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An Application of Triangular Fuzzy Numbers to Learning Assessment

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ABSTRACT

Assessment cases appear frequently in our everyday life involving a degree of uncertainty and (or) ambiguity. Fuzzy logic, due to its nature of characterizing such cases with multiple values, offers rich resources for dealing with them. Fuzzy Numbers play a fundamental role in fuzzy mathematics, analogous to the role played by the ordinary numbers in classical mathematics. In the present paper we utilize the simplest form of Fuzzy Numbers i.e. the Triangular Fuzzy Numbers, for assessing student learning skills. The concept of learning is fundamental for the study of human cognitive action and very many theories have been developed through the years by psychologists and education researchers for the description and explanation of its mechanisms. Therefore the above fuzzy assessment approach has an increased interest. Our results are illustrated by an application on learning mathematics, in which the use of Fuzzy Numbers as an assessment tool is validated through the comparison with assessment methods of the bivalent and fuzzy logic already established by the author in earlier works.

Keywords: Learning assessment; GPA index; Fuzzy logic; Fuzzy assessment models; Fuzzy numbers; Triangular fuzzy numbers

1. Introduction

The concept of learning is fundamental for the study of human cognitive action. But, while everyone knows empirically what learning is, the understanding of its nature has proved to be complicated. This happens because it is very difficult to understand the way in which the human mind works, and therefore to describe the mechanisms of the acquisition of knowledge by the individual. The problem is getting even harder by taking into account that these mechanisms, although they appear to have some common general characteristics, they actually differ in their details from person to person.

In 1956 Benjamin Bloom with collaborators Max Englehart, Edward Furst, Walter Hill, and David Krathwohl published a framework for learning, teaching, and assessing, the *Taxonomy of Educational Objectives* [1]. The Bloom's Taxonomy has been applied in the USA by generations of teachers and college instructors in the teaching process. A revised version of the taxonomy was created by Lorin Anderson, former student of Bloom [2]. Since the taxonomy reflects different forms of thinking and thinking is an active process, in the revised version the names of its six major levels were changed from noun to verb forms. These levels, moving through the lowest order processes to the higher are: *Knowing - Remembering, Organizing - Understanding, Applying, Analyzing, Generating*

- *Evaluating* and *Integrating-Creating.* The three upper levels are considered to be parallel to each other, in contrast to the lower three levels, where the success to one of them requires the earlier success in the previous levels (for more details see [3]).

There are very many theories and models in general, developed through the years by psychologists and education researchers, for the description and explanation of the mechanisms of learning. Voss [4], adopting a Ferguson's [5] hypothesis, has developed an argument that learning is a specific case of the general class of the *transfer of knowledge* (i.e. the use of the existing knowledge to produce new knowledge) and therefore any instance of learning involves the use of already existing knowledge. Thus, *learning consists of successive problem solving activities*, in which the input information is represented of existing knowledge, with the solution occurring when the input is appropriately interpreted.

According to Voss [4] and many other researchers the learning process involves the following main steps: *Representation* of the input data, *interpretation* of this data in order to produce the new knowledge, *generalization* of the new knowledge to a variety of situations and *categorization* of the knowledge. More explicitly, the representation of the stimulus input relies upon the individual's ability to use contents of his/her memory in order to find information that will facilitate a solution development. Learning consists of developing an appropriate number of interpretations and generalizing them to a variety of situations. When the knowledge becomes substantial, much of the process involves categorization, i.e. the input information is interpreted in terms of the classes of the existing knowledge. Thus the individual becomes able to relate the new information to his/her knowledge structures that have been variously described as schemata, or scripts, or frames.

When placed in this relationship with transfer, learning takes a level of complexity greater than that of a simple extension of knowledge resulting from generalization, which involves efficient execution of awareness, schema induction and automation of problem solving operations; *low – road* and *high – road* transfer respectively according to the Salomon's & Perkins's [6] terminology.

Voskoglou ([7] and [8]: Section 2.3) developed a *stochastic model* to describe mathematically the process of learning in the classroom by introducing a finite Markov chain on the steps of the Voss's [4] framework for learning. However, the knowledge that students have about various concepts is usually imperfect, characterized by a different degree of depth. On the other hand, from the teacher's point of view vagueness usually exists for the degree of his/her students' success at each step of the learning process. All the above gave us the motive to introduce principles of fuzzy logic for a more realistic representation of the process of learning. Namely, we have represented the main steps of the learning process, presented above, as *fuzzy sets on a set of linguistic labels (grades)* characterizing the learner's performance at each step [9] and later we have used the corresponding system's *uncertainty* for measuring learning skills ([8], [10], etc). Meanwhile, Subbotin, Badkoobehi and Bilotckii [11], based on Voskoglou's [9] fuzzy model for the process of learning, introduced the idea of applying *the Center of Gravity (COG) defuzzification technique* to learning assessment (see also [12]: Section 2). Recently, two, equivalent to each other, variations of the COG technique, initiated by Subbotin, have been developed treating better the ambiguous assessment cases being at the boundaries between two successive characterizations (grades) of the individual's performance: The *Triangular* (e.g. [13]) and the *Trapezoidal* (e.g. [12]: Section 3) *Fuzzy Assessment Models*. Some more details about all the above fuzzy assessment methods will be presented in the next Section of this work.

Our main target in the present paper is to introduce an alternative fuzzy assessment method for the learning skills by utilizing the *Triangular Fuzzy Numbers* (*TFNs*) as assessment tools. In fact, there exists a strong logical pro argument for employing this approach: Roughly speaking, a TFN (a, b, c), with a, b and c real numbers such that $a< b < c$, actually means *"approximately equal to b"* or, if you prefer, *"the value of b lies in the real interval [a, c]"*, expressions that constitute the basis for a fuzzy assessment.
The rest of the paper is organized as follows: In the second Section we summarize the assessment methods (traditional and fuzzy) that we have already applied in earlier works. In the third Section we introduce the notion of *Fuzzy Numbers (FN)*, while in the fourth Section we present the TFNs, the arithmetic operations defined on them and basic properties of them, to be used later in the paper. In the fifth Section we describe the use the TFNs for assessing learning skills and we discuss the advantages and disadvantages of this method with respect to the already established in earlier works assessment methods. Finally, the last Section 6 is devoted to our conclusions and a brief discussion of the perspectives of future research on the subject.

2. Assessment methods: a summary of our previous researches

The assessment of a system's effectiveness (i.e. of the degree of attainment of its targets) with respect to an action performed within the system (e.g. problem-solving, decision making, learning, etc) is a very important task that enables the correction of the system's weaknesses resulting to the improvement of its general performance. In particular, the social demand not only to educate, but also to classify students according to their qualifications, makes the student assessment one of the most important components of the educational practice and research. Furthermore, the teacher obtaining through the student assessment an overall view of his (her) students' progress, is helped to suitably adapt his (her) teaching methods and plans aiming to the best possible result.

2.1. Traditional assessment methods

The assessment methods commonly used in practice are based on the principles of the *Boolean logic* (yes-no). In case of group assessment the majority of these methods focus on the group's *mean performance*, the most typical example being the calculation of the ***mean value*** of the individual performances of all the group's members. However, some other methods focus on the group's *quality performance* by assigning greater coefficients (weights) to the higher performances of the group's members, a characteristic example being the very popular in the USA *Grade Point Average (GPA) index* (e.g. see [12]: Section 4.1) .

The GPA index is a weighted average of a group's performance. For calculating it, the individual performance of each group's member is characterized by one of the grades A (85-100%) = excellent, B (75-84%) = very good, C (60-74%) = good, D (50-59%) = fair and F (< 50%) = unsatisfactory. Notice that the above percentages assigned to each grade are indicativeonly, which means that they may differ (slightly) from case to case. Now, if n is the total number of the group's members and n_A, n_B, n_C, n_D, n_F denote the numbers of them obtaining the grades A, B, C, D and F respectively, the GPA index is calculated by the formula: GPA = $\frac{0n_F + 1n_D + 2n_C + 3n_B + 4n_A}{n}$ (1) In the worst case ($n_F = n$) formula (1) gives that GPA = 0, while in the ideal case ($n_A = n$) it gives that GPA = 4. Therefore, we have in general that $0 \leq$ GPA ≤ 4.

2.2. Fuzzy assessment methods

Fuzzy logic, the development of which is based on fuzzy sets theory[1] [14], provides a rich and meaningful addition to the standard (Boolean) logic. Unlike Boolean logic, which has only two states, true or false, fuzzy logic deals with truth values which range continuously from 0 to 1. Thus something could be *half true* 0.5 or *very likely true* 0.9 or *probably not true* 0.1, etc. In this way fuzzy logic allows one to express knowledge in a rule format that is close to a natural language expression and therefore it opens the door to construction of mathematical solutions of computational problems which are imprecisely defined (e.g. [3, 8 - 13, 15 - 17], etc).

In particular, assessment cases frequently appear in our everyday life involving a degree of uncertainty and (or) ambiguity. Fuzzy logic, due to its nature of characterizing such cases with multiple values, offers rich resources for dealing with them. This was our motive in earlier works for using a number of fuzzy methods for the assessment of several human activities, like learning (see our Introduction), problem-solving, decision making, etc (e.g. [8, 12, 13]), but also for the assessment of the effectiveness of Case-Based Reasoning Systems [15]. Below we summarize the most important of these methods:

(i) Measurement of the Uncertainty: As it is well known from the classical *Information Theory* [18], the reduction of a *system's uncertainty* with respect to an action performed within the system is connected to the new information obtained by this action: The greater is the reduction of the uncertainty, the more the new information obtained. The reduction of the uncertainty (and therefore the information connected to it) is calculated by the classical *Shannon's formula* [18] , better known as the *Shannon's entropy*, which is based on principles of *Probability Theory.* This formula has been properly adapted for use in a fuzzy environment (17: p. 20). However, Schackle [20] and many other researchers after him believe that the human behaviour can be better represented by the *Possibility* rather [21] than by the Probability Theory. This gave us in earlier works (e.g. [8, 10], etc) the idea of utilize a system's *total possibilistic uncertainty* ([19]: p. 28) for assessing

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1 Let U denote the universal set of the discourse. Then, we recall that a *fuzzy set* A *on* U (or otherwise a *fuzzy subset of* U), is a set of ordered pairs of the form A = {(x, m_A(x)): x $\in$ U}, where $m_A : U \longrightarrow [0,1]$ is its *membership function* that assigns to each element x of U a real value from the interval [0,1]. The value m_A(x), called the *membership degree (or grade) of x in A,* expresses the degree to which x verifies the characteristic property of A. Thus, the nearer is m_A(x) to 1, the better x satisfies this property. For reasons of simplicity many authors identify a fuzzy set with its membership function. A fuzzy set A is also frequently represented either by a symbolic sum (finite or infinite) of the form $\sum_{x \in U} m_A(x)/x$ or, if U has the power of the continuous, by a symbolic integral of the form $\int_U m_A(x)dx$. Obviously each classical (crisp) subset A of U can be considered as a fuzzy set on U, with m_A(x) = 1, if x$\in$ U and m_A(x)=0, if x$\notin$ U. Most of the concepts of crisp sets are extended to fuzzy sets. For general facts on fuzzy sets and the uncertainty connected to them we refer to the book of Klir and Folger [21].

its effectiveness with respect to an action performed within it. The main disadvantage of this (fuzzy) assessment method, which focuses on the *mean system's performance*, is that it requires laborious calculations. Therefore we are not going to present any more details of this method here.

(ii) The COG Defuzzification Technique: When reasoning with fuzzy rules, the initial numeric data values are *fuzzified*, that is they are turned into fuzzy values using the proper membership functions. These values are combined using fuzzy logic operators. The result is a single fuzzy set, which then must be *defuzzified* to return to a crisp output value. There are several defuzzification techniques in use, the most popular being probably the *Centre of Gravity (COG)* technique (e.g. see [22]). For applying the COG technique one corresponds to each x of the universal set U an interval of real values taken from a prefixed numerical distribution (i.e. it replaces U with a set of real intervals), which enables to construct the graph of the membership function involved. Then, according to the principles of the COG technique, the final fuzzy outcome is represented by the coordinates of the COG of the level's section contained between this graph and the X- axis.

In earlier papers (e.g. [12]: Section 2, etc) we have properly adapted the COG defuzzification technique for use as an assessment method. For this, we have expressed the group G under assessment as a fuzzy set on the set U = {A, B, C, D, F} of the grades characterizing its members' individual performance (see GPA index above) and we have replaced U with a set of real intervals as follows: F $\rightarrow$ [0, 1), D $\rightarrow$ [1, 2), C $\rightarrow$ [2, 3), B $\rightarrow$ [3, 4), A $\rightarrow$ [4, 5]. Then the graph of the membership function of G takes the form of a bar graph consisting of five rectangles, each one of them corresponding to the grades F = x_1, D = x_2, C = x_3, B = x_4 and A = x_5 respectively (see Figure 1 of [12]). The side of each rectangle lying on the X-axis, has length equal to 1 metric unit, while the other side has length equal to $y_i = m(x_i)$, for i=1,2, 3, 4, 5 respectively, where $y=m(x)$ is the corresponding membership function. Then, using well known from Mechanics formulas - see formulas (4) below in the proof of Proposition 2 - it is straightforward to calculate the coordinates of the COG of the resulting scheme and further to obtain a criterion for comparing the performance of two (or more) groups (e.g. see [12]: Section 2). However, as said in our Introduction, we recently have developed two (equivalent to each other) variations of the COG technique treating better the ambiguous assessment cases. Consequently, here we shall focus on these variations rather, than on the COG technique.

(iii) The Trapezoidal Fuzzy Assessment Model (TpFAM): The central idea of the TpFAM is the replacement of the five rectangles appearing in the COG's scheme by five isosceles trapezoids sharing common parts and corresponding to the grades F, D, C, B and A respectively (see Figure 2 of [12]). The heights of the trapezoids have lengths equal to the percentages y_i , i = 1, 2, 3, 4, 5 of the members of G obtaining the corresponding grade, while their common parts correspond to the ambiguous cases being at the boundaries between two successive grades (e.g. something like 84-85% being at the boundaries between A and B, etc). It is logical to consider that all the ambiguous cases *belong to both of the corresponding grades* and consequently the common parts of the adjacent trapezoids must be included twice in the whole area of the TpFAM's scheme, which is therefore equal to the sum of the areas of the five trapezoids. Thus, the COG of the whole area is *the resultant of the system of the GOC's of the five trapezoids*. Then, it is straightforward to

check ([12]: Section 3) that the coordinates (X_c, Y_c) of the COG of the TpFAM's scheme are calculated by the formulas: $X_c = (7\sum_{i=1}^{5} iy_i) - 2$, $Y_c = \frac{3}{7}\sum_{i=1}^{5} y_i^2$ (2).

Further, using elementary algebraic inequalities and by simple geometric observations it is straightforward to verify ([12]: Section 3) that *the greater the value of X_c, the better the corresponding group's performance*. Also, *if two groups have equal values for X_c then: a) If $X_c \geq 19$, the group with the greater Y_c demonstrates the better performance, b) If $X_c < 19$, the group with the smaller Y_c demonstrates the better performance*.

(iv) The Triangular Fuzzy Assessment Model (TFAM): The corresponding idea of the TFAM is the use of isosceles triangles instead of the trapezoids of the TpFAM (see Figure 2 of [13]). Then, following a similar procedure with TpFAM ([13]: Section 3), one finds that the coordinates (X_c, Y_c) of the COG of the TFAM's scheme are calculated by the formulas:

$$X_c = (7\sum_{i=1}^{5} iy_i) - 2, \quad Y_c = \frac{1}{5}\sum_{i=1}^{5} y_i^2 \qquad (3).$$

The same with the TpFAM criterion is also obtained for comparing the performance of two (or more) groups. Observing formulas (2) and (3) one can immediately see that the only difference between the TpFAM and the TFAM is in the values of Y_c, but this does not affect the assessment of the group performance. Therefore *the above two fuzzy assessment models (TFAM and TpFAM) are equivalent to each other* in the sense that through them one obtains exactly the same assessment results.

3. Fuzzy numbers (FNs)

3.1. Definitions

A *Fuzzy Number* (FN) is a special form of fuzzy set on the set ***R*** of real numbers. FNs play a fundamental role in fuzzy mathematics, analogous to the role played by the ordinary numbers in classical mathematics. For general facts on FNs we refer to Chapter 3 of the book of Theodorou [23], which is written in Greek language, and also to the classical on the subject book of Kaufmann and Gupta [24].

For introducing the notion of a FN, it becomes necessary first to give the following three introductory definitions:

Definition 1: A fuzzy set A on U with membership function $y = m(x)$ is said to be *normal*, if there exists x in U, such that $m(x) = 1$.

Definition 2: Let A be a fuzzy set in U, and let x be a real number of the interval [0, 1]. Then the x-*cut* of A, denoted by A^x, is defined to be the set $A^x = \{y \in U: m(y) \geq x\}$.

Definition 3: A fuzzy set A on ***R*** is said to be *convex*, if its x-cuts A^x are ordinary closed real intervals, for all x in [0, 1].

For example, for the fuzzy set A whose membership function's graph is represented in Figure 1, we observe that $A^{0.4} = [5, 8.5] \cup [11, 13]$ and therefore A is not a convex fuzzy set.

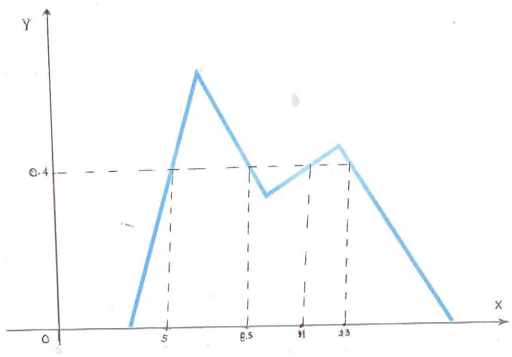

Figure 1: Graph of a non convex fuzzy set

We are ready now to give the definition of a FN:

Definition 4: A FN is a normal and convex fuzzy set A on ***R*** with a piecewise continuous membership function.

Figure 2 represents the graph of a FN expressing the fuzzy concept: "*The real number x is approximately equal to 5*". We observe that the membership function of this FN takes constantly the value 0 outside the interval [0, 10], while its graph in [0, 1] is a parabola.

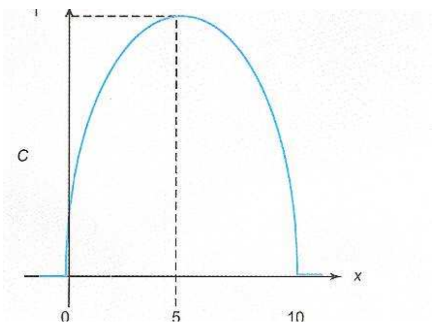

Figure 2: Graph of a fuzzy number

Since the x-cuts A^x of a FN A are closed real intervals, we can write $A^x = [\, A_l^{\,x}, A_r^{\,x} \,]$ for each x in [0, 1], where $A_l^{\,x}, A_r^{\,x}$ are real numbers depending on x. The following statement defines a *partial order* in the set of all FNs:

Definition 5: Given the FNs A and B we write $A \le B$ (or $\ge$) if, and only if, $A_l^{\,x} \le B_l^{\,x}$ and $A_r^{\,x} \le B_r^{\,x}$ (or $\ge$) for all x in [0, 1]. Two FNs for which the above relations hold are called *comparable*, otherwise they are called *non comparable*.

3.2. Arithmetic operations on FNs

The basic arithmetic operations on FNs are defined in general in two alternative ways,

equivalent to each other:
(i) *With the help of their x-cuts and the Representation-Decomposition Theorem* for fuzzy sets[2]: In fact, if *A* and *B* are given FNs, and “*” denotes an arithmetic operation (addition, subtraction, multiplication or division) between them, then applying the above theorem for the fuzzy set *A * B* we find that $A * B = \sum_{x\in[0,1]} x(A*B)^x$. But the x-cuts of the FNs are ordinary closed real intervals, therefore, if we define that $(A * B)^x = A^x * B^x$ (where, for reasons of simplicity, “*” in the second term of the last equation denotes the corresponding operation between closed real intervals), the *fuzzy arithmetic is turned to the well known arithmetic of the closed real intervals* [3].
(ii) By applying the *Zadeh's extension principle* ([21]: Section 1.4, p.20), which provides the means for any function f mapping the crisp set X to the crisp set Y to be generalized so that to map fuzzy subsets of X to fuzzy subsets of Y.
In practice the above two general methods of the fuzzy arithmetic, requiring laborious calculations, are rarely used in applications, where the utilization of simpler forms of FNs is preferred.

4. Triangular fuzzy numbers (TFNs)

4.1. Definition and basic properties of TFNs

The membership function's graph of the TFN (*a, b, c*), where $a< b < c$ are given real numbers, is represented in Figure 3. We observe that the membership function *y=m(x)* of it takes constantly the value 0, if x is outside the interval [*a, c*], while its graph in the interval [*a, c*] is the union of two straight line segments forming a triangle with the X-axis.

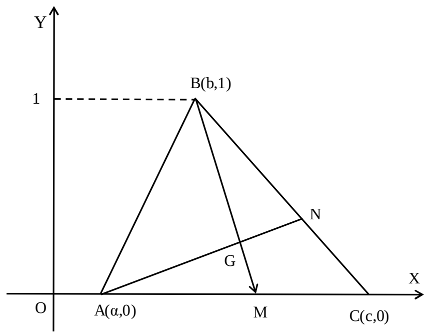

Figure 3: Graph and COG of the TFN (a, b, c)

Therefore, the analytic definition of a TFN is given as follows:

[2] The Representation-Decomposition Theorem of Ralesscou-Negoita ([25]: Theorem 2.1, p. 16) states that a fuzzy set A can be completely and uniquely expressed by the family of its x-cuts in the form
$A = \sum_{x\in[0,1]} xA^x$.

[3] We recall that an arithmetic operation “*” between closed real intervals is defined by the general rule
$[a, b] * [a_1, b_1] = \{x * y: x, y \in \boldsymbol{R}, a \leq x \leq a_1, b \leq y \leq b_1\}$ [24].

Definition 6: Let *a*, *b* and *c* be real numbers with *a* < *b* < *c*. Then the *Triangular Fuzzy Number (TFN)* A = (*a*, *b*, *c*) is a FN with membership function:

$$y = m(x) = \begin{cases} \dfrac{x-a}{b-a}, & x \in [a,b] \\ \dfrac{c-x}{c-b}, & x \in [b,c] \\ 0, & x < a \text{ or } x > c \end{cases}$$

In the above definition we obviously have that *m(b)=1*, while *b* need not be in the "middle" of *a* and *c*.

The following two Propositions refer to basic properties of TFNs that we are going to use later in this paper:

Proposition 1. The x-cuts A^x of a TFN A = (*a*, *b*, *c*), x∈ [0, 1], are calculated by the formula A^x = [A_l^x, A_r^x] = [*a* + x(*b* - *a*), *c* - x(*c* - *b*)] .

Proof: Since A^x = {*y*∈ ***R***: *m(y*≥ x}, Definition 6 gives for the case of A_l^x that $\frac{y-a}{b-a}$ = x ⇔ *y* = *a* + x(*b* – *a*). Similarly for the case of A_r^x we have that $\frac{c-y}{c-b}$ = x ⇔ *y* = *c* - x(*c* - *b*).

Proposition 2. The coordinates (*X*, *Y*) of the COG of the triangle forming the graph of the TFN (*a*, *b*, *c)* are calculated by the formulas $X = \frac{a+b+c}{3}$, $Y = \frac{1}{3}$.

Proof: The graph of the TFN (*a*, *b, c)* is the triangle ABC of Figure 3, with A (*a*, 0), B (*b*, 1) and C (*c*, 0). Then, the COG, say G, of ABC is the intersection point of its medians AN and BM, where N ($\frac{b+c}{2}, \frac{b}{2}$) and M ($\frac{a+c}{2}$, 0). Therefore the equation of the straight line on which AN lies is $\dfrac{x-a}{\frac{b+c}{2}-a} = \dfrac{y}{\frac{1}{2}}$, or *x* + *(2a - b- c)y = a* (4).

In the same way one finds that the equation of the straight line on which BM lies is
2*x + (a + c +2b)y = a + c* (5)

Since D = $\begin{vmatrix} 2 & a+c-2b \\ 1 & 2a-b-c \end{vmatrix} = 3(a-c) \neq 0$, the linear system of (4) and (5) has a unique solution with the respect to the variables *x* and *y* determining the coordinates of the triangle's COG.

The proof of the Proposition is completed by observing that

D_x= $\begin{vmatrix} a+c & a+c-2b \\ a & 2a-b-c \end{vmatrix} = a^2 - c^2 + ba - bc = (a+c)(a-c) + b(a-c)$

= $(a-c)(a+c+b)$ and D_y = $\begin{vmatrix} 1 & a+c \\ 2 & a \end{vmatrix} = c-a$.

4.2. Arithmetic operations on TFNs

It can be shown that the two general methods for defining arithmetic operations on FNs presented in the previous Section lead to the following simple rules for the *addition* and *subtraction* of TFNs:

Let A = (a, b, c) and B = (a_1, b_1, c_1) be two TFNs. Then

- The sum A + B = ($a+a_1, b+b_1, c+c_1$).
- The difference A - B = A + (-B) = ($a-c_1, b-b_1, c-a_1$), where –B = ($-c_1, -b_1, -a_1$) is defined to be the *opposite* of B [4].

In other words, the opposite of a TFN, as well as the sum and the difference of two TFNs are also TFNs.

On the contrary, the product and the quotient of two TFNs, although they are FNs, *they are not always TFNs*. However, in the special case where a, b, c, a_1, b_1, c_1 are in $\boldsymbol{R}^+$, it can be shown that the fuzzy operations of *multiplication* and *division* of TFNs can be *approximately performed* by the rules:

- The product A . B = (aa_1, bb_1, cc_1).
- The quotient A : B = A . B^{-1} = $(\frac{a}{a_1}, \frac{b}{b_1}, \frac{c}{c_1})$, where B^{-1} = $(\frac{1}{a_1}, \frac{1}{b_1}, \frac{1}{c_1})$ is defined to be the *inverse* of B.

In other words, in $\boldsymbol{R}^+$ the inverse of a TFN, as well as the product and the division of two TFNs can be approximately considered to be TFNs too.

Further, one can define the following two *scalar operations*:

- k + A= (k+a, k+b, k+c), k$\in \boldsymbol{R}$
- kA = (ka, kb, kc), if k>0 and kA = (kc, kb, ka), if k<0.

We close with the following definition, which will be proved useful in the next Section of this paper for the assessment of learning skills using the TFNs :

Definition 7: Let A_i (a_i, b_i, c_i) be n TFNs, where n is a non negative integer, $n \geq 2$. Then we define the *mean value* of the above TFNs to be the TFN A = $\frac{1}{n}$ ($A_1 + A_2 + \ldots + A_n$).

5. Use of the TFNs for student assessment

In this Section we utilize the TFNs as an alternative tool for student assessment. The effectiveness of this approach is validated by comparing the results obtained with the corresponding results of other assessment methods already established in earlier works (see the second Section of this paper). All these are materialized through the following classroom application on learning mathematics:

5.1. The classroom application

Mathematical activity is an original and natural element of the human cognition.

[4] Obviously A + (-A) = (a-c, 0, c-a) $\neq$ O = (0, 0, 0), where the TFN O is defined by O(x) = 1, if x = 0 and O(x)=0, if x $\neq$ 0

Therefore, it is of great importance to experiment on effective ways of evaluating the student skills for learning mathematics. This gave us the impulse to perform the following classroom experiment, which is based on the traditional and fuzzy assessment methods for learning presented in this paper.

The experiment took place recently at the Graduate Technological Educational Institute (T. E. I.) of Western Greece and it was related to the teaching (by the same instructor) of the definite integral to the students of two different Departments of the School of Management and Economics within their common course “Mathematics for Economists I” of their first term of studies.

The duration of the corresponding lecture was three hours for each Department, but the teaching methods followed were different: In fact, for the first Department (*control group*) the lecture was performed in the classical way on the board, starting with the presentation of the relevant theoretical results (the details of some proofs were omitted), which was followed by the detailed solution of a number of suitably chosen exercises and problems ([26]: Chapter 17). The students were able to ask questions, but not to participate in the solutions’ procedure. In this way the instructor saved time resulting to the solution of more exercises and problems on the board.

On the contrary, for the second Department (*experimental group*), the instructor followed the process of *rediscovery* [27], keeping in mind what Polya [28] says for *active learning:* “For an effective learning the learner discovers alone the biggest possible, under the circumstances, part of the new information”. Thus, in his short introduction he presented the concept of the definite integral through the need of calculating the area under a curve, but he stated the fundamental theorem of the integral calculus - connecting the indefinite (that have been already taught earlier) with the definite integral of a continuous in a closed interval function - without proof. Then he left students to work alone on their drafts and he was inspecting their efforts and reactions, giving to them from time to time suitable hints or instructions. His intension was to help students to understand the basic methods of calculating a definite integral in terms of the already known corresponding methods for the indefinite integral; step of interpretation of the Voss’s [4] framework for learning; see our Introduction.

Next, the instructor gave to students for solution a number of exercises involving calculation of improper integrals as limits of definite integrals and also calculation of the area under a curve, or among curves. In this way he wanted to help students to generalize the new information to a variety of situations (step of generalization). Finally, integrating his lecture, he presented for solution a number of composite problems involving applications to economics, such as calculation of the present value in cash flows, of the consumer’s and producer’s surplus resulting from the change of prices of a given good, of probability density functions, etc (cf. [26], chapter 17). In this way he wanted to help students to relate the new information to their existing knowledge structures (step of categorization). Obviously the teaching method followed for the experimental group was consuming time, which means that part of the above composite problems was left to students as homework. The lectures for the definite integral were followed (the next week) by a written test (exam) for checking the student progress. Students achieved the following scores (in a climax from 0 to 100) in this test:
First Department (D_1): 100(2 times), 99(3), 98(5), 95(8), 94(7), 93(1), 92 (6), 90(5), 89(3), 88(7), 85(13), 82(6), 80(14), 79(8), 78(6), 76(3), 75(3), 74(3), 73(1), 72(5), 70(4),

68(2), 63(2), 60(3), 59(5), 58(1), 57(2), 56(3), 55(4), 54(2), 53(1), 52(2), 51(2), 50(8), 48(7), 45(8), 42(1), 40(3), 35(1).
Second Department (D_2) *:* 100(1), 99(2), 98(3), 97(4), 95(9), 92(4), 91(2), 90(3), 88(6), 85(26), 82(18), 80(29), 78(11), 75(32), 70(17), 64(12), 60(16), 58(19), 56(3), 55(6), 50(17), 45(9), 40(6).

5.2. Assessment of the application's data

Traditional Methods: Calculating the means of the above scores, one approximately finds the values $\frac{12314}{170} \approx 72.44$ for D_1 and $\frac{18369}{255} \approx 72.04$ for D_2 respectively, showing that D_1 demonstrated a slightly better *mean performance* than D_2.
Next, summarizing the student scores presented above with respect to the grades (linguistic labels) A, B, C, D and F defined earlier (see paragraph for the GPA index), one forms Table 1 as follows:

TFN	D_1	D_2
A	60	60
B	40	90
C	20	45
D	30	45
F	20	15
Total	170	255

Table 1: Students' performance in terms of the linguistic grades

Replacing the data of Table 1 in formula (1) one finds for D_1 the value GPA = $\frac{30+2*20+3*40+4*60}{170} = \frac{430}{170} \approx 2.529$ and similarly the same value for D_2. This means that both Departments demonstrated the same *quality performance*, which can be characterized as more than satisfactory, since the value 2.529 found for the GPA index is greater than the half of its maximal possible value (4:2=2).
The TpFAM/TFAM methods: From Table 1 one easily calculates the percentages of the students of D_1 who obtained the grades F, D, C, B and A respectively, which are the following: $y_1 = y_3 = \frac{2}{17}$, $y_2 = \frac{3}{17}$, $y_4 = \frac{4}{17}$, $y_5 = \frac{6}{17}$. Replacing these values in the first of formulas (2) or (3) one finds that the x-coordinate of the COG of the corresponding scheme for D_1 is equal to $X_c = 7(\frac{2+2*3+3*2+4*4+5*6}{17}) - 2 =$

$\frac{386}{17}\approx$ 22.7. Working similarly one also finds the same value of X_c for D_2. Therefore, in order to compare the two Departments' performance one must also calculate the y-coordinates Y_c of the corresponding COGs. This is done by replacing the values of the y_i, for i=1, 2, 3, 4, 5, in the second of formulas (2) or (3). For example, the second of (3) gives for D_1 that

$Y_c = \frac{1}{5}[(\frac{2}{17})^2+(\frac{3}{17})^2+(\frac{2}{17})^2+(\frac{4}{17})^2+(\frac{6}{17})^2]=\frac{69}{1445}$. In the same way one finds for D_2 the value $Y_c=\frac{71}{1445}$. But $X_c \approx 22.7 > 19$, therefore, according to the corresponding criterion (see paragraph for the TpFAM), D_2 demonstrated a slightly better *quality performance* than D_1.

Notice also that in case of the ideal performance ($y_5 = 1$, $y_1 = y_2 = y_3 = y_4 = 0$) the first of formulas (2) or (3) give that $X_c = 33$. Therefore, since the value of $X_c \approx 22.7$ found for both Departments is greater than the half of its value corresponding to the ideal performance (33:2 = 16.5), the quality performance of the two Departments can be characterized as more than satisfactory.

Finally we observe that, although according to the GPA index the two Departments demonstrated the same quality performance, the TpFAM/TFAM methods have shown that D_2 demonstrated a slightly better than D_1 quality performance. In order to explain this difference, observe first that formula (1) calculating the GPA index can be written in terms of the student percentages in the form GPA = $y_2 + 2y_3 + 3y_4 + 4y_5$. Then, a simple observation of the last formula and the first of formulas (2) or (3) combined with the corresponding criterion for the comparison of the group performance (depending on the values of X_c), shows that the TpFAM/TFAM methods assign greater coefficients (weights) to the higher scores than the COG index. In other words, *the TpFAM/TFAM methods are more sensitive than the GPA index to the higher scores* and this explains the above difference.

Use of the TFNs: Let us now come to the core of this section, which is the use of TFNs as an alternative tool for learning assessment. For this, we assign to each linguistic grade a TFN (denoted, for simplicity, by the same letter) as follows: A= (85, 92.5, 100), B = (75, 79.5, 84), C = (60, 67, 74), D= (50, 54.5, 59) and F = (0, 24.5, 49). Namely, the middle entry of each TFN is equal to the mean value of the student scores previously assigned to the corresponding linguist label (grade). In this way a TFN corresponds to each student assessing his (her) *individual performance*. The replacement of the linguistic grades by TFNs for the individual student assessment *have the advantage of determining numerically the scores assigned to each grade*, which, as we have already seen, are not standard, since they may slightly differ from case to case.

It is of worth to notice here that in an earlier work [29] an assessment of the student individual performance in problem solving was attempted by assigning to each student an *ordered triple of linguistic grades* characterizing his (her) performance in the three main steps of the problem solving process. In the same work it was shown that this approach is equivalent to the A. Jones method [30] of assessing a student's knowledge in terms of his (her) *fuzzy deviation with respect to the teacher*.

The same approach can be also applied here for assessing the individual student learning skills. For example, the ordered triple (A, B, C) could be assigned to a student who demonstrated an excellent performance at the step of interpretation, a very good performance at the step of generalization and a good performance at the step of categorization. However, in this way the overall performances of two different students are not always comparable. For example this happens with two students with profiles (A, B, C) and (B, B, B) respectively. Mathematically speaking, this approach defines a *partial order* only on the student individual performances; e.g. a student with profile (A, B, C) demonstrates a better performance than one with profile (B, B, D), etc. Further, this approach is laborious requiring an independent evaluation of the student performance at *each step* of the learning process, which could not be practically possible, since the boundaries between these steps are not always clear.

After this parenthesis, let us return to the TFNs. We observe that in Table 1 we actually have 170 TFNs representing the individual performance of the students of D_1 and 255 TFNs representing the individual performance of the students of D_2. Therefore, it is logical to accept that the overall performance of each Department can be represented by the corresponding mean values of the above TFNs (see Definition 7). For simplifying our notation, let us denote the above means by the letter of the corresponding Department. Then, making straightforward calculations, one finds that

$$D_1 = \frac{1}{170}.(60A+40B+20C+30D+20F) \approx (63.53, 71.74, 83.47) \text{ and}$$

$$D_2 = \frac{1}{255}.(60A+90B+45C+45D+15F) \approx (65.88, 72.63, 79.53).$$

The above TFNs (mean values) give us the following information:

(i) The overall performance of D_1 is characterized numerically by a score lying in the interval [63.53, 83.47], i.e. from good (C) to very good (B). Similarly, the performance of D_2 is characterized by a score lying in the interval [65.88, 79.53].

(ii) The middle entries 71.74 and 72.63 of the two TFNs give a *rough approximation* (C=good) of the scores characterizing numerically the performance of D_1 and D_2 respectively.

But, let us explain why we have characterized the values of the middle entries of the TFNs D_1 and D_2 as been rough approximations of the corresponding scores. We observe first that these values *do not calculate the mean performances of the two Departments*. In fact, calculating the means of the student scores in the classical way we found above (see Traditional Methods) the values 72.44 and 72.04 respectively, demonstrating a slightly better mean performance for D_1. Let us now go back to the definition of the TFNs A, B, C, D and F. The middle entries of these TFNs were chosen to be equal to the means of the scores assigned to each of the corresponding linguistic grades. Therefore the middle entries of the TFNS D_1 and D_2 are actually *equal to the mean values of these means*, which justifies completely the characterization "rough" given to them.

Thus, the question is how one can compare the overall performances of the two Departments. If the TFNS D_1 and D_2 are comparable (see Definition 5), the answer to this question is easy. For example, if $D_1 < D_2$, then D_2 demonstrates a better performance than D_1. Therefore, it becomes necessary to check if the TFNs D_1 and D_2 obtained above are comparable or not.

For this, by Proposition 1 one finds that the x-cuts of the two TFNs are D_1^x = [63.53+8.21x, 83.47-11.73x] and D_2^x = [65.88+6.75x, 79.53-6.9x] respectively for all x in [0, 1]. Further, we have that $63.53+8.21x \leq 65.88+6.75x \Leftrightarrow 1.46x \leq 2.35 \Leftrightarrow x \leq 1.61$, which is true for all x in [0, 1]. But $83.47-11.73x \leq 79.53-6.9x \Leftrightarrow 3.94 \leq 4.83x \Leftrightarrow 0.82 \leq x$, which does not hold for all x in [0, 1]. Therefore, according to Definition 5, the TFNs D_1 and D_2 are not comparable, which means that one *can not immediately decide which of the two Departments demonstrates the better performance.* A good way to overcome this difficulty is to *defuzzify our fuzzy outputs*, i.e. the TFNs D_1 and D_2. For this, we apply the COG defuzzification technique. In fact, by Proposition 2, the COGs of the triangles forming the graphs of the TFNs D_1 and D_2 have x-coordinates equal to $X = \frac{63.53+71.74+83.47}{3} \approx 72.91$ and $X' = \frac{65.88+72.63+79.53}{3} \approx 72.68$ respectively.

Observe now that the GOGs of the graphs of D_1 and D_2 lie in a rectangle with sides of length 100 units on the X-axis (student scores from 0 to 100) and one unit on the Y-axis (normal fuzzy sets). Therefore, *the nearer the x-coordinate of the COG to 100, the better the corresponding Department's performance*, Thus, since $X > X'$, D_1 demonstrates a better overall performance than D_2.

6. Discussion and conclusions

In the present paper we used the TFNs as a tool for student assessment. The main advantage of this approach is that in case of *individual assessment* leads to a numerical result, which is more indicative than the qualitative results obtained in earlier works by applying alternative fuzzy assessment methods. On the contrary, in case of *group assessment* this approach *initially leads to a linguistic characterization of the corresponding group's overall performance, which is not always sufficient for comparing the performances of two different groups*, as our fuzzy assessment methods applied in earlier works do. This is due to the fact that the inequality between TFNs defines on them a relation of partial order only. In such cases *some extra calculations are needed* in order to obtain the required comparison by defuzzifying the resulting TFNs. This could be considered a disadvantage of this approach, although the extra calculations needed are very simple.

Concerning our classroom experiment on measuring the student learning skills, notice first that the student scores obtained in the Panhellenic Exam for entering the Tertiary Education were at the same level for both Departments. This means that the potential of the two Departments concerning their student competencies on the secondary mathematics was almost the same. Therefore, since the *mean performance* of D_1 as well as its *overall performance* assessed using the TFNs were proved to be better than D_2, it seems that the students of D_1 (control group) were helped better in general by the application of the classical method of teaching the definite integral, since in this way they had the opportunity to see on the board more applications solved in detail by the instructor. However, according to the TpFAM/TFAM model D_2 (control group) demonstrated a slightly better quality performance than D_1 (the quality performances were proved to be identical according to the GPA index), which could mean that the good students of D_2 (higher scores) were benefit by the application of the rediscovery method. At any case, all the above are weak indications only, since the performance differences were very small in all cases. In concluding, more experimental research is needed for obtaining statistically safer conclusions about the effectiveness of rediscovery as a teaching method for mathematics,

Further, our new method of using the TFNs for learning assessment is of general character, which means that it could be utilized in future for assessing other human (or machine) activities too. Further, the utilization of other types of FNs as assessment tools could be of particular interest. For example, *trapezoidal FNs* [24] of the form (a, b, c, d) could be used in cases where one wants to assess the possibility of a value to be approximately in the interval [b, c]. All the above constitute targets of our future research on the subject.

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