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A Coupled Coincidence Point Theorem in a G-Complete Fuzzy Metric Space

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ABSTRACT
There is a large interest in coupled fixed point and related problems in different spaces. In this paper a coupled coincidence point theorem is established in a fuzzy metric space which is assumed to be G-complete. Open problems are given.

Keywords: Fuzzy metric space, G-completeness, t-norm, Coupled coincidence point

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1. Introduction

Fuzzy concepts, after being introduced by Zadeh[18] made quick headways into different branches of mathematics. Particularly, a version of fuzzy metric space was introduced in [10] which was modified by George et al in [8]. The space introduced in [8] served a fertile field for the growth of fixed point theory. Some examples of works in fixed point and related problem are noted in [2,3,4,6,12,13]. One of the reasons for this is that the topology in this space is Hausdorff which is essential for the development of metric fixed point theory. The purpose of the paper is to establish some coupled coincidence point result in the space mentioned above. We begin with the following mathematical preliminaries.

Definition 1.1. [15] A binary operation $*: [\mathbf{0},\mathbf{1}]^{\mathbf{2}} \to [\mathbf{0},\mathbf{1}]$ is called a *t*-norm if the following properties are satisfied:

(i) $*$ is associative and commutative,
(ii) $a * 1 = 1$ for all $a \in [\mathbf{0},\mathbf{1}]$,
(iii) $a * b \le c * d$ whenever $a \le c$ and $b \le d$, for all $a,b,c,d \in [\mathbf{0},\mathbf{1}]$.

Examples of continuous t--norms are $a *_1 b = \min\{a,b\}$, $a *_2 b = \frac{ab}{Max\{a,b,\lambda\}}$ for $0 < \lambda < 1$ and $a *_3 b = \text{ab}$ for all $a,b \in [\mathbf{0},\mathbf{1}]$.

Definition 1.2.Fuzzy metric space in the sense of Kramosil and Michalek [10] The 3-tuple $(X,M,*)$ is called a fuzzy metric space if $\boldsymbol{X}$ is an arbitrary non-empty set, $*$ is a *t*-norm and M is a fuzzy set on $X^2 \times [0,\infty)$ satisfying the following conditions:

(i) $M(x,y,0) = 0$,
(ii) $M(x,y,t) = 1$ for all $t > 0$ if and only if $x = y$,

(iii) $M(x, y, t) = M(y, x, t)$,
(iv)$M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ and
(v)$M(x, y, .) : [0, \infty) \longrightarrow [0,1]$ is left-continuous , where $t, s > 0$ and $x, y, z \in X$.

George and Veeramani [8] after modification of the above definition introduced the following one. We call it simply as a fuzzy metric space and we consider only this space here.

Definition 1.3.[8] The 3-tuple $(X, M, *)$ is called a fuzzy metric space if X is an arbitrary non-empty set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions for all $x, y, z \in X$ and $t, s > 0$:
$(i)$$M(x, y, t) > 0$,
$(ii)$$M(x, y, t) = 1$ if and only if $x = y$,
(iii) $M(x, y, t) = M(y, x, t)$,
(iv)$M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ and
(v) $M(x, y, .) : (0, \infty) \longrightarrow [0,1]$ is continuous.

Let $(X, M, *)$ be a fuzzy metric space. For $t > 0$,the open ball $B(x, r, t)$ with center $x \in X$ and radius r, $0 < r < 1$ is defined by

$$B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}.$$

A subset $A \subset X$ is called open if for each $x \in A$, there exist $t > 0$ and $0 < r < 1$ such that $B(x, r, t) \subset A$. Let τ denote the family of all open subsets of X. Then τ is called the topology on X induced by the fuzzy metric M. This topology is Hausdorff and first countable [8].

Example 1.4. [8] Let $\boldsymbol{X} = \mathbb{R}$. Let $a * b = a \,.\, b$ for all $a, b \in [0,1]$. For each $t \in (0, \infty)$, let

$$M(x, y, t) = \frac{t}{t + |x - y|}$$

for all $x, y \in X$. Then $(\mathbb{R}, M, *)$ is a fuzzy metric space.

Example 1.5. Let (X, d) be a metric space and ψ be an increasing and a continuous function of $\mathbb{R}_+$ into $(0,1)$ such that $\lim_{t\to\infty} \psi(t) = 1$. Three generic examples of these functions are $(t) = \frac{t}{t+1}$, $\psi(t) = \sin\left(\frac{\pi t}{2t+1}\right)$ and $\psi(t) = 1 - e^{-t}$. Let $*$ be any continuous t-norm. For each $t \in (0, \infty)$, let $M(x, y, t) = \psi(t)^{d(x,y)}$ for all $x, y \in X$. Then, $(X, M, *)$ is a fuzzy metric space.

Definition 1.6. [8, 11] Let $(X, M, *)$ be a fuzzy metric space.
(i) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if $\lim_{n\to\infty} M(x_n, x, t) = 1$ for each $t > 0$.
(ii) A sequence $\{x_n\}$ in X is called a Cauchy sequence if for each $0 < \varepsilon < 1$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m \geq n_0$.

(iii)A fuzzy metric space in which every Cauchy sequence is convergent, is said to be complete.

(iv)A sequence $\{x_n\}$ in X is a G-Cauchy sequence if $\lim_{n\to\infty} M(x_n, x_{n+p}, t) = 1$ for all t > 0 and fixed positive integer p.

(v)X is G- complete if every G- Cauchy sequence is convergent.

We use the following lemma in our main result.

Lemma 1.7. [14] $\boldsymbol{M}$ is a continuous function on $X^2 \times (0,\infty)$.

As mentioned earlier, the purpose here is to prove a coupled coincidence point result in fuzzy metric spaces. Such types of results have occupied recent interest of mathematicians, exampled of these works being [1,5,7,11] in metric spaces and [3,9, 16] in fuzzy metric spaces. The following are some definitions related to it.

Definition 1.8. [1] An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $F: X \times X \to X$ if $F(x, y) = x, F(y, x) = y$.

Further Lakshmikantham and Ćirić have introduced the concept of coupled coincidence point.

Definition 1.9. [11] An element $(x, y) \in X \times X$ is called a coupled coincidence point of the mappings $F: X \times X \to X$ and $g : X \to X$ if $F(x, y) = g(x)$, $F(y, x) = g(y)$.
If, in particular, $x = g(x) = F(x, y)$ and $y = g(y) = F(y, x)$, then (x, y) is a coupled common fixed point of g and F .

Definition 1.10. [11] Let $\boldsymbol{X}$ be a non-empty set and $F: X \times X \to X$ and $g : X \to X$. We say that F and g commute if $g\big(F(x, y)\big) = F(g(x), g(y))$ for all $x, y \in X$.

Lemma 1.11.[17] If for a sequence $\{x_n\}$, $M(x_n, x_{n+1}, t) \to 1$ $as\ n \to \infty$, then $\{x_n\}$ in X is a convergent sequence.

2. Main results

Theorem 2.1. Let $(X, M,*)$ be a G-complete fuzzy metric space where $*$ satisfies $a * b \geq a.b$ for all $a, b \in [0,1]$. Let there be functions $F: X \times X \to X$ and $g : X \to X$ such that

$$\tfrac{1}{2}[M(F(x,y),F(u,v),t)+ M(F(y,x), F(v,u),t)] \geq \gamma(\tfrac{1}{2}(M(g(x),g(u),t)+ M(g(y),g(v),t))) \quad (2.1)$$

for either $x \neq y$ or $u \neq v$.
$x, y, u, v \in X, t > 0$ where $\gamma : [0,1] \to [0,1]$ is a monotone increasing continuous function such that $\gamma(a) > \sqrt{a}$ for each $a \in (0,1)$. Let g be continuous, commute with F and is such that $F(X \times X) \subseteq g(X)$. Then g and F have a coupled coincidence point.

Proof. Let $x_0, y_0 \in X$. Since $F(X \times X) \subseteq g(X)$, we can choose $x_1, y_1 \in X$ such that $g(x_1) = F(x_0, y_0)$ and that $g(y_1) = F(y_0, x_0)$ for some $x_0, y_0 \in X$. Again we can choose $x_2, y_2 \in X$ such that $g(x_2) = F(x_1, y_1)$ and $g(y_2) = F(y_1, x_1)$

Continuing this process, we construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that
$g(x_{n+1}) = F(x_n, y_n)$ *and* $g(y_{n+1}) = F(y_n, x_n)$ for all $n \geq 0$ (2.2)

If $x_n = x_{n+1}$, $y_n = y_{n+1}$, for some $n \geq 0$ then existence of coupled coincidence point is established. So we assume either $x_n \neq x_{n+1}$ or $y_n \neq y_{n+1}$
Let, for all $t > 0$, $n \geq 0$,

$$\delta_n(t) = \frac{1}{2}[M(g(x_n), g(x_{n+1}), t) + M(g(y_n), g(y_{n+1}), t)]. \tag{2.3}$$

From (2.1) and (2.2), for all $t > 0$, $n \geq 1$, with $x = x_{n-1}$, $y = y_{n-1}$, $u = x_n$, $v = y_n$ we have

$$\frac{1}{2}[M(F(x_{n-1},y_{n-1}),F(x_n,y_n),t)+M(F(y_{n-1},x_{n-1}),F(y_n,x_n),t)]$$
$$\geq \gamma(\tfrac{1}{2}(M(g(x_{n-1}),g(x_n),t)+M(g(y_{n-1}),g(y_n),t))) ,$$

that is, $$\frac{1}{2}[M(g(x_n),g(x_{n+1}),t)+M(g(y_n),g(y_{n+1}),t)]$$
$$\geq \gamma(\tfrac{1}{2}(M(g(x_{n-1}),g(x_n),t)+ M(g(y_{n-1}),g(y_n),t))) ,$$

that is,

$$\delta_n(t) \geq \gamma(\delta_{n-1}(t))$$

Then $\delta_n(t) \geq \delta_n(t) * \delta_n(t) \geq \gamma(\delta_{n-1}(t)) * \gamma(\delta_{n-1}(t)) \geq (\gamma(\delta_{n-1}(t)))^2 > \delta_{n-1}(t)$
(2.4)

(Since $a * b \geq ab, \gamma(a) > \sqrt{a}$). for 0<a<1

Thus for each $t > 0, \{\delta_n(t); n \geq 0\}$ is an increasing sequence in [0,1] and hence tends to a limit $a(t) \leq 1$. We claim that $a(t) = 1$ for all $t > 0$. If there exists t> 0 such that $a(t) < 1$, then taking limit as $n \to \infty$ for $t = t_0$ in (2.4), and using the properties of γ, we get $a(t) \geq (\gamma(a(t)))^2 > a(t)$, which is a contradiction. Hence $a(t) = 1$ for every $t > 0$, that is, for all $t > 0$,

$$\lim_{n\to\infty} \frac{1}{2}[M(g(x_{n-1}), g(x_n), t) + M(g(y_{n-1}), g(y_n), t)] = 1$$

which implies that

$$\lim_{n\to\infty} M(g(x_{n-1}), g(x_n), t) = \lim_{n\to\infty} M(g(y_{n-1}), g(y_n), t) = 1 \tag{2.5}$$

Then, by an application of lemma1.11, $\{g(x_n)\}$ and $\{g(y_n)\}$ are convergent sequences. Therefore, there exist $x, y \in X$ such that

$$\lim_{n\to\infty} g(x_n) = x \text{ and } \lim_{n\to\infty} g(y_n) = y . \tag{2.6}$$

From (2.6), and the continuity of g, we obtain
$\lim_{n\to\infty} g(g(x_n)) = g(x)$ and $\lim_{n\to\infty} g(g(y_n)) = g(y)$ (2.7)

From (2.2) and the commutativity of g and F , for all $n \geq 0$, we have

$$g(g(x_{n+1})) = g(F(x_n, y_n)) = F(g(x_n), g(y_n)), \tag{2.8}$$

and

$$g(g(y_{n+1})) = g(F(y_n, x_n)) = F(g(y_n), g(x_n)), \tag{2.9}$$

Taking $n \to \infty$ in (2.8) and (2.9), using (2.7) and the continuity of F, we obtain

$$g(x) = F(x,y) \text{ and } g(y) = F(y,x).$$

Then (x,y) is a coupled coincidence point of g and F. This completes the proof of the theorem.

Open problems: It may be investigated whether the result obtained here is also true in complete fuzzy metric spaces. Also whether the result is valid under more general conditions is also a point to be investigated.

REFERENCES

1. T. G. Bhaskar, V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, *Nonlinear Anal.*, 65 (2006)1379-1393.
2. B. S. Choudhury, K. P. Das, S. K. Bhandari and P. Das, Tripled coincidence point results in partially ordered probabilistic metric spaces, *Journal of Physical Sciences*, 16 (2012) 9-22.
3. B. S. Choudhury, K. P. Das, P. Das, Coupled coincidence point results for compatible mappings in partially ordered fuzzy metric spaces, *Fuzzy Sets and Systems*,222(2013) 84-97.
4. B.S. Choudhury and P.N. Dutta, Fixed point result for a sequence of mutually contractive self-mappings on fuzzy metric spaces, *The Journal of Fuzzy Math.*, 13(2005) 723-730.
5. B.S. Choudhury and A. Kundu, A coupled coincidence point result in partially ordered metric spaces for compatible mappings, *Nonlinear Anal.*, 73 (2010) 2524-2531.
6. L. Ćirić, Some new results for Banach contractions and Edelstein contractive mappings on fuzzy metric spaces, *Chaos, Solitons and Fractals*, 42(2009) 146-154.
7. L. Ćirić, V. Lakshmikantham, Coupled random fixed point theorems for nonlinear contractions in partially ordered metric spaces, *Stoc. Anal. Appl.*, 27 (2009) 1246-1259.
8. A. George and P. Veeramani, On some result in fuzzy metric space, *Fuzzy Sets and Systems*, 64 (1994) 395-399.
9. X.Q. Hu, Common coupled fixed point theorems for contractive mappings in fuzzy metric spaces, *Fixed Point Theory Appl.*, 2011 (2011), 8. doi:10.1155/2011/363716. Art. ID 363716.
10. I. Kramosil and J. Michalek, Fuzzy metric and statistical metric spaces, *Kybernetica.*, 11 (1975) 326-334.
11. V. Lakshmikantham, L. Ćirić, Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, *Nonlinear Anal.*,70(2009) 4341-4349.
12. D. Mihet, On fuzzy contractive mappings in fuzzy metric spaces, *Fuzzy Sets and Systems*, 158(2007) 915-921.

13. A. Razani, A contraction theorem in fuzzy metric space, *Fixed Point Theory Appl.*, 2005 (2005) 257-265.
14. J. Rodríguez López, S. Ramaguera, The Hausdorff fuzzy metric on compact sets, *Fuzzy Sets and Systems*, 147 (2004) 273-283.
15. B. Schweizer and A. Sklar, *Probabilistic Metric Spaces*, Elsevier, North-Holland, 1983.
16. S. Sedghi, B.S. Choudhury and N. Shobe, Strong common coupled fixed point result in fuzzy metric spaces, *Journal of Physical Sciences*, 17 (2013) 1-9.
17. C. Vetro, D. Gopal, M. Imdad, Common Fixed point theorems for (φ,ψ)- weak contractions in fuzzy metric spaces, *Indian J. Math*., 52(2010) 573-590.
18. L. A. Zadeh, Fuzzy sets, *Inform and Control*, 8 (1965) 338-353.