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Nonlocal Cauchy Problem for Sobolev Type Mixed Volterra-Fredholm Functional Integrodifferential Equation

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ABSTRACT

In this paper we prove the existence, uniqueness of a mild solution of mixed Volterra-Fredholm functional integrodifferential equation of Sobolev type with nonlocal condition. The results are established by using the semigroup theory and the Banach fixed point theorem.

***Keywords**:* C_0 semigroup, Nonlocal condition, Mixed Volterra-Fredholm functional integrodifferential equation, Banach fixed point theorem.

1. Introduction

Byszewski and Acka [6] established the existence, uniqueness and continuous dependence of a mild solution of a semilinear functional differential equation with nonlocal condition of the form

$$u'(t) + Au(t) = f(t, u_t), \quad t \in [0, a],$$

$$u(s) + \left[g\left(u_{t_1}, \dots, u_{t_p}\right)\right](s) = \emptyset(s), \quad s \in [-r, 0],$$

where $0 < t_1 < \cdots < t_p \leq a \ (p \in N)$, $-A$ is the infinitesimal generator of a C_0 semigroup of operators on a general Banach space, f, g and $\emptyset$ are given functions and $u_t(s) = u(t+s)$ for $t \in [0, a], s \in [-r, 0]$.

In this paper, we shall prove the existence and uniqueness of a mild solution for a mixed Volterra-Fredholm functional integrodifferential equation of Sobolev type with nonlocal condition of the form

$$\left(Bu(t)\right)' + Au(t) = f\left(t, u_t, \int_0^t k(t, s, u_s)ds, \int_0^a h(t, s, u_s)ds\right), t \in [0, a], \tag{1}$$

$$u(s) + \left[g\left(u_{t_1}, \dots, u_{t_p}\right)\right](s) = \emptyset(s), \quad s \in [-r, 0], \tag{2}$$

where B and A are linear operators with domains contained in a Banach space Q and range contained in a Banach space E, $\emptyset \in C([-r,0],E)$, $f: J \times X \times X \times X \to E$, $g: X^p \to X$ and $k,h: J \times J \times X \to X$.

The work on abstract nonlocal semilinear initial value problems was initiated by Byszewski [7, 8]. Such problems with nonlocal conditions have been extensively studied in the literature [1, 3, 4, 9, 10, 11, 14]. Sobolev type equations arise in various applications such as in the flow of fluid through fissured rocks, thermodynamics and shear in the second order fluids. For more details, we refer to [5, 11, 12]. Recently, Xiaoping Xu [13] studied the existence for delay integrodifferential equations of sobolev type with nonlocal conditions by using the theory of semigroup and the method of fixed points. Balachandran and Park [2] established the existence and uniqueness of a mild solution of a functional integrodifferential equation of Sobolev type with nonlocal condition using the theory of semigroup and the Banach fixed point principle. In this paper, we generalize the results of Balachandran and Park [2] for a mixed Volterra-Fredholm functional integrodifferential equation of Sobolev type with nonlocal condition.

2. Preliminaries

In order to prove our main theorem we consider some conditions on the operators A and B. Let Q and E be Banach space with norm $|.|$ and $\|.\|$ respectively. The operators $A: D(A) \subset Q \to E$ and $B: D(B) \subset Q \to E$ satisfy the assumptions which are given below:

(A_1) A and B are closed linear operators,

(A_2) $D(B) \subset D(A)$ and B is bijective,

(A_3) $B^{-1}: E \to D(B)$ is continuous.

From the above fact and the closed graph theorem imply the boundedness of the linear operators $AB^{-1}: E \to E$. Again $-AB^{-1}$ generates a uniformly continuous semigroup $S(t), t \geq 0$ and so $\max_{t\in[0,a]}\|S(t)\|$ is finite. In this continuation the operator norm $\|.\|_{B(E)}$ will be denoted by $\|.\|$. Consider $J_0 = [-r,0], J = [0,a]$ and $X = C([-r,0],E)$, $Y = C([-r,a],E)$, $Z = C([0,a],E)$. We denote $M = \max_{t\in[0,a]}\|B^{-1}S(t)B\|$, and $R = \|B^{-1}S(t)\|$. We make the following hypothesis:

(H_1) For every $u,v,w \in Y$ and $t \in [0,a]$, $f(.,u_t,v_t,w_t) \in Z$.

(H_2) There exists a constant $L > 0$ such that

$$\|f(t,x_t,y_t,z_t) - f(t,u_t,v_t,w_t)\| \leq L\left(\|x-u\|_{C([-r,t],E)} + \|y-v\|_{C([-r,t],E)} + \|z-w\|_{C([-r,t],E)}\right)$$

for $x,y,z,u,v,w \in Y, t \in [0,a]$.

(H_3) There exists a constant $K > 0$ such that

$$\|k(t,s,u_s) - k(t,s,v_s)\| \leq K\|u-v\|_{C([-r,s],E)}, \text{ for } u,v \in Y, s \in [0,a].$$

(H_4) There exists a constant $H > 0$ such that

$\|h(t,s,u_s)-h(t,s,v_s)\| \le H\|u-v\|_{C([-r,s],E)}$, for $u,v \in Y, s \in [0,a]$.

(H_5) Let $g: X^p \to X$ and there exists a constant $G > 0$ such that

$$\left\| \left[g\left(u_{t_1},\dots,u_{t_p}\right)\right](s) - \left[g\left(v_{t_1},\dots,v_{t_p}\right)\right](s) \right\| \le G\|u-v\|_X,$$

for $u,v \in Y, s \in [-r,0]$.

(H_6) $MG + RLa + RLKa^2 + RLHa^2 < 1$.

A function $u \in Y$ satisfying

(i) $$u(t) = B^{-1}S(t)B\emptyset(0) - B^{-1}S(t)B\left[g\left(u_{t_1},\dots,u_{t_p}\right)\right](0) + \int_0^t B^{-1}S(t-s)f\left(s,u_s,\int_0^s k(s,\xi,u_\xi)d\xi,\int_0^a h(s,\xi,u_\xi)d\xi\right)ds, t \in [0,a],$$

(ii) $$u(s) + \left[g\left(u_{t_1},\dots,u_{t_p}\right)\right](s) = \emptyset(s), \quad s \in [-r,0]$$

is called a mild solution of the nonlocal Cauchy problem (1) – (2).

3. Existence of a mild solution

Theorem 3.1: *Consider that the assumptions $(A_1)-(A_2)$ holds and the functions f,g,h and k satisfy the conditions $(H_1)-(H_6)$. Then the nonlocal Cauchy problem (1) – (2) has a unique mild solution.*

Proof: Define an operator F on the Banach space Y by the formula

$$(Fu)(t) = \begin{cases} \emptyset(t) - \left[g\left(u_{t_1},\dots,u_{t_p}\right)\right](t), t \in [-r,0] \\ B^{-1}S(t)B\emptyset(0) - B^{-1}S(t)B\left[g\left(u_{t_1},\dots,u_{t_p}\right)\right](0) \\ + \int_0^t B^{-1}S(t-s)f\left(s,u_s,\int_0^s k(s,\xi,u_\xi)d\xi,\int_0^a h(s,\xi,u_\xi)d\xi\right)ds, t \in [0,a] \end{cases} \quad (3)$$

where $u \in Y$. It is easy to see that F maps Y into itself. Now, we will show that F is contraction on Y.

Consider the following two differences

$$(Fu)(t) - (Fv)(t) = \left[g\left(u_{t_1},\dots,u_{t_p}\right)\right](t) - \left[g\left(v_{t_1},\dots,v_{t_p}\right)\right](t), \quad (4)$$

for $u,v \in Y, t \in [-r,0]$ and

$$(Fu)(t)-(Fv)(t)=B^{-1}S(t)B\left[\left(g\left(u_{t_1},\ldots,u_{t_p}\right)\right)(0)-\left(g\left(v_{t_1},\ldots,v_{t_p}\right)\right)(0)\right]$$
$$+\int_0^t B^{-1}S(t-s)\left[f\left(s,u_s,\int_0^s k(s,\xi,u_\xi)d\xi,\int_0^a h(s,\xi,u_\xi)d\xi\right)\right.$$
$$\left.-f\left(s,v_s,\int_0^s k(s,\xi,v_\xi)d\xi,\int_0^a h(s,\xi,v_\xi)d\xi\right)\right]ds,$$
$$\text{for } u,v\in Y,\qquad t\in[0,a]. \tag{5}$$

From (4) and (H_5), we have

$$\|(Fu)(t)-(Fv)(t)\|\le G\|u-v\|_Y,\qquad for\ u,v\in Y,t\in[-r,0] \tag{6}$$

Moreover by (5),$(H_2)-(H_6)$,

$$\|(Fu)(t)-(Fv)(t)\|\le\|B^{-1}S(t)B\|\left\|\left(g\left(u_{t_1},\ldots,u_{t_p}\right)\right)(0)-\left(g\left(v_{t_1},\ldots,v_{t_p}\right)\right)(0)\right\|$$
$$+\int_0^t\|B^{-1}S(t-s)\|\left\|f\left(s,u_s,\int_0^s k(s,\xi,u_\xi)d\xi,\int_0^a h(s,\xi,u_\xi)d\xi\right)\right.$$
$$\left.-f\left(s,v_s,\int_0^s k(s,\xi,v_\xi)d\xi,\int_0^a h(s,\xi,v_\xi)d\xi\right)\right\|ds$$

$$\le MG\|u-v\|_Y+RL\int_0^t\left[\|u-v\|_{C([-r,s],E)}+\int_0^s\|k(s,\xi,u_\xi)-k(s,\xi,v_\xi)\|d\xi\right.$$
$$\left.+\int_0^a\|h(s,\xi,u_\xi)-h(s,\xi,v_\xi)\|d\xi\right]ds$$

$$\le MG\|u-v\|_Y+RL\int_0^t\left[\|u-v\|_{C([-r,s],E)}+K\int_0^s\|u-v\|_{C([-r,\xi],E)}d\xi\right.$$
$$\left.+H\int_0^a\|u-v\|_{C([-r,\xi],E)}d\xi\right]ds$$

$$\le MG\|u-v\|_Y+RL\|u-v\|_Y\int_0^t\left[1+K\int_0^s d\xi+H\int_0^a d\xi\right]ds$$

$$\le[MG+RLa+RLKa^2+RLHa^2]\|u-v\|_Y. \tag{7}$$

From the equation (6) and (7) we get

$$\|(Fu)(t)-(Fv)(t)\| \le q\|u-v\|_Y, \text{ for } u,v \in Y, \tag{8}$$

where $q = MG + RLa + RLKa^2 + RLHa^2$. Since, $q < 1$ then equation (8) shows that F is a contraction on Y. Consequently, the operator F satisfies all the assumptions of the Banach contraction mapping theorem. Therefore, in space Y there is a unique fixed point for F and this point is the mild solution of the considered problem (1) – (2).

4. Continuous dependence of mild solution

Theorem 4.1: *Assume that the assumptions $(A_1)-(A_3)$ hold and that the function f, g, k and h satisfy the hypothesis $(H_1)-(H_6)$. Then for each $\emptyset_1, \emptyset_2 \in X$ and for the corresponding mild solutions u_1, u_2 of the problems*

$$\left(Bu(t)\right)' + Au(t) = f\left(t, u_t, \int_0^t k(t,s,u_s)ds, \int_0^a h(t,s,u_s)ds\right), \quad t \in [0,a], \tag{9}$$

$$u(s) + \left[g\left(u_{t_1}, \dots, u_{t_p}\right)\right](s) = \emptyset_i(s), \qquad s \in [-r,0], (i = 1,2) \tag{10}$$

the following inequality

$$\|u_1-u_2\|_Y \le Me^{aRL(1+Ka)}[\|\emptyset_1-\emptyset_2\|_X + (G+LHa^2)\|u_1-u_2\|_Y] \tag{11}$$

is true. Additionally, if $(G+LHa^2) < \frac{1}{M}e^{-aRL(1+Ka)}$ then,

$$\|u_1-u_2\|_Y \le \frac{Me^{aRL(1+Ka)}}{[1-M(G+LHa^2)e^{aRL(1+Ka)}]}\|\emptyset_1-\emptyset_2\|_X. \tag{12}$$

Proof: Suppose that $\emptyset_i$ $(i = 1,2)$ be an arbitrary functions belonging to X and suppose u_i $(i = 1,2)$ be the mild solutions of the problem (9) - (10). Consequently,

$$\begin{aligned} u_1(t)-u_2(t) &= B^{-1}S(t)B[\emptyset_1(0)-\emptyset_2(0)] \\ &\quad -B^{-1}S(t)B\left[\left(g\left((u_1)_{t_1},\dots,(u_1)_{t_p}\right)\right)(0)-\left(g\left((u_2)_{t_1},\dots,(u_2)_{t_p}\right)\right)(0)\right] \\ &\quad +\int_0^t B^{-1}S(t-s)\left[f\left(s,(u_1)_s,\int_0^s k\big(s,\xi,(u_1)_\xi\big)d\xi,\int_0^a h\big(s,\xi,(u_1)_\xi\big)d\xi\right)\right. \\ &\quad \left.-f\left(s,(u_2)_s,\int_0^s k\big(s,\xi,(u_2)_\xi\big)d\xi,\int_0^a h\big(s,\xi,(u_2)_\xi\big)d\xi\right)\right]ds, \quad t \in J, \end{aligned} \tag{13}$$

and for $t \in J_0$ we have

$$u_1(t)-u_2(t) = [\emptyset_1(t)-\emptyset_2(t)]$$

$$-\left[\left(g\left((u_1)_{t_1},\ldots,(u_1)_{t_p}\right)\right)(t)-\left(g\left((u_2)_{t_1},\ldots,(u_2)_{t_p}\right)\right)(t)\right]. \qquad (14)$$

By our assumptions,

$$\|u_1(\delta)-u_2(\delta)\| \le M\|\emptyset_1-\emptyset_2\|_X + MG\|u_1-u_2\|_Y$$
$$+\int_0^\delta \|S(\delta-s)\| \left\| f\left(s,(u_1)_s,\int_0^s k(s,\xi,(u_1)_\xi)d\xi,\int_0^a h(s,\xi,(u_1)_\xi)d\xi\right) - f\left(s,(u_2)_s,\int_0^s k(s,\xi,(u_2)_\xi)d\xi,\int_0^a h(s,\xi,(u_2)_\xi)d\xi\right)\right\| ds$$

$$\le M\|\emptyset_1-\emptyset_2\|_X + MG\|u_1-u_2\|_Y + RL\int_0^\delta \Bigg[\|u_1-u_2\|_{C([-r,\delta],E)}$$
$$+\int_0^s \left\|k(s,\xi,(u_1)_\xi)-k(s,\xi,(u_2)_\xi)\right\| d\,\xi + \int_0^a \left\|h(s,\xi,(u_1)_\xi)-h(s,\xi,(u_2)_\xi)\right\| d\xi\Bigg] ds$$

$$\le M\|\emptyset_1-\emptyset_2\|_X + MG\|u_1-u_2\|_Y + RL\int_0^\delta \Bigg[\|u_1-u_2\|_{C([-r,\delta],E)}$$
$$+K\int_0^s \|u_1-u_2\|_{C([-r,\xi],E)} d\,\xi + H\int_0^a \|u_1-u_2\|_{C([-r,\xi],E)} d\xi\Bigg] ds$$

$$\le M\|\emptyset_1-\emptyset_2\|_X + MG\|u_1-u_2\|_Y + RLHa^2\|u_1-u_2\|_Y$$
$$+RL\int_0^\delta \left[\|u_1-u_2\|_{C([-r,\delta],E)} + Ka\|u_1-u_2\|_{C([-r,\xi],E)}\right] ds$$

$$\le M\|\emptyset_1-\emptyset_2\|_X + (MG+RLHa^2)\|u_1-u_2\|_Y$$
$$+RL(1+aK)\int_0^t \|u_1-u_2\|_{C([-r,s],E)} ds, \qquad \text{for } 0\le\xi\le\delta\le t\le a.$$

Therefore,

$$\sup_{\delta\in[0,t]}\|u_1(\delta)-u_2(\delta)\| \le M\|\emptyset_1-\emptyset_2\|_X + (MG+RLHa^2)\|u_1-u_2\|_Y$$
$$+RL(1+aK)\int_0^t \|u_1-u_2\|_{C([-r,s],E)} ds, \;\; t\in[0,a] \qquad (15)$$

From (H_5) and (14) we have

$$\|u_1(t) - u_2(t)\| \leq \|\emptyset_1 - \emptyset_2\|_X + G\|u_1 - u_2\|_Y \quad \text{for } t \in J_0. \tag{16}$$

Since, $M \geq 1$, (15) and (16) imply that

$$\|u_1(t) - u_2(t)\|_{C([-r,t],E)} \leq M\|\emptyset_1 - \emptyset_2\|_X + (MG + RLHa^2)\|u_1 - u_2\|_Y$$
$$+RL(1 + aK)\int_0^t \|u_1 - u_2\|_{C([-r,s],E)}ds, \qquad \text{for } t \in J. \tag{17}$$

By Gronwall's inequality, we have

$$\|u_1(t) - u_2(t)\|_Y \leq [M\|\emptyset_1 - \emptyset_2\|_X + (MG + RLHa^2)\|u_1 - u_2\|_Y]e^{aRL(1+Ka)}.$$

and therefore inequality (11) is true. Finally, inequality (12) is a consequence of inequality (11). Thus, the proof is complete.

REFERENCES

1. K. Balachandran, D.G. Park and Y.C. Kwun, Nonlinear integrodifferential equations of Sobolev type with nonlocal conditions in Banach spaces, Commun. Korean Math. Soc., 14 (1999), 223-231.
2. K. Balachandran and J.Y. Park, Nonlocal Cauchy problem for Sobolev type functional integrodifferential equation, Bull. Korean Math. Soc., 39(4) (2002), 561-569.
3. K. Balachandran and J.Y. Park, Existence of a mild solution of a functional integrodifferential equation with nonlocal condition, Bull. Korean Math. Soc., 38(1) (2001) 175-182.
4. K. Balachandran and K. Uchiyama, Existence of solutions of nonlinear integrodifferential equations of Sobolev type with nonlocal condition in Banach spaces, Proc. Indian Acad. Sci. (Math. Sci.), 110 (2000) 225-232.
5. H. A. Brill, Semilinear Sobolev evolution equation in Banach space, J. Differential Equation, 24 (1977) 412-425.
6. L. Byszewski and H. Akca, On a mild solution of a semilinear functional differential evolution nonlocal problem, J. Appl. Math. Stoch. Anal., 10 (1997) 265-271.
7. L. Byszewski, Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem, J. Math. Anal. Appl., 162 (1992) 494-505.
8. L. Byszewski and V. Lakshmikantham, Theorem about the existence and uniqueness of solutions of a nonlocal Cauchy problem in a Banach space, Appl. Anal., 40 (1990) 11-19.
9. M.B. Dhakne and H.L. Tidke, Existence and uniqueness of solutions of nonlinear mixed integrodifferential equations with nonlocal condition in Banach spaces, EJDE 2011(31) (2011) 1-10.
10. K. Kumar, R. Kumar and R.K. Shukla, Existence of solutions of nonlinear delayed integrodifferential equation with nonlocal condition, in the Proceedings Advance in

Applied Mathematics and Computational Physics (Eds. Rakesh Kumar et.al.), World Education Publishers, Delhi, India (2012), 227-235.

11. J.H. Lightbourne, III and S.M. Rankin, III, A partial functional differential equation of Sobolev type, J. Math. Anal. Appl., 93 (1983) 328-337.
12. R.E. Showalter, Existence and representation theorem for a semilinear Sobolev equation in Banach space, SIAM J. Math. Anal., 3 (1972) 527-543.
13. X. Xiaoping, Existence of delay integrodifferential equations of Sobolev type with nonlocal conditions, International Journal of Nonlinear Science, 12(3) (2011) 263-269.
14. Yu Tian, Existence of nonoscillatory solution of higher-order neutral difference equations with continuous argument, Journal of Physical Sciences, 10 (2006) 1-17.