\begin{aligned} L_S(a,b) \propto{}& \exp\{-d_1(a+b\varphi_1) - d_2(a+b\varphi_2) - \exp\{-a-b\varphi_1\}(W_1+n_2\tau)\} \\ &\times\exp\{-\exp\{-a-b\varphi_2\}W_2\}, \end{aligned} \tag{7}$$

where $W_1 = \sum_{j=1}^{d_1}(1+C_j)y_j, W_2 = \left[\sum_{j=1}^{d_2}(1+C_{d_1+j})(y_{d_1+j}-\tau) + C^*_{d+1}(T-\tau)\right]$. So the MLEs $\hat{a}_C$ and $\hat{b}_C$ in CSALT, $\hat{a}_S$ and $\hat{b}_C$ in SSALT of parameters a and b are obtained by equations (6) and (7), which are respectively expressed as

$$\begin{cases} \hat{a}_C = [(\ln T_1 - \ln d_1)\varphi_2 - (\ln \mathrm{T}_2 - \ln d_2)\varphi_1]/[\varphi_2 - \varphi_1] \\ \hat{b}_C = [(\ln T_1 - \ln d_1)\varphi_2 - (\ln \mathrm{T}_2 - \ln d_2)]/[\varphi_2 - \varphi_1] \end{cases} \tag{8}$$

and

$$\begin{cases} \hat{a}_S = [[\ln(W_1+n_2\tau) - \ln d_1]\varphi_2 - (\ln W_2 - \ln d_2)\varphi_1]/[\varphi_2 - \varphi_1] \\ \hat{b}_S = [[\ln(W_1+n_2\tau) - \ln d_1] - (\ln W_2 - \ln d_2)]/[\varphi_2 - \varphi_1] \end{cases} \tag{9}$$

3.2. Expected Fisher information matrix

From equations (6) and (7), we can obtain the expressions of the expected Fisher information matrix I_C and I_S respectively based on the constant-stress and step-stress accelerated life models under Type-I PHCS. Therefore, the expected Fisher information matrix I_C for CSALT is given by

$$I_C = \begin{pmatrix} \sum_{i=1}^{2}\theta_i^{-1}ET_i & \sum_{i=1}^{2}\theta_i^{-1}\varphi_i ET_i \\ \sum_{i=1}^{2}\theta_i^{-1}\varphi_i ET_i & \sum_{i=1}^{2}\theta_i^{-1}\varphi_i^2 ET_i \end{pmatrix} \triangleq \begin{pmatrix} A_c & B_c \\ B_c & C_c \end{pmatrix} \tag{10}$$

where $\theta_i = \exp\{a + b\varphi_i\}, i = 1, 2$ and so I_S for SSALT has the following expression

$$I_S = \begin{pmatrix} \theta_1^{-1}E(W_1 + n_2\tau) + \theta_2^{-1}EW_2 & \theta_1^{-1}E(W_1 + n_2\tau) + \theta_2^{-1}EW_2 \\ \theta_1^{-1}E(W_1 + n_2\tau) + \theta_1^{-1}EW_2 & \theta_1^{-1}E(W_1 + n_2\tau) + \theta_2^{-1}EW_2 \end{pmatrix} \triangleq \begin{pmatrix} A_s & B_s \\ B_s & C_s \end{pmatrix} \tag{11}$$

Because of the random failure numbers d_1 and d_2 in CSALT, we have $E(d_1) = r_1$ and $E(d_2) = r_2$. Similarly, we have $E(d_1) = nF_1(\tau)$ and $E(d_2) = r - nF_1(\tau)$ in SSALT. Finally, we have following expectations

$$ET_i = \sum_{l=1}^{r_i - 1} E(T_i \mid d_i = l)P_{C,i}(d_i = l) + E(T_i \mid d_i = r_i)P_{C,i}(d_i = r_i),$$

$$E(W_1 + n_2\tau) = \sum_{j=1}^{r}(1 + C_j)E[y_j I(y_j < \tau) + \tau\bar{I}(y_j < \tau)],$$

$$EW_2 = \sum_{l=1}^{r - \bar{r}_1 - 1} E(W_2 \mid d_2 = l)P_S(d_2 = l) + E[W_2 \mid d_2 = r - \bar{r}_1]P_S(d_2 = r - \bar{r}_1), \tag{12}$$

where $\bar{r}_1 = nF_1(\tau)$, $\bar{I}(y_j < \tau) = 1 - I(y_j < \tau)$, $I(y_j < \tau)$ is the indicator function. For $\mathrm{i} = 1, 2,$ the probability mass function (pmf) of d_i in CSALT is $P_{C,i}(d_i = 1, 2, ..., r_i)$ and the pmf of d_2 under the stress S_2 in SSALT is $P_S(d_2 = 1, 2, ..., r - \bar{r}_1)$.

3.3. Conditional probability density function

The likelihood function is important to calculate the MLEs, regardless of the coefficients. Virtually, we can find that there are two independent life tests with progressively Type-I hybrid censoring data in CSALT from the likelihood function (6). Correspondingly, from equation (7) in SSALT, the life test under stress S_1 is equal to that with progressive Type-I censoring scheme. As for the life test under S_2 based on progressively Type-I censoring data in SSALT, it is independent with that under stress S_1. Therefore, the explicit probability density functions of order statistics and probability mass functions of discrete random variables in CSALT and SSALT can be derived and the expectation values in equation (12) also can be computed.

In order to calculate the expectation values, firstly present two lemmas. The proof of Lemma 3.3.1 can be found in [18] and Lemma 3.3.2 was proved by Childs et al. [11].

Lemma 3.3.1. (a) Let $f(x)$ and $F(x)$ denote the pdf and the cdf of an absolutely continuous random variable X and let $a_j > 0$ for $j = 1, 2, \ldots, r$. Then for $r \geq 1$, we have

$$\int_T^{x_{r+1}} \cdots \int_T^{x_3}\int_T^{x_2} \prod_{j=1}^{r} f(x_j)\{1 - F(x_j)\}^{a_j - 1}dx_1 dx_2 ... dx_r = \sum_{i=0}^{r} c_{i,r}(\vec{a}_r)\{1 - F(x_{r+1})\}^{b_{i,r}(\vec{a}_r)}\{1 - F(T)\}^{\sum_{j=1}^{r-i} a_j}, \tag{13}$$

where $\vec{a}_r = (a_1, a_2, ..., a_r)$, $c_{i,r}(\vec{a}_r) = (-1)^i / \{\prod_{j=1}^{i} \sum_{k=r-i+1}^{r-i+j} a_k\} \{\prod_{j=1}^{r-i} \sum_{k=j}^{r-i} a_k\}$, $b_{i,r}(\vec{a}_r) = \sum_{j=r-i+1}^{r} a_j$ in which we adopt the usual conventions that $\prod_{j=1}^{0} d_j \equiv 1$ and $\sum_{j=i}^{i-1} d_j \equiv 0.$

(b) Let $f(x)$ and $F(x)$ denote the pdf and the cdf of an absolutely continuous random variable X. The density function of $X_s (1 \le s \le m \le n)$ with R_s removal units in progressive censoring scheme is given by

$$f_{X_s}(x_s) = c'(n,s) \sum_{i=0}^{s-1} c_{i,s-1}(R_1+1, ..., R_{s-1}+1) f(x_s) \{1 - F(x_s)\}^{\sum_{j=s-i}^{m}(1+R_j)} \tag{14}$$

where $-\infty < x_s < \infty$ and $\sum_{j=1}^{m} (1+R_j) = n.$

Considering the Type-I PHCS, n units are put on test with censoring scheme $(R_1, R_2, \ldots, R_m)$ and the test is terminated at the time $\min\{X_m, T\}$ with the ordered failure time $X_1 \le X_2 \le \ldots \le X_m$. Suppose the final failure number is D, then the $R^*_{D+1} = \sum_{j=D+1}^{m} (1+R_j)$ units are censored at time T. Therefore we have the Lemma 3.3.2.

Lemma 3.3.2. (a) For $d = 1, \ldots, m-1,$ the conditional joint density of $X_1, X_2, \ldots, X_m$, given $D = d$, is

$$f(x_1, x_2, ..., x_d \mid D = d) = [c'(n,d)\{1 - F(T)\}^{R^*_{d+1}} / P(D = d)] \prod_{j=1}^{d} f(x_j)\{1 - F(x_j)\}^{R_j}, \quad -\infty < x_1 < x_2 < ... < x_d < T, \tag{15}$$

where $c'(n,d) = \prod_{j=1}^{d} \sum_{k=j}^{m} (R_k + 1)$ for $d = 1, 2, \ldots, m.$

(b) The conditional joint density of $X_1, X_2, \ldots, X_m$, given $D = m$, is

$$f(x_1, x_2, ..., x_m \mid D = m) = [c'(n,m) / P(D = m)] \prod_{j=1}^{m} f(x_j)\{1 - F(x_j)\}^{R_j}, \quad -\infty < x_1 < x_2 < ... < x_m < T. \tag{16}$$

Theorem 3.3.1. (a) For $d = 1, \ldots, m-1,$ given $D = d$, the conditional marginal density function of $X_j, 1 < j < d$, is

$$f_{X_j}(x_j \mid D = d) = [c'(n,d)\{1 - F(T)\}^{R^*_{d+1}} / P(D = d)] \times \sum_{i_1=0}^{d-j} \sum_{i_2=0}^{j-1} P(i_1, i_2)\{1 - F(T)\}^{b_{i_1,d-j}(\vec{a}_{d-j})} f(x_j)\{1 - F(x_j)\}^{K(i_1,i_2)}, \tag{17}$$

where $0 < x_j < T$, $\vec{a}_{d-j} = (R_{j+1}+1, R_{j+2}+1, ..., R_d+1)$, $\vec{a}_{j-1} = (R_1+1, R_2+1, ..., R_{j-1}+1)$,

$$b_{i_1,d-j}(\vec{a}_{d-j}) = \sum_{l=d-i_1-1}^{d} (R_l + 1)$$

$$P(i_1, i_2) = c_{i_1,d-j}(R_{j+1}+1, R_{j+2}+1, ..., R_d+1) \times c_{i_2,j-1}(R_1+1, R_2+1, ..., R_{j-1}+1),$$

$$K(i_1, i_2) = R_j + \sum_{l=j+1}^{d-i_1} (R_l + 1) + \sum_{l=j-i_2}^{j-1} (R_l + 1).$$

Especially, given $D = d$, the conditional marginal density functions of X_1 and X_d are expressed as, respectively.

$$f_{X_1}(x_1 \mid D=d) = [c'(n,d)/P(D=d)]\sum_{i=0}^{d-1} c_{i,d-1}(R_2+1,\ldots,R_d+1)\{1-F(T)\}^{R_{d+1}^*+\sum_{l=d-i+1}^{d}(R_l+1)}$$
$$\times f(x_1)\{1-F(x_1)\}^{R_1+\sum_{l=2}^{d-i}(R_l+1)},$$
$$f_{X_d}(x_d \mid D=d) = [c'(n,d)\{1-F(T)\}^{R_{d+1}^*}/P(D=d)] \tag{18}$$
$$\times\sum_{i=0}^{d-1} c_{i,d-1}(R_1+1,\ldots,R_{d-1}+1) f(x_d)\{1-F(x_f)\}^{R_d+\sum_{l=d-i}^{d-1}(R_l+1)},$$

where $0 < x_1 < T$ and $0 < x_d < T$.

(b) Given $D=m$, the conditional marginal density function of $X_j, 1<j<d$, is

$$f_{X_j}(x_j \mid D=m) = [c'(n,d)/P(D=m)]$$
$$\times\sum_{i_1=0}^{m-j}\sum_{i_2=0}^{j-1} P(i_1,i_2)\{1-F(T)\}^{b_{i_1,m-j}(\vec{a}_{m-j})} f(x_j)\{1-F(x_j)\}^{K(i_1,i_2)},\ 0<x_j<T, \tag{19}$$

Also, $f_{X_1}(x_1 \mid D=m)$ and $f_{X_m}(x_m \mid D=m)$ can be formed similarly with equation (18) as $R_{m+1}^*=0$.

Proof. Using Lemma 3.3.1(a), Part (a) is obtained by firstly integrating out the variables $X_{j+1}, X_{j+2}, \ldots, X_d$ in equation (15) with the support $\{x_j < x_{j+1} < \ldots < x_d\}$. That is

$$f(x_1,x_2,\ldots,x_j \mid D=d) = [c'(n,d)\{1-F(T)\}^{R_{d+1}^*}/P(D=d)]\int_{x_j}^{T}\cdots\int_{x_j}^{x_{j+3}}\int_{x_j}^{x_{j+2}}\prod_{j=1}^{d} f(x_j)\{1-F(x_j)\}^{R_j}dx_{j+1}dx_{j+2}\ldots dx_d,$$
$$=[c'(n,d)\{1-F(T)\}^{R_{d+1}^*}/P(D=d)]\left[\sum_{i_1=0}^{d-j} c_{i_1,d-j}(\vec{a}_{d-j})\{1-F(T)\}^{b_{i_1,d-j}(\vec{a}_{d-j})}\{1-F(x_j)\}^{\sum_{l=j+1}^{d-i_1}(R_l+1)}\right]$$
$$\times\prod_{l=1}^{j} f(x_l)\{1-F(x_l)\}^{R_l},\quad 0<x_1<x_2<\ldots<x_j<T. \tag{20}$$

When $j=1$ in equation (20), $f_{X_1}(x_1 \mid D=d)$ in equation (18) is proved.

Secondly to integrate out the remaining variables $X_1, X_2, \ldots, X_{j-1}$ in equation (20) with the support $\{0<x_1<x_2<\ldots<x_j\}$. Then we have

$$f_{X_j}(x_j \mid D=d) = c'(n,d)\{1-F(T)\}^{R_{d+1}^*}/P(D=d)$$
$$\times\sum_{i_1=0}^{d-j} c_{i_1,d-j}(\vec{a}_{d-j})\{1-F(T)\}^{b_{i_1,d-j}(\vec{a}_{d-j})} f(x_j)\{1-F(x_j)\}^{R_j+\sum_{l=j+1}^{d-i_1}(R_l+1)} \tag{21}$$
$$\times\int_0^{x_j}\cdots\int_0^{x_3}\int_0^{x2}\prod_{l=1}^{j-1} f(x_l)\{1-F(x_l)\}^{R_l}dx_1dx_2\ldots dx_{j-1}, 0<x_j<T.$$

When $j=d$ in equation (21), $f_{X_d}(x_d \mid D=d)$ in equation (18) holds. Taking the result in Lemma 3.3.1(a) into equation (21), Part (a) is proved. Part (b) is similarly straightforward to be obtained from equation (16). Finally, Theorem 3.3.1 holds.

Theorem 3.3.2. As D is a random variable with possible values $\{1,2,\ldots,m\}$ in Type-I PHCS, the probability mass function of D is

$$P(D=d) = c'(n,d)\sum_{i=0}^{d} c_{i,d}(R_1+1,R_2+1,\ldots,R_d+1)\{1-F(T)\}^{R_{d+1}^*+\sum_{l=d-i+1}^{d}(R_l+1)},\ d=1,2,\ldots,m-1,$$
$$P(D=m) = c'(n,m)\sum_{i=0}^{m} c_{i,m}(R_1+1,R_2+1,\ldots,R_m+1)\{1-F(T)\}^{\sum_{l=m-i+1}^{m}(R_l+1)}. \tag{22}$$

Proof. In Lemma 3.3.2, we have the property of equation (15) that is expressed as

$$\int_0^T \cdots \int_0^{x_3} \int_0^{x_2} f(x_1, x_2, ..., x_d \mid D=d) dx_1 dx_2 ... dx_d = 1.$$

Then

$$P(D=d) = c'(n,d)\{1-F(T)\}^{R^*_{d+1}} \int_0^T \cdots \int_0^{x_3} \int_0^{x_2} \prod_{j=1}^{d} f(x_j)\{1-F(x_j)\}^{R_j} dx_1 dx_2 ... dx_d. \qquad (23)$$

The Lemma 3.3.1(a) gives the expression of multiple integration in equation (23). In the similar way, $P(D=m)$ can be obtained as equation (22). Therefore, Theorem 3.3.2 is proved.

As for the exponential distribution $F(x)$ and exponential density $f(x)$ with mean life θ of the jth order statistics in Type-I PHCS, we have the following results, according to equations (17), (18) and (22),

$$E(X_j \mid D=d) = c'(n,d)\exp\{-\theta^{-1}T\} / P(D=d)$$
$$\times \sum_{i_1=0}^{d-j} \sum_{i_2=0}^{j-1} \frac{P(i_1,i_2)\theta\{1-\exp\{-\theta^{-1}[1+K(i_1,i_2)]T\} - \theta^{-1}[1+K(i_1,i_2)]T\exp\{-\theta^{-1}[1+K(i_1,i_2)]T\}\}}{[1+K(i_1,i_2)]^2 \exp\{\theta^{-1}[R^*_{d+1} + \sum_{l=d-i_1-1}^{d}(R_l+1)]T\}},$$

$$P(D=d) = c'(n,d)\sum_{i=0}^{d} c_{i,d}(R_1+1, R_2+1, ..., R_d+1)\exp\{-\theta^{-1}[R^*_{d+1} + \sum_{l=d-i-1}^{d}(R_l+1)]T\}. \qquad (24)$$

Then, in progressive Type-I censoring scheme with the censored time τ, from Lemma 3.3.1 (b), to estimate $E(W_1 + n_2\tau)$, we have the result

$$E(X_{s:m:n}) = c'(n,s)\sum_{i=0}^{s-1} c_{i,s-1}(R_1+1, ..., R_{s-1}+1)\frac{1-\exp\{-\theta^{-1}[1+\sum_{j=s-i}^{m}(1+R_j)]\tau\}}{\theta^{-1}[1+\sum_{j=s-i}^{m}(1+R_j)]^2}. \qquad (25)$$

When $D=m$, $R^*_{d+1}=0$ in equation (24). Substituting equations (24) and (25) into equation (12), the expected information matrix I_C and I_S can be calculated.

3.4. V-Optimality

To determine the optimum ALT under Type-II PHCS, we minimize the asymptotic variance of mean life under the use stress. In spite of the expected information matrix I_C and I_S, the asymptotic variance values V_C and V_S of mean life under the use stress S_0 in CSALT and SSALT are respectively given by

$$V_C(\pi_1) = Var(\ln\theta_0) = (1 \quad \varphi_0) I_C^{-1} \begin{pmatrix} 1 \\ \varphi_0 \end{pmatrix} = [A_c\varphi_0^{\ 2} - 2B_c\varphi_0 + C_c] / [A_cC_c - B_c^{\ 2}],$$

$$V_S(\tau) = Var(\ln\theta_0) = (1 \quad \varphi_0) I_S^{-1} \begin{pmatrix} 1 \\ \varphi_0 \end{pmatrix} = [A_s\varphi_0^{\ 2} - 2B_s\varphi_0 + C_s] / [A_sC_s - B_s^{\ 2}]. \qquad (26)$$

Ultimately, find the optimal π_1^* in CSALT and τ^* in SSALT to satisfy $V_C(\pi_1^*) = \min_{0<\pi_1<1} V_C(\pi_1)$ and $V_S(\tau^*) = \min_{0<\tau<T} V_S(\tau)$. In order to illustrate the procedure of optimum test design, a numerical example is given in the following section.

4. Numerical example

As the asymptotic variance values of mean life under the use stress in CSALT and SSALT cannot be explicitly given, it is necessary to present the numerical results and make comparison between the simple constant-stress and step-stress accelerated life models under Type-I PHCS. Considering the stress condition is the temperature in assumption A3, that is, $\varphi(S)=S^{-1}$. Suppose that $a=2, b=20$, two stress levels $S_1=200^\circ\mathrm{C}, S_2=435^\circ\mathrm{C}$ and the use stress level $S_0=80^\circ\mathrm{C}$ in simple accelerated life test with n test units and prefixed censored time T. In simple CSALT, the allocated rate to S_1 is $\pi_1\in(0,1)$. In simple SSALT, the changing time from S_1 to S_2 is $\tau\in(0,T)$. In addition, for $i=1,2,$ we assume n_i test units put into the accelerated stress level S_i and r_i failures take place with the expected removal scheme in PHCS determined by $R_{i,l}=[(n_i-r_i)/r_i], l=1,2,...,r_i$.

Finally, for π_1 and τ with uniformly discrete values, by calculating the expected Fisher information matrix and the asymptotic variance values, the optimal allocated rate π_1^* and changing time τ^* for different n and T can be obtained in Table 1. The numerical results in Table 1 show that the asymptotic variance in SSALT is smaller than the variance in CSALT.

	CSALT		SSALT	
(n,T)	π_1^*	$V_C(\pi_1^*)$	τ^*	$V_S(\tau^*)$
(20, 0.8)	0.5070	5.1131	0.3064	5.3150
(20, 1.5)	0.3500	4.8308	0.3587	4.6712
(30, 2.0)	0.2300	9.9352	0.3771	3.1859
(40, 2.0)	0.7000	2.0679	0.2000	2.8431

Table 1: Optimal CSALT and SSALT for different n and T

5. Conclusions

In this paper, we investigate the optimum design for simple accelerated life test. When the failure time follows exponential distribution, the exact conditional joint and marginal density functions and expectations of order statistics under progressively Type-I hybrid censoring scheme are given by explicit expressions. Using variance optimality, the optimum allocated rate in CSALT and the changing time in SSALT can be numerically calculated. The numerical results show that SSALT is a better choice.

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