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Semiprime Γ-rings with Jordan Derivations

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ABSTRACT

Let M be a 2-torsion free semiprime Γ-ring and d:M→M a Jordan left derivation. We find the existence of a positive integer n with the condition $(d(x)\alpha)^n d(x)=0$, for all x∈ M and α∈ Γ, which gives d=0. With the help of this assertion, we show that the presence of Jordan derivations d and g on the 2-torsion free Γ-ring M such that $d^2(x)=g(x)$, for all x∈ M implies d=0.

Keywords: n-torsionfree, Jordan derivation, Jordan left derivation, commutativity, semiprime Γ-rings.

1. Introduction

Let M and Γ be additive abelian groups. M is said to be a Γ-ring if there exists a mapping M×Γ×M→M (sending (x,α,y) into xαy) such that
(a) (x+y)αz=xαz+yαz ,
x(α+β)y=xαy+xβy,
xα(y+z)=xαy+xαz,
(b) (xαy)βz= xα(yβz), for all $x, y, z \in M$ and α,β ∈ Γ.

A subset A of a Γ-ring M is a left(right) ideal of M if MΓA(AΓM) is contained in A. A ideal P of a Γ-ring M is prime if P≠M and for any ideals A and B of M , AΓB⊆P, then A⊆P or B⊆P. M is prime if aΓMΓb=0 with $a, b \in M$, then $a = 0$ or $b = 0$. M is semiprime if aΓMΓa=0 with $a \in M$, then $a = 0$. M is $n-$torsion free if na=0 for a ∈ M implies a=0,where n is an integer. We denote the commutator aαb-bαa by $[a, b]_\alpha$ for all a,b∈ M and ∈ Γ. A Γ-ring M is commutative if aαb=bαa, for all a∈ M and α∈ Γ. A Γ-ring M is non-commutative if it is not commutative. An element a of a Γ-ring M is nilpotent if $(a\alpha)^n a=0$, for all α∈ Γ and for some positive integer n. An ideal I of a Γ-ring M is nilpotent if$(I\Gamma)^n I=0$, for some positive integer n. A Γ-ring M is nil if every element of M is nilpotent. An additive mapping d:M→M is a derivation if d(aαb)=aαd(b)+d(a)αb, a left derivation if d(aαb)=aαd(b)+bαd(a),a Jordan derivation if d(aαa)=aαd(a)+d(a)αa and a Jordan left derivation if d(aαa)=2aαd(a), for all $a, b \in M$ and α∈Γ.

Ceven [4] worked on Jordan left derivations on completely prime Γ-rings. He investigated the existence of a nonzero Jordan left derivation on a completely prime Γ-ring that makes the Γ-ring commutative with an assumption. With the same assumption, he showed that every Jordan left derivation on a completely prime Γ-ring is a left derivation on it. In this paper, he gave an example of Jordan left derivation for Γ-ring. Mustafa Asci and Sahin Ceran [7] studied on a nonzero left derivation d on a prime Γ-ring M for which M is commutative with the conditions d(U)⊆U and d^2(U)⊆Z, where U is an ideal of M and Z is the centre of M. They also proved the commutativity of M by the nonzero left derivation d_1 and right derivation d_2 on M with the conditions d_1(U)⊆ U and $d_1 d_2$(U)⊆ Z.

In [9], Sapanci and Nakajima defined a derivation and a Jordan derivation on Γ-rings and investigated a Jordan derivation on a certain type of completely prime Γ-ring which is a derivation. They also gave examples of a derivation and a Jordan derivation of Γ-rings.

Bresar and Vukman [2] proved that a Jordan derivation on a prime Γ-ring is a derivation.Furthermore, in [3], Bresar and Vukman showed that the existence of a nonzero Jordan left derivation of R into X implies R is commutative, where R is a ring and X is a 2-torsion free and 3-torsion free left R-module. In [6], Jun and Kim proved their results without the property 3-torsion free. Qing Deng [5] worked on Jordan left derivationd of prime ring R of characteristic not 2 into a nonzero faithful and prime left R-module X. He proved the commutativity of R with Jordan left derivation d.

JosoVukman [10] studied on Jordan left derivations on semiprimerings. He investigated a Jordan left derivation d on a 2-torsion free semiprime ring R such that d=0 with the condition $(d(x))^n$=0,for all x∈M. He also showed that d=0 if d and g are Jordan derivations on a 2-torsion free and 3-torsion free semiprime ring R with the condition d^2(x)=g(x),for all x∈M.

In this present study, we motivate the results of Joso Vukman [10] in Γ-rings. We show that the existence of a positive integer n makes the Jordan left derivation d on a 2-torsion free semiprimeΓ-ring M zero with $(d(x)\alpha)^n$d(x)=0, for all x∈ M and α∈Γ. We also investigate the Jordan derivations d and g on M with the conditiond^2(x)=g(x), for all x∈M, which gives d=0.

Throughout this work, we denote the condition $a\alpha b\beta c = a\beta b\alpha c$,for all a,b,c∈ M and α,β∈ Γ, by (*) for convenience.

2. Supporting lemmas

Lemma 2.1. Let M be a Γ-ring satisfying (*). If d:M→M is a Jordan left derivation then
(a) d(xαy+yαx)=2xαd(y)+2yαd(x),
(b) d(xαyβx)=xαxβd(y)+3xαyβd(x)-yαxβd(x), for all x,y∈ M and α,β∈ Γ.
The proof is given in Y.Ceven [4].

Lemma 2.2. Let M be a 2-torsion free and 3-torsion free Γ –ring satisfying (*), and d:M→M a Jordan left derivation then. If $(d[[d(x),x]_\alpha,x]_\beta)$=0 holds for all x∈ M and α,β∈ Γ the $[d(x),x]_\alpha\beta$d(x)=0, for all x∈ M and α,β∈ Γ.

Proof: Suppose that for all x∈ M and α,β∈ Γ,
0= $(d[[d(x),x]_\alpha,x]_\beta)$ =6 $[d(x),x]_\alpha\beta$d(x), by Lemma 2.1 and (*). Since M is 2-torsion free and 3-torsion free, $[d(x),x]_\alpha\beta$d(x)=0, for all x∈ M and α,β∈ Γ.

For our work, we only state the following lemma

Lemma 2.3. Let M be a non-commutative prime Γ-ring of characteristic not 2, and d:M→M a Jordan left derivation. Then d=0.

Lemma 2.4. Let M be a semi-prime Γ-ring. Then M contains no nonzero nilpotent ideals.
Proof: Let I be a nilpotent ideal of M. Then $(I\Gamma)^n I=0$, for some positive integer n. Let us assume that n is minimal. Now, suppose that n>1. Since I ΓI⊆I, we have $(I\Gamma)^{n-1}I\Gamma M\Gamma(I\Gamma)^{n-1}I \subseteq (I\Gamma)^{n-1}I\Gamma(I\Gamma)^{n-1}I= (I\Gamma)^n I\Gamma(I\Gamma)^{n-2}I=0$.Hence by thr semi-primeness of M,we have $(I\Gamma)^{n-1}I=0$, a contradiction to the minimality of n.Therefore, n=1.This implies that IΓI=0. Then we get IΓMΓI ⊆IΓI=0.But, since M is semiprime, it yields I=0.

Lemma 2.5. Let M be a Γ-ring. Then the following conditions are equivalent.
(a) M has no nonzero nilpotent elements.
(b) For every a∈ M and α∈ Γ, $(a\alpha)^n a=0$ implies a=0,for some positive integern.
Proof: Let a≠0, then a is a nonzero nilpotent element of M, which is a contradiction. Hence (a) implies (b). Let (a≠0) ∈ M be a nilpotent element. Then $(a\alpha)^m a=0$,for every α∈ Γ and for some positive integer m. Suppose that m is minimal. If n<m, then n is the degree of nilpo- tency, a contradiction. If n=m, then by hypothesis a is a zero nilpotent element, which is also a contradiction. If n>m, say n=m+k, k≥1. Then we have $(a\alpha)^m((a\alpha)(a\alpha)\ldots(a\alpha))_{k-factors}a=0$. This gives $(a\alpha)^{m+k}a=0$. This implies that $(a\alpha)^n a=0$.By hypothesis, a=0, a contradiction. Hence M has no nonzero nilpotent elements. Thus (b) implies (a).

3. Main theorems

Theorem 3.1. Let M be a 2-torsion free semi-prime Γ-ring satisfying (*) and d:M→M a Jordan left derivation . If there exists a positive integer n such that $(d(x)\alpha)^n d(x)=0$, for all x∈ M and α∈ Γ , then d=0.
Proof: Since M is semi-prime, ∩P=(0),where the intersection runs over all prime ideals P of M. We need to show that d(P)⊆P, for every prime ideal P of M. Let $a \in P, x \in M$. Then by Lemma2.1(a),We have

$0=d(a\alpha x+x\alpha a)\alpha d(a\alpha x+x\alpha a) =2^2(a\alpha d(x)\alpha a\alpha d(x)+a\alpha d(x)\alpha x\alpha d(a)+x\alpha d(a)\alpha a\alpha d(x)+x\alpha d(a)\alpha x\alpha d(a))$.

Since M is 2-torsion free, aαd(x)∈ P and xαd(a) ∈M,
$(x\alpha d(a)\alpha)(x\alpha d(a)) \equiv 0(mod)P$, for all α ∈Γ. Also,

$0=d(a\alpha x+x\alpha a)\alpha d(a\alpha x+x\alpha a)\alpha d(a\alpha x+x\alpha a)$
$=2^3(a\alpha d(x)\alpha a\alpha d(x)\alpha a\alpha d(x) + x\alpha d(a)\alpha a\alpha d(x)\alpha a\alpha d(x)+ a\alpha d(x)\alpha x\alpha d(a)\alpha a\alpha d(x)$
$+ x\alpha d(a)\alpha x\alpha d(a)\alpha a\alpha d(x)+ a\alpha d(x)\alpha a\alpha d(x)\alpha x\alpha d(a) + x\alpha d(a)\alpha a\alpha d(x)\alpha x\alpha d(a)+$
$a\alpha d(x)\alpha x\alpha d(a)\alpha x\alpha d(a)+ x\alpha d(a)\alpha x\alpha d(a)\alpha x\alpha d(a))$.

Since M is 2-torsion free, aαd(x)∈ P and xαd(a) ∈M,
$(x\alpha d(a)\alpha)^2(x\alpha d(a))\equiv 0(,mod)P$, for all α ∈Γ.
Proceeding in this way, we have
$(x\alpha d(a)\alpha)^n(x\alpha d(a))\equiv 0(,mod)P$, for all α ∈Γ.

Thus, in the prime Γ-ring, $M'=M/P$, we have $(x'\alpha d(a)'\alpha)^n x'\alpha d(a)'=0$, for all $x' \in M'$ andα ∈Γ. By Lemma 2.5, $x'\alpha d(a)'=0$, for all $x' \in M'$ andα ∈Γ. Since M' is prime, d(a)=0. This gives d(a) ∈ P, and so d(P) ⊆P. Therefore, d(P) ⊆P, for all prime ideals P of M, and so d induces a Jordan left derivation d' on the prime Γ- ring, $M'=M/P$. Let us first assume that M' is commutative. In this case, d' is a derivation and we also have $(d'(x')\alpha)^n d'(x')=0$,which follows that $d'=0$. In case, M' is non-commutative, it follows by Lemma 2.3 that $d'=0$. Thus, in any case, $d'(M')=0$,that is, $d(M)\subseteq P$, for all prime ideals P of M. Since $\cap P=(0)$,we obtain d(M)=0, and hence d=0.

Theorem 3.2. Let M be a 2-torsion free and 3-torsion free semi-prime Γ-ring satisfying (*). If d:M→M and g:M→M are Jordan derivations such that $d^2(x)=g(x)$, for all x∈ M, then d=0.

Proof: We have, $d^2(x)=g(x)$, for all x∈ M. (1)

Replacing xαx for x in (1), and then using the condition that M is 2-torsion free, we get $d(x\alpha d(x))=x\alpha g(x)$, for all x∈ M and α∈ Γ. (2)

Then by Lemma 2.1(a), and using (1) and (2), we have

$d(d(x)\alpha x)=2d(x)\alpha d(x)+x\alpha g(x)$, for all x∈ M and α∈ Γ (3)

Taking (3)-(2), we get $d([d(x),x]_\alpha)=2d(x)\alpha d(x)$, (4)

for all x∈ M and α∈ Γ.

Replacing x by x+y in (4), we have $d([d(x),y]_\alpha+[d(y),x]_\alpha)=2d(x)\alpha d(y)+2d(y)\alpha d(x)$.

Replacing xαx for y in the above equation and then using Lemma 2.1(a) and the condition that M is 2-torsion free, we have

$d(x\alpha[d(x),x]_\alpha)=d(x)\alpha x\alpha d(x)+x\alpha d(x)\alpha d(x)$, for all x∈ M and α∈ Γ. (5)

By Lemma 2.1(a), equation (4) and equation (5), we have

$d([d(x),x]_\alpha\alpha x)=d(x)\alpha x\alpha d(x)+x\alpha d(x)\alpha d(x)$, for all x∈ M and α∈ Γ. (6)

Taking (6)-(5) and then applying Lemma 2.4, we get

$[d(x),x]_\alpha\alpha d(x)=0$, for all x∈ M and α∈ Γ. (7)

Using Lemma 2.1(a),(1),(4) and (7), we obtain

$d(d(x)\alpha[d(x),x]_\alpha)=4d(x)\alpha d(x)\alpha d(x)+2[d(x),x]_\alpha\alpha g(x)$, for all x∈ M and α∈ Γ. (8)

By Lemma 2.1(b),(7)and (1), we have

$d(x\alpha d(x)\alpha d(x))=d(x)\alpha d(x)\alpha d(x)+3d(x)\alpha x\alpha g(x)-x\alpha d(x)\alpha g(x)$, for all x∈ M and α∈ Γ. (9)

By using (1),(9) and Lemma 2.1, we have

$d(d(x)\alpha d(x)\alpha x)=d(x)\alpha d(x)\alpha d(x)+5x\alpha d(x)\alpha g(x)-3d(x)\alpha x\alpha g(x)$, for all x∈ M and α∈ Γ.(10)

Applying (10)-(9),we have

$d(d(x)\alpha d(x)\alpha x-x\alpha d(x)\alpha d(x))=-6[d(x),x]_\alpha\alpha g(x)$, for all x∈ M and α∈ Γ. (11)

On the other hand, by (7), we obtain

$d(d(x)\alpha d(x)\alpha x-x\alpha d(x)\alpha d(x))=d(d(x)\alpha[d(x),x]_\alpha)$, for all x∈ M and α∈ Γ. (12)

From (11) and (12), we have

$d(d(x)\alpha[d(x),x]_\alpha)=-6[d(x),x]_\alpha\alpha g(x)$, for all x∈ M and α∈ Γ. (13)

Combining (8) and (13) and using the condition that M is 2-torsion free, we have

$d(x)\alpha d(x)\alpha d(x)\alpha d(x)+2d(x)\alpha[d(x),x]_\alpha\alpha g(x)=0$, for all x∈ M and α∈ Γ. (14)

By Lemma 2.1(b),(1) and (4), we have

$2d(x)\alpha d(x)\alpha d(x)\alpha d(x)+3d(x)\alpha[d(x),x]_\alpha\alpha g(x)=0$, for all x∈ M and α∈ Γ. (15)

Now, taking (15)×2-(14)×3, we have

$d(x)\alpha d(x)\alpha d(x)\alpha d(x)=0$, for all x∈ M and α∈ Γ. That is $(d(x)\alpha)^3 d(x)=0$, for all x∈ M and α∈ Γ.

Finally by Theorem 3.1, we have d=0.

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