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Paracompact U-spaces

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ABSTRACT
This is the second in a series of papers on U- spaces. Here paracompactness has been introduced for U- spaces and many topological theorems related to paracompactness have been generalized to U- spaces, as an extension of study of supratopological spaces.

Keywords: Supratopology, U-space, paracompact, locally finite refinement, barycentric refinement.

1. Introduction

In a previous paper [1] we have introduced U- spaces and studied some of their properties. In this paper we use the terminology of [1]. Some study of these spaces was done previously in ([2,3,9,11]) in less general form, and the spaces were called supratopological spaces.

The concept of paracompactness for topological spaces was defined by Dieudonne [4]. This concept has been proved to be very important and useful. In this paper the notion of a paracompact U- space has been introduced and a number of sufficient conditions for paracompactness for such spaces have been established.

In connection with paracompactness of U- spaces. We have generalized the concepts of refinement, locally finite, countably locally finite, star and barycentric refinements in U- spaces and proved the U- space- versions of a few theorems concerning paracompact topological spaces (see [5,8,10]). A few relevant examples have been provided.

2. Paracompact U-spaces

We start with a few necessary definitions in U- spaces which generalize the corresponding topological concepts.

Definition 2.1. Let $\mathcal{G}$ be a collection of subsets of the U- space X. A collection $\mathcal{B}$ of subsets of X is said to be a U- refinement of $\mathcal{G}$ (or is said to refine $\mathcal{G}$) if for each element B of $\mathcal{B}$, there is an element $G\in \mathcal{G}$, such that $B\subseteq G$. If the elements of $\mathcal{B}$ are open sets, we call $\mathcal{B}$ a U-open refinement of $\mathcal{G}$; if they are closed sets, we call $\mathcal{B}$ a U- closed refinement of $\mathcal{G}$.

Definition 2.2. A collection $\mathcal{G}$ of subsets of a U- space X is locally finite if every point of X has a neighborhood that intersects only finitely many members of $\mathcal{G}$.

Thus, for a U-space X and a collection $\{A_\alpha\}$ of subsets of X, $\{A_\alpha\}$ is locally finite if, for each $x\in X$, there exists a U-open set G containing x such that $G\cap A_\alpha \neq \Phi$, for only a finite number of α's.

Locally finite collections are also called neighborhood- finite.

Example 2.1. Let $X = \mathbb{N}$ and let $\mathcal{U}$ consist of X, Φ and all subsets of $\mathbb{N}$ of the form $G_n = \{n, n + 1, n + 2, n + 3\}$ and their unions. Then $(X, \mathcal{U})$ is a proper U- space, since $G_1 \cap G_2 = \{2, 3, 4\} \notin \mathcal{U}$. Let $\mathcal{C}$ denote the family of sets $C_k = \{n\in \mathbb{N} \mid n \geq k\}$, $k\in \mathbb{N}$. Let $x\in X$. Then $x = n_0$, for some $n_0 \in \mathbb{N}$. For the neighborhood, $G_{n_o} = \{ n_0, n_0 +1, n_0 + 2, n_0 + 3\}$ of x, $G_{n_o} \cap C_k \neq \Phi$, only for $k = 1, 2, 3, \cdots, n_0 + 3$.

Hence $\mathcal{C}$ is locally finite.

Definition 2.3. A collection $\mathcal{C}$ of subsets of a U- space X is said to be countably locally finite if $\mathcal{C}$ is a countable union of locally finite collections $\mathcal{C}_n$ i.e., $\mathcal{C} = \bigcup_{n=1}^{\infty} \mathcal{C}_n$.

Example 2.2. Let $(X, \mathcal{U})$ be the proper U- space of example 2.1. For each positive integer k and m, let $\mathcal{C}_{k,m} = \{n\in \mathbb{N} \mid n \geq \frac{k}{m}\}$, Let $\mathcal{C}_m = \{ \mathcal{C}_{k,m} \}_{k\in N}$. For each m, $\mathcal{C}_m$ is locally- finite. Therefore $\mathcal{C} = \bigcup_m \mathcal{C}_m$ is countably locally- finite.

Definition 2.4. A U-space X is paracompact if X is Hausdorff and every U- open cover $\mathcal{C}$ of X has a locally finite U-open refinement of $\mathcal{C}$ that covers X.

Clearly, any compact Hausdorff U-space is paracompact.

We now give a non- trivial example of a paracompact U- space.

Example 2.3. Let $X = \mathbb{Z}$ and $\mathcal{U}$ = The collection of all A_n's and their unions, where for $n\in \mathbb{Z}$, $A_n =\{x\in X : n \leq x \leq n + 3\}$. Then, $\mathcal{U}$ is a U- structure but not a topology, since $A_1 \cap A_2 = \{x\in X : 2 \leq x \leq 4\}$ which does not belong to $\mathcal{U}$. Also, $(X,\mathcal{U})$ is Hausdorff. For, if $m, n\in \mathbb{Z}$, $m \neq n$, then let $m < n$, $m \in A_{m-3}$, $n \in A_n$, and $A_{m-3} \cap A_n = \Phi$.

We shall now show that every U-open cover of X has a locally finite refinement. Let $\mathcal{C}$ be a U-open cover of X. For each $x\in X$, $x\in A_{n,x} \subseteq G_x$, for some $A_{n,x}\in A_n$, where G_x is a member of $\mathcal{C}$.(Such $A_{n,x}$ and G_x exist. G_x exists because $\mathcal{C}$ is a U- open cover of X. And, by definition, G_x is a union of a class of A_n's at least one of which must contain x. Call this A_n $A_{n,x}$).

Let $\mathcal{A} = \{A_{n,x}: x\in X\}$. Then $\mathcal{A}$ is a refinement of $\mathcal{C}$ which covers X. Let $x_0\in X$ and let $G = A_{n, x_0}$. Then G is a U- open set containing x_0 and G intersects only seven members of $\mathcal{A}$, viz, A_{n-3, x_0}, A_{n-2, x_0}, A_{n-1, x_0}, A_{n, x_0}, A_{n+1, x_0}, A_{n+2, x_0}, A_{n+3, x_0}.

Thus, $\mathcal{A}$ is locally finite refinement of $\mathcal{C}$ which covers X.

Hence X is a paracompact U- space which is not a topological space.

It is clear that an infinite number of such proper paracompact U- spaces can be similarly constructed.

Our next example is a proper U- space which is not paracompact but in which every U-open cover has a locally finite refinement that covers X.

Example 2.4. Let X $=\mathbb{Z}$, fix $x_o \in \mathbb{Z}$. For each $x \in \mathbb{Z}$, let $A_x = \{x_o, x, x+1, x+2\}$. Let $\mathcal{U}$ be the collection of Φ, all A_x's, $x \in \mathbb{Z}$ and their unions. Then (X, $\mathcal{U}$) is U- space, but not a topological space. Since $A_{x+1} \cap A_{x+5} = \{x_o\} \notin \mathcal{U}$.

Let $\mathcal{G}$ be an U- open cover of X. Let $x \in X$. Then there is a $G_x \in \mathcal{G}$ such that $x \in G_x$ and so $x \in A_y$, for some $y \in X$. Let $\mathcal{D}$ be the collection of all sets B_y's such that for some y_o, $B_{y_o} = A_{y_o}$, and for each $y \neq y_o$, $B_y = A_y - \{x_o\}$.

Then $\mathcal{D}$ is a refinement of $\mathcal{G}$ and covers X. Now A_y is a U–open set containing x and it is clear that A_y intersects only a finite number of B_y's.
Thus $\mathcal{D}$ is a locally finite refinement of $\mathcal{G}$.
We now note that (X, $\mathcal{U}$) is not Hausdorff, since for each $x, y \in \mathbb{Z}$, $A_x \cap A_y \neq \Phi$. Hence X is not paracompact.

We recollect that the usual U- space $\mathbb{R}$ is $\mathbb{R}$ with the U – structure consisting of all subsets of $\mathbb{R}$ of the forms $(-\infty,a)$ and (b, ∞) and their unions.

Remark 2.1. $\mathbb{R}$ with the usual topology is paracompact. But the usual U- space $\mathbb{R}$ is not paracompact. We prove its truth below:

For $\mathcal{G} = \{(-\infty,a) | a \in \mathbb{R}\}$ is an open cover of $\mathbb{R}$. If $x \in \mathbb{R}$, and $x \in G$ with G is U-open, then G is the form $\bigcup_{i,j}[(-\infty,a_i) \cup (b_j,\infty)]$, for some a_i's, b_j's, and x belongs to some $(-\infty,a_i)$ or, some (b_j,∞).

If $\mathcal{D}$ is a refinement of $\mathcal{G}$ which covers $\mathbb{R}$, then $\mathcal{D}$ is a collection of sets of the form $(-\infty, c)$, where $c < a$, for each a with $(-\infty, a) \in \mathcal{G}$. Clearly, $\mathcal{D}$ is an infinite collection of U- open sets, and G meets infinitely many members of $\mathcal{D}$. So $\mathcal{D}$ is not locally finite.
Thus $\mathcal{G}$ has no locally finite refinement.

Let (X, $\mathcal{U}$) be a U- space and $\mathcal{T} = \mathcal{T}_{\mathcal{U}}$ be the topology generated by $\mathcal{U}$ on X. Then we have the following theorem.

Theorem 2.1. If (X, $\mathcal{U}$) is paracompact, then (X, $\mathcal{T}_{\mathcal{U}}$)= (X, $\mathcal{T}$) is paracompact.
Proof: Clearly (X, $\mathcal{T}$) is Hausdorff if (X, $\mathcal{U}$) is Hausdorff. Let $\mathcal{G}$ be an open cover of X in (X, $\mathcal{T}$). For each $x \in X$, there exists G_x in $\mathcal{G}$ such that $x \in G_x$. Then G_x contains a set H_x such that $x \in H_x$ and H_x is the intersection of a finite collection of sets $U_{1,x}, U_{2,x}, \ldots, U_{r,x}$ in $\mathcal{U}$. Choose any $U_{i,x}$ and call it U_x. Let $\mathcal{D} = \{U_x : x \in X\}$. Then, $\mathcal{D}$ is a U - cover of X.

Since (X, $\mathcal{U}$) is paracompact, $\mathcal{D}$ has a locally finite refinement say $\mathcal{D}'$ which covers X. For each y in X, let $y \in V_y \in \mathcal{D}$. Let $H_y = G_y \cap V_y$.

Then $\mathcal{C}' = \{H_y : y \in X\}$ is a open cover of X. $\mathcal{C}'$ is a locally finite refinement of $\mathcal{C}$, since $\mathcal{D}'$ is a locally finite refinement of $\mathcal{D}$. Thus $(X, \mathcal{T}_{\mathcal{U}})$ is paracompact.

Our next theorems are generalizations of Theorems in [8] (p. 160-161).

Theorem 2.2. Every paracompact U-space X is normal.

Proof: Let X be a paracompact U-space. Firstly, we shall show that X is regular. Let $x \in X$ and B be a U –closed subset of X, where $x \notin B$. since X is Hausdorff, for every $b \in B$ there exist two disjoint U- open sets U_b, V_b such that $x \in U_b$ and $b \in V_b$. So $x \notin \overline{V_b}$ .Then $\mathcal{C} = \{V_b\}_{b \in B} \cup \{X - B\}$ is a U- open cover of X. Since X is paracompact, there exists a locally finite refinement $\mathcal{D}$ of $\mathcal{C}$ which is a U-open cover of X. Let $\mathcal{E}$ be the sub collection of $\mathcal{D}$ consisting of all those members of $\mathcal{D}$ which intersect B. Then $\mathcal{E}$ is a U-open cover of B. Since for every $b \in B$, $x \notin \overline{V_b}$, so for every $E \in \mathcal{E}$, $x \notin \overline{E}$.

Let $W = \bigcup_{E \in \xi} E$, then W is U- open set of B. We shall now show that $\overline{W} = \bigcup_{E \in \xi} \bar{E}$ .Obviously, $\bigcup_{E \in \xi} \bar{E} \subseteq \overline{W}$. If possible, suppose $x \in \overline{W}$. Then for every U- open set G containing x, $G \cap W \neq \Phi$. Since E is locally finite, G intersects only a finite number of members say E_1, E_2, E_3,------,E_r of E.

Let $W_1 = E_1 \cup E_2 \cup E_3 \cup \text{-------} \cup E_r$ and $W_2 = \bigcup_{E \in \xi} E \quad {}_{E \neq E_1, E_2, E_3, \ldots\ldots\ldots, E_r}$.

So, $G \cap W_2 = \Phi$. This implies that $x \notin \overline{W}_2$. Since $\overline{W} = \overline{W}_1 \cup \overline{W}_2$, $x \in \overline{W}_1 = \overline{E}_1 \cup \overline{E}_2 \cup \overline{E}_3 \cup \text{-------} \cup \overline{E}_r$. So, $\overline{W} \subseteq \bigcup_{E \in \xi} \bar{E}$.

Thus, $\overline{W} = \bigcup_{E \in \xi} \bar{E}$. But this is a contradiction, since $x \notin \overline{E}$ for each E. So $x \in \overline{W}$ ..

Hence $\overline{W'}$ is a U- open set containing x. Therefore X is regular.

Now let A and B be two U- closed subsets. Since X is regular, for every $a \in A$ and for B there exist disjoint U- open set U_a and V_a such that $a \in U_a$ and $B \subseteq V_a$. One merely repeats the same argument, there exists a U- open set $W = \bigcup_{E \in \xi} E$ containing A, where (i) $\mathcal{E}$ is a locally finite U- open cover of A and (ii) Every $\overline{E} \cap B = \Phi$ .Since $\mathcal{E}$ is U-locally finite, $\overline{W} = \bigcup_{E \in \xi}^{-} \overline{E}$, and $B \subseteq \overline{W}$. Hence X is normal.

Theorem 2.3 Every U- closed subspace of a paracompact U- space is paracompact.

Proof: Let X be a paracompact U-space and Y be a U-closed subspace of X. Obviously, Y is Hausdroff. Let $\mathcal{C}^{/} = \{C^{/}_{\alpha}\}$ be a U-open cover set of Y. Then for each $C^{/}_{\alpha} = C_{\alpha} \cap Y$, where C_{α} is a U-open set of X. Now supposes $\mathcal{C} = \{C_{\alpha}\}$.

Then $\mathcal{D} = \{c_{\alpha}\} \cup \{Y^{c}\}$ is a U-open cover of X. Since X is a paracompact, there exists locally finite U- open cover $\mathcal{E}$ which is a refinement of $\mathcal{D}$. $\mathcal{G}$ is the sub collection of members of $\mathcal{E}$ whose members are not subsets of Y^c and $\mathcal{G}$ is refinement of $\mathcal{C}$. For this reason $\mathcal{G}$ is a U-open cover of Y and a refinement of $\mathcal{C}^{/}$. Since $\mathcal{E}$ is locally finite, $\mathcal{G}$ is locally finite.
Hence Y is paracompact**.**

Remark 2.2. A U- subspace of a paracompact U- space need not be paracompact.
Since this statement is true about topological spaces (see [8], p.161), it is also true about U- spaces.

For proof, we need to define a special U- structure on $\mathbb{R}$ which is called the lower limit U- structure. This U- space is denoted by $\mathbb{R}_l$.

Definition 2.5. Let $\mathcal{C}$ be the collection of subsets of the form [a, b) = {n $\mid a \leq x < b$ }, where a < b, the U- structure generated by $\mathcal{C}$ is called the lower limit U- structure on $\mathbb{R}$.

Theorem 2.4. Product of two paracompact U-spaces need not be paracompact.
Proof: As for topological spaces one can be shown that the U- space $\mathbb{R}_l$ is paracompact, but $\mathbb{R}_l \times \mathbb{R}_l$ is not normal, and hence, not paracompact.

We now generalize the theorems and lemma in [8](p. 162 - 166). The proofs are almost the same as those for topological spaces.

Theorem 2.5. Let X be a regular U-space and let $\mathcal{C}$ be a U- open cover of X. Consider the following conditions on $\mathcal{C}$: $\mathcal{C}$ has a refinement which is

(i) a U- open cover of X and count ably locally finite,
(ii) a cover of X and locally finite,
(iii) a U-closed cover of X and locally finite,
(iv) a U- open cover of X and locally finite.

Among the above four conditions on $\mathcal{C}$, the following implications hold; (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i) $\Rightarrow$ (ii).
Proof: It is trivial that (iv) $\Rightarrow$ (i).
(i) $\Rightarrow$ (ii) Let $\mathcal{G}$ be a U- open cover of X. Let $\mathcal{B}$ be an U-open refinement of $\mathcal{G}$ that covers X and is countably locally finite i.e. $\mathcal{B} = \cup \mathcal{B}_n$, where $\mathcal{B}$ is a locally finite. Let $V_i = \bigcup_{U \in B} G$ and for each n $\in \mathbb{N}$ and each G$\in \mathcal{B}$, define $S_n(G) = G - \bigcup_{i<n} V_i$. Let $\mathcal{C}_n = \{S_n(G) \mid G \in \mathcal{B}_n\}$. Since $S_n(G) \subseteq G$, then $\mathcal{C}_n$ is refinement of $\mathcal{B}_n$, because $S_n(G) \subseteq G$,

for each G$\in \mathcal{B}_n$.Let $\mathcal{C} = \cup \mathcal{C}_n$. We shall show that $\mathcal{C}$ is a locally finite collection refinement of $\mathcal{G}$, covers X. Suppose $x \in X$. We shall show that for any one $S_n(G)$, $x\in S_n(G)$ a neighborhood of x that intersects finite elements $\mathcal{C}$. Since $\mathcal{B}$ covers X, there is a smallest positive integer number n_0 such that $x \in G\in \mathcal{B}_{n_0}$. Since x does not belong to any member of $\mathcal{B}_i$ for $i < n_0$, $x \in S_{n_0}(G) \in \mathcal{C}$. Since each collection $\mathcal{B}_n$ is locally finite, we can choose for each n = 1, 2, 3, ...n_0 a neighborhood W_n of x that intersects only finitely many members of $\mathcal{B}_n$ .Now if W_n intersects the member $S_n(V)$ of $\mathcal{C}$, W_n must intersect the member V of $\mathcal{B}_n$, since $S_n(V)\subset V$.

Therefore, W_n intersects only finitely many members of $\mathcal{C}$. Furthermore, because $G\in \mathcal{B}_n$, G does not intersect any element of $\mathcal{C}_n$,for $n > n_0$. As a result, the neighborhood $W_1\cap W_2\cap W_3\cap \text{-----------} \cap W_{n_0}\cap G$ of x intersects only finitely many elements of $\mathcal{C}$.

(iii) $\Rightarrow$ (iv)

Let be $\mathcal{G}$ a U-open cover of X. Using (iii) Choose $\mathcal{B}$ be a refinement of $\mathcal{G}$ that is locally finite and a U- closed cover of X. Now we consider for every $B\in \mathcal{B}$ a U-open set $D(B) \supseteq B$ that the collection $\{D(B)|B\in \mathcal{B}\}$is also locally finite and refinement of $\mathcal{G}$. Since B is locally finite. For every $x\in X$, there exist a neighborhood N_x of x that intersect finite members of $\mathcal{B}$. Then $\{N_x \mid x\in X\}$ is a U-open cover of X.

According to (iii) there is a collection $\mathcal{C}$ refinement of $\{N_x \mid x\in X\}$ that is U-closed cover of X. Clearly for every $C\in \mathcal{C}$ intersects finite members $B\in \mathcal{B}$. For each $B\in \mathcal{B}$, let $\mathcal{C}(B) = \{C : C \in \mathcal{C} \text{ and } C \subseteq X - B\}$.

Again let, $E(B) = X- \bigcup_{C\in\xi(B)} C$. By lemma-"Let $\{A_\alpha\}$ be locally finite collection of subsets of X. Then (a) Any sub collection of $\{A_\alpha\}$ is locally finite. (b) $\{\overline{A_\alpha}\}$ is locally finite.(c) $\overline{\bigcup_\alpha A_\alpha} = \bigcup_\alpha \overline{A_\alpha}$." $\bigcup_{C\in\xi(B)} C$ is U- closed. So E(B) is an U-open set. According to the definition $E(B)\supseteq B$. The collection $\{E(B)\}$is a U- open cover of X. For each $B\in \mathcal{B}$, $F(B)\in \mathcal{G}$, where $F(B)\supseteq B$.

Let $\mathcal{D} = \{E(B) \cap F(B)|B\in \mathcal{B}\}$. Then the collection $\mathcal{D}$ is refinement of $\mathcal{G}$ and U-open cover of X. Since $B\subseteq E(B) \cap F(B)$ and B is a U-open cover of X. Suppose $x\in X$. Now we shall show that $\mathcal{D}$ is locally finite. Since $\mathcal{C}$ is locally finite, there exists a neighborhood W of x that intersects only finite members of C, (say) C_1, C_2, C_3, ------, C_n. Since $\mathcal{C}$ is U-cover of X, so $W\subseteq C_1\cup C_2 \cup C_3\cup \text{------}\cup C_n$.

Now if any member C of $\mathcal{C}$ intersects the set $E(B) \cap F(B)$, then $C \not\subset X- B$. Therefore C intersects B. Since C intersect finite members B, so C will intersect maximum members of $\mathcal{D}$. Therefore W will also intersect finite members of $\mathcal{D}$.

Now if we write $E(B) \cap F(B)$, the collection $\mathcal{D} = \{D(B)|B\in \mathcal{B}\}$ is refinement of $\mathcal{G}$ and is a locally finite U-open cover of X.

Comment 2.1. [5] The properties (i) - (iv) of the above Theorem 2.5 can also be stated as :

(a) Each U- open covering of X has a U-open refinement that can be decomposed into an at most countable collection of locally finite families of U-open sets.

(b) Each U- open covering of X has a locally finite refinement, consisting of sets not necessarily either U- open or U-closed.

(c) Each U- open covering of X has a U- closed locally finite refinement.

(d) X is paracompact.

We now generalize the theorems in [8] (p. 165-166).

Theorem 2.6. If a locally compact Hausdorff U- space X is a countable union of compact U-spaces then X is paracompact.

Proof: Let X be a locally compact Hausdorff U-space and X = $\bigcup_n C_n$, where C_n is compact. Let for each n, $C_n \subseteq C_{n+1}$ (We can assume this, for otherwise we can consider C'_n instead of C_n where $C'_n = \bigcup_{i=1}^{n} C_i$). At first we shall show that X = $\cup W_n$, where W_n is U-open, $\overline{W_n}$ is compact and $\overline{W_n} \subseteq$ Wn+1. Let $x \in C_1$ and let G_x be a neighborhood of x, where $\overline{G_x}$ is compact. Then $\{G_x\}_{x\in C_1}$ is a U - open cover of C_1**.** Since C_1 is compact**,** there is a finite U-open subcover **{**$G_{x_1}, G_{x_2}, \text{------}, G_{x_n}$**}** of C_1. Let $W_1 = \bigcup_{i=1}^{n} G_{x_i}$.

Therefore $\overline{W_1}$ is compact, this implies that $C_2 \cup \overline{W_1}$ is compact. Suppose W_2 is a U-open set of $C_2 \cup \overline{W_1}$ obtained in the same way as the U-open set W_1 of C_1. So $\overline{W_2}$ is compact, $C_2 \subseteq W_2$ and $\overline{W_1} \subseteq W_2$. Let, for each m< n, the U-open set W_m be defined in a similar member such that $C_m \subseteq W_m$, $\overline{W_m}$ is compact and $\overline{W_m} \subseteq W_{m+1}$. Proceeding as before we get for each positive integer $n \geq 2$ a U-open set W_n of $C_n \bigcup \overline{W_{n-1}}$ **,** where $\overline{W_n}$ is compact and $\overline{W_{n-1}} \subseteq W_n$.

Let $\mathscr{W} = \{G_\alpha\}$ be a U-open cover of X and $K_n = \overline{W_n} - W_{n-1}$. Then K_n is compact. Now for every $x \in K_n$, there is a neighborhood V_x of x such that for any α, $V_x \subseteq G_\alpha$. Assume that $V_x \subseteq W_{n+1}$, since $\overline{W_n} \subseteq W_{n+1}$ and $V_x \cap W_{n-2} = \Phi$, since $\overline{W_{n-2}} \subseteq W_{n-1}$. Since K_n is compact, so there is a finite cover $\mathscr{D}_n$ = $\{V_{x_1}, V_{x_2} \text{-----} V_{x_n}\}$ of K_n. We denote by $\mathscr{V}$ the union of the finite covers $\mathscr{D}_n$ of K_n for all n. Then $\mathscr{V}$ is a U- open cover of X and since $V_x \in \mathscr{V}$ is contained in a $G_\alpha \in \mathscr{W}$. $\mathscr{V}$ is refinement of $\mathscr{W}$. Suppose $x \in X$. Then there exists a least natural number n such that $x \in \overline{W_n}$. Since $x \notin W_{n-1}$, so, $x \in K_n$. As a result there is a neighborhood $V \in \mathscr{V}$

which intersect only finite member of those members of $\mathscr{V}$ which covers K_{n-2}, K_{n-1},K_n and K_{n+1}.

Theorem 2.7. A locally compact Hausdorff U-space with a countable basis is paracompact.

Proof: Let X be a locally compact Hausdorff U-space with a countable basis and let $\{B_n\}$ be a countable basis of X. Let x$\in$ X. Then there exists a neighborhood V_x of x, such that $\overline{V_x}$ is compact. Again since $\{B_n\}$ is basis, x$\in$ $B_n(x)$ $\subseteq V_x$, for some n. Since $\overline{V_x}$ is compact, every $\overline{B_n(x)}$ is compact. So X a is union of $\overline{B_n(x)}$. Hence X is paracompact.

The following five theorems are the U- space generalization of those in [5].

To prove the next theorem we need a lemma.

Lemma 2.1. If X, Y are U- spaces with X normal, and p: X$\to$ Y is a U-continuous U-closed surjection, then Y is too normal.

Proof: Let A and B be two disjoint U- closed sets in Y. Since p is U- continuous, p^{-1}(A) and p^{-1}(B) are disjoint U- closed sets in X. X being normal, there are disjoint U-open sets G and H in X such that p^{-1}(A) $\subseteq$G, p^{-1}(B) $\subseteq$H. Since p is U- closed, p(G) and p(H) are disjoint U- open sets in Y with A$\subseteq$p(G), B$\subseteq$p(H). Thus Y is normal.

We now generalize the theorems in [5]

Theorem 2.8. Every U-continuous closed image of a paracompact U-space is paracompact.

Proof: Let X and Y be U- spaces with X paracompact, and let p: X$\to$Y be U- continuous U-closed surjection mapping. Let $\{G_\alpha \mid \alpha \in \mathscr{A}\}$ be any U- open covering of Y. Since X is normal and p is U-continuous, U-closed and surjection, Y is normal. By Theorem 2.5 and comment 2.1 it suffices to show that $\{G_\alpha \mid \alpha \in \mathscr{A}\}$ has an U- open refinement which can be decomposed into at most countably many locally finite families. We assume $\mathscr{A}$ is well-ordered and begin by constructing a U-open covering $\{V_{\alpha,n} \mid (\alpha,n) \in \mathscr{A} \times Z^+\}$ of X such that:

(i) For each n, $\{\overline{V}_{\alpha,n} \mid \alpha \in \mathscr{A}\}$ is a U-covering of X and a precise locally finite refinement of $\{p^{-1}(G_\alpha) \mid \alpha \in \mathscr{A}\}$.

(ii) If $\beta > \alpha$ then $p(\overline{V}_{\beta,\, n+1}) \cap p(\overline{V}_{\alpha,n}) = \Phi$.

Proceeding by induction, we take a precise U-open locally finite refinement of $\{p^{-1}(G_\alpha)\}$ and shrink it by normality of X to get $\{\overline{V}_{\alpha,1}\}$. Assuming $\{V_{\alpha,i}\}$ to be defined for all i$\leq$ n, let W$_{\alpha,n+1}$ $= p^{-1}(G_\alpha) - p^{-1}p(\bigcup_{\lambda<\alpha}\overline{V}_{\lambda,n})$. Each W$_{\alpha,n+1}$ is U- open, since by local finiteness $\bigcup_{\lambda<\alpha}\overline{V}_{\lambda,n}$ is U- closed and p is a U- closed map.

Furthermore,{ W$_{\alpha,n+1}$ $|\alpha\in \mathscr{A}$} is a U- covering of X: given x$\in$ X, let α_o be the first index for which x$\in$ $p^{-1}(G_\alpha)$; then x $\in$ W$_{\alpha_o,n+1}$, since p^{-1}p $(\overline{V}_{\lambda,n}) \subset p^{-1}(G_\lambda)$ for each λ. Taking a precise, U- open locally finite refinement of { W$_{\alpha,n+1}$ $|\alpha\in \mathscr{A}$ }, shrink it to get {$\overline{V}_{\alpha,\, n+1}$}. Clearly, condition (i) holds, and since $\overline{V}_{\beta,\, n+1}$ is not in the inverse image of any p($\overline{V}_{\alpha,\, n}$) for $\alpha<\beta$, condition (ii) is also satisfied.

For each n and α, let H$_{\alpha,n}$ = Y – p($\bigcup_{\beta\neq\alpha}V_{\beta,n}$) which is an U- open set. We have

(a) H$_{\alpha,n}$ $\subset$p($\overline{V}_{\alpha,\, n}$)$\subset$G$_\alpha$ for each n and α. Indeed,

p^{-1}(H$_{\alpha,n}$) = X - p^{-1}p($\bigcup_{\beta\neq\alpha}V_{\alpha,n}$)$\subset$X - p^{-1}p(X - $\overline{V}_{\alpha,\, n}$)$\subset$ $\overline{V}_{\alpha,\, n}$ $\subset$ p^{-1}(G$_\alpha$).

(b). H$_{\alpha,n}$ $\cap$ H$_{\beta,n}$ = Φ for each n whenever $\alpha\neq\beta$. In fact, y $\in$ H$_{\alpha,n}$ $\Rightarrow y\in\ p$ ($\overline{V}_{\alpha,\, n}$) and is in no other p($\overline{V}_{\beta,\, n}$).

(c). {H$_{\alpha,n}$ $|(\alpha,n)\in \mathscr{A}\times$ Z$^+$} is an U- open covering of Y. Let y$\in$ Y be given; for each fixed n there is, because of (i), a first α_n with y$\in$ p($\overline{V}_{\alpha_n,\, n}$); choosing now $\alpha_k = \min\{\alpha_n\ |n\in Z^+\}$, we have y$\in$ p($\overline{V}_{\alpha_k,\, k}$). If $\beta<\alpha_k$, then the definition of α_k shows y$\notin$ p($\overline{V}_{\beta,\, k+1}$); if $\beta>\alpha_k$, then by (ii), we find that y$\notin$ p($\overline{V}_{\beta,\, k+1}$); therefore we conclude that y$\in$ H$_{\alpha_k,k+1}$.

To complete the proof, we need only modify the H$_{\alpha,n}$ slightly to assure locally finiteness for each n. Choose a precise U- open locally finite refinement of { p^{-1} (H$_{\alpha,n}$)$|(\alpha,n)\in \mathscr{A}\times$ Z$^+$ }, and shrink it to get an U- open locally finite covering {K$_{\alpha,n}$} satisfying p($\overline{K}_{\alpha,\, n}$)$\subset$H$_{\alpha,n}$. For each n, let S$_n$ = {y $|$some nbd of y intersects at most one H$_{\alpha,n}$ }; S$_n$ is U- open and contains the U-closed $\bigcup_\alpha p\,(\overline{K}_{\alpha,\, n})$ = p ($\bigcup_\alpha \overline{K}_{\alpha,\, n}$), so by normality of Y we find an U-open G$_n$ with $\bigcup_\alpha p\,(\overline{K}_{\alpha,\, n})$ $\subset$

$G_n \subset \overline{G_n} \subset S_n$. The U-open covering $\{ G_n \cap H_{\alpha,n} \mid (\alpha,n) \in \mathscr{A} \times Z^+ \}$, with the decomposition $\{ G_n \cap H_{\alpha,n} \mid \alpha \in \mathscr{A} \}$ for n = 1, 2, 3, ------- satisfies the conditions of Theorem 2.5 and comment 2.1 for the given $\{G_\alpha\}$.

Definition 2.6. Let $\mathscr{G} = \{G_\alpha \mid \alpha \in \mathscr{A}\}$ be a covering of U- space X. For any $B \subset X$ the set $\cup \{ G_\alpha \mid B \cap G_\alpha \neq \Phi \}$ is called the U-**star of B** with respect to $\mathscr{G}$, and is denoted by St (B, $\mathscr{G}$).

Definition 2.7. A U-covering $\mathscr{B}$ is called a U- barrycentric refinement of a U-covering $\mathscr{G}$ whenever the covering $\{St (x, \mathscr{B}) \mid x \in X \}$ refines $\mathscr{G}$.

Theorem 2.9. Let X be normal U- space, and $\mathscr{G} = \{G_\alpha \mid \alpha \in \mathscr{A}\}$ a locally finite U-open covering. Then $\mathscr{G}$ has an U- open barrycentric refinement.

Proof: Shrink $\mathscr{G}$ to an U- open covering $\mathscr{B} = \{V_\alpha \mid \alpha \in \mathscr{A}\}$ such that $\overline{V}_\alpha \subset G_\alpha$ for each α; clearly, $\mathscr{B}$ is also locally finite. For each $x \in X$, define W(x) = $\bigcap \{ G_\alpha \mid x \in \overline{V}_\alpha \} \cap \bigcap \{ \complement \overline{V}_\beta \mid x \in \overline{V}_\beta \}$.

We show that $\mathscr{B}^* = \{W(x) \mid x \in X\}$ is the required U- open covering. Note that each W(x) is U- open: the locally finiteness of $\mathscr{B}$ assures that the first term is a finite intersection and that the last term, $\complement \cup \overline{V}_\beta$ is a U- open set. Next, $\mathscr{B}^*$ is a U- covering, since $x \in W(x)$ for each $x \in X$. Finally, fix any $x_o \in X$ and choose a $\overline{V}_\alpha$ containing x_o. Now, for each x such that $x_o \in W(x)$, we must have $x \in \overline{V}_\alpha$ also, otherwise $W(y) \subset \complement \overline{V}_\alpha$; and because $x \in \overline{V}_\alpha$, we conclude that $W(x) \subset G_\alpha$. Thus, $St(x_o, \mathscr{B}^*) \subset G_\alpha$, and the proof is complete.

Definition 2.8. A **U-** covering $\mathscr{B} = \{V_\beta \mid \beta \in \mathscr{B}\}$ is called a U- star refinement of the U- covering $\mathscr{G}$ whenever the U- covering $\{St (V_\beta, \mathscr{B} \mid \beta \in \mathscr{B}\}$ refines $\mathscr{G}$.

Theorem 2.10. A U- barrycentric refinement $\mathscr{B}^*$ of a U- barrycentric refinement $\mathscr{B}$ of $\mathscr{G}$ is a U- star refinement of $\mathscr{G}$.

Proof: Given $W_o \in \mathscr{B}^*$, choose a fixed $x_o \in W_o$. For each $W \in \mathscr{B}^*$ such that $W \cap W_o \neq \Phi$, choose a $z \in W \cap W_o$; then $W \cup W_o \subset St(z, \mathscr{B}^*) \subset$ some $V \in \mathscr{B}$. Because each such V contains x_o, we conclude that $St(W_o, \mathscr{B}^*) \subset St(x_o, \mathscr{B}) \subset$ some $G \in \mathscr{G}$. Since it is clear that a U- barycentric refinement of any refinement of $\mathscr{G}$ is also a U-barycentric refinement of $\mathscr{G}$, it follows from Theorem-2.9 that each U- open covering of a paracompact U- space has an U- open barycentric, and an U- open star, refinement.

Much more important, however, is that this property characterizes the paracompact U-spaces, not only among the Hausdorff U-spaces, but in fact also among the T_1-U- spaces.

Theorem 2.11. A T_1-U- space X is paracompact if and only if each U- open covering has an U- open barycentric refinement.

Proof: Only the sufficiency requires proof. We first show that any U- open covering $\mathcal{G} = \{G_\alpha \mid \alpha \in \mathcal{A}\}$ has a refinement as in Theorem-2.5 and comment 2.1.

Let $\mathcal{G}^*$ be an U- open star refinement of $\mathcal{G}$ and let $\{\mathcal{G}_n \mid n \geq 0\}$ be a sequence of U- open coverings, where each $\mathcal{G}_{n+1}$ **U**-star refines $\mathcal{G}_n$ and $\mathcal{G}_0$ U-Star refines $\mathcal{G}^*$ .Define a sequence of U-covering inductively by $\mathcal{B}_1 = \mathcal{G}_1$, $\mathcal{B}_2 = \{St(V, \mathcal{G}_2) \mid V \in \mathcal{B}_1\}$......, $\mathcal{B}_n = \{St(V, \mathcal{G}_n) \mid V \in \mathcal{B}_{n-1}\}$, ...

Each $\mathcal{B}_n$ is an U-open refinement of $\mathcal{G}_o$; in fact, each covering $\{St(V, \mathcal{G}_n) \mid V \in \mathcal{B}_n\}$ refines $\mathcal{G}_o$: this is true for n = 1 and, proceeding by induction, if it is true for n = k – 1, its truth for n = k follows by noting that whenever $V = St(V_o, \mathcal{G}_k)$ for some $V_o \in \mathcal{B}_{k-1}$, then $St(V_o, \mathcal{G}_k) = St[St(V_o, \mathcal{G}_k), \mathcal{G}_k] \subset St(V_o, \mathcal{G}_{k-1})$ because $\mathcal{G}_k$ is a U-star refinement of $\mathcal{G}_{k-1}$.

Now well-order X and for each $(n, x) \in Z^+ \times X$ define $E_n(x) = St(x, \mathcal{B}_n) - \cup \{St(z, \mathcal{B}_{n+1}) \mid z \text{ precedes } x\}$. Then $\mathcal{D} = \{E_n(x) \mid (n, x) \in Z^+ \times X\}$ is a U-covering: given $p \in X$, the set $A = \{z \mid p \in \bigcup_{i=1}^{\infty} St(z, \mathcal{B}_i)\}$ is not empty, since $p \in A$; if x is the first member of A, then $p \in St(x, \mathcal{B}_n)$ for some $n \in Z^+$ and $p \in St(z, \mathcal{B}_{n+1})$ for all z preceding x, so $p \in E_n(x)$. Moreover, since $\mathcal{B}_n$ refines $\mathcal{G}_0$, we find that $\mathcal{D}$ refines $\mathcal{G}^*$.

Each $G \in \mathcal{G}_{n+1}$ can meet at most one $E_n(x)$: for, if $G \cap E_n(x) \neq \Phi$, then there is a $V \in \mathcal{B}_n$ with $x \in V$ and $V \cap G \neq \Phi$, so $x \in V \cup G \subset V_o \in \mathcal{B}_{n+1}$ and $G \subset St(x, \mathcal{B}_{n+1})$. Thus, if $E_n(x)$ is the first set G meets, it cannot meet any $E_n(p)$ for p following x.

Now let $W_n(x) = St(E_n(x), \mathcal{G}_{n+1})$. Then $\mathcal{B}^* = \{W_n(x) \mid (n, x) \in Z^+ \times X\}$ is clearly an U- open covering of X. Furthermore, $\mathcal{B}^*$ refines $\mathcal{G}$ because $\mathcal{D}$ refines $\mathcal{G}^*$. Finally, for each fixed $n \in Z^+$, the family $\{W_n(x) \mid x \in X\}$ is locally finite: indeed, each $G \in \mathcal{G}_{n+2}$ can meet at most one $W_n(x)$, because $G \cap W_n(x) \neq \Phi$, if, and only if, $E_n(x) \cap St(G, \mathcal{G}_{n+2}) \neq \Phi$ and $St(G, \mathcal{G}_{n+2})$ is contained in some $G_o \in \mathcal{G}_{n-1}$ which we know can meet at most one $E_n(x)$.

The theorem will follow from Theorem 2.5 and comment 2.1, once we show that X is regular U- space. To this end, let $B \subset X$ be U-closed and $x \in B$. Since in a T_1-U-space each point is a U- closed set, $\mathcal{G} = \{X - x, \complement B\}$ is an U- open covering. Let $\mathcal{B}$ be an

U- open star refinement. Then St(x, $\mathcal{B}$) and St(B, $\mathcal{B}$) are the required disjoint neighborhoods of x and B: for if there were a V containing x and a $V^{/}$ meeting B such that $V \cap V^{/} \neq \Phi$, then St(V, $\mathcal{B}$) would contain x and points of B, which is impossible. The theorem is proved.

Definition 2.9. Let $\mathcal{G} = \{G_\alpha \mid \alpha \in \mathcal{A}\}$ be an U- open covering of X. A sequence $\{\mathcal{G}_n \mid n \in Z^+\}$ of U- open coverings is called U- locally starring for $\mathcal{G}$ if for each $x \in X$ there exists an nbd V(x) and $n \in Z^+$ such that $St(V, \mathcal{G}_n) \subset$ some G_α.

Theorem 2.12. A T_1-U-space is paracompact if and only if each U- open covering $\mathcal{G}$ there exists a sequence $\{\mathcal{G}_n \mid n \in Z^+\}$ of U- open coverings that is U- locally starring for $\mathcal{G}$.

Proof: "Only if" is trivial. "If": We can assume that $\mathcal{G}_{n+1} \prec \mathcal{G}_n$ for each $n \in Z^+$. Let $\mathcal{B}$ = {V open in X $\mid \exists\, n : [V \subset G \in \mathcal{G}_n] \wedge [St(V, \mathcal{G}_n) \subset$ some G_α]}. For each $V \in \mathcal{B}$, let n(V) be the smallest integer satisfying the condition. Because $\{\mathcal{G}_n \mid n \in Z^+\}$ is locally starring for $\mathcal{G}$, it follows that $\mathcal{B}$ is a U- open covering; we will show that $\mathcal{B}$ is in fact a U-barrycentric refinement of $\mathcal{G}$.

Let $x \in X$ be fixed, let n(x) = min{n(V) $\mid (x \in X) \wedge (V \in \mathcal{B})$}, and let $V_o \in \mathcal{B}$ be a set containing x such that $n(V_o) = n(x)$.

For any $V \in \mathcal{B}$ containing x, we have $n(V) \geq n(x)$, and consequently $St(x, \mathcal{B}) \subset \bigcup \{St(x, \mathcal{G}_i) \mid i \geq n(x)\}$. Since $\mathcal{G}_{i+1} \prec \mathcal{G}_i$ for each i, this shows $St(x, \mathcal{B}) \subset St(x, \mathcal{G}_{n(x)}) \subset St(V_o, \mathcal{G}_{n(V_o)}) \subset$ some G_α. By Theorem-2.11, X is therefore paracompact.

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