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Fuzzy Approach to Interest Rate in Mathematics of Actuarial Science and Finance

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ABSTRACT

Since state of economy is uncertain, deterministic system of interest rate model is not the satisfactory tool to use in calculating the worth of future financial decision. Interest rate plays a vital rule in study of financial mathematics. In this study usual mathematics of actuary and finance has been derived and investigated taking the interest rate $\tilde{i}$ as a triangular fuzzy number. The method of $\alpha - cuts$ and interval arithmetic are used to renovate the usual mathematics of actuary and finance into a fuzzy form. The computation has been performed using MATLAB. The main outcome obtained from this study is that computational results using fuzzy interest contains the exact and it is found by taking a rate with grade 1.

Keywords: Actuarial mathematics, annuity, equation of value, interval arithmetic, fuzzy interest rate, fuzzy term structure interest rate

1. Introduction

Financial contracts are either short-term or long-term in nature. Fixed term interest rate model for the contract period may result in dissatisfaction. Interest rates historically have been volatile. Long-term borrowings on a floating rate loan such as in project finance transactions may expose the sponsors to unexpected risk if interest rates rise to levels that seriously affect their borrowing or repayment capability. Many factors that cause the interest rate to vary are expectations, liquidity, inflations, taxes [10,16].

The study on the application of fuzzy sets in finance and economics has been carried out by [6,7]. However there are still scopes to extend and improve the application technique of fuzziness in actuarial and financial mathematics. The objective of this study is to apply the fuzziness to interest rate in the usual mathematics of actuarial science. If the endeavor is successful, it will increase the confidence in the application of fuzzy mathematics to the area of actuarial science, finance and economics.

The interest rate is considered as a Triangular fuzzy number $\tilde{i} = (a,b,c)$. Triangular fuzzy numbers are used because of their convenient properties [7,8,15]. The usual equations and formulas of finance in [8,11,12,14,15] are modified by the interest rate $\tilde{i}$, which is a triangular fuzzy number. By taking alpha-cuts and using interval arithmetic these equations have been solved. The result obtained is also a fuzzy number. The computations are performed by MATLAB.

This paper is based on the methods given in [9,15]. In all the cases the amount invested and the duration of investment are considered crisp numbers except the interest rates.

2. Preliminaries

Fuzzy mathematics is now well established among the researchers of various subjects. The preliminaries of fuzzy sets, fuzzy numbers, $\alpha - cuts$ and interval arithmetic are defined in many books and articles [8, 15].

3. Interval arithmetic

Interval arithmetic in its modern form was introduced by [20]. In this study we discussed interval arithmetic based on arithmetic conducted on closed sets of real numbers [1]and[19].

Theorem 3.1. In [19] If $A=[\underline{a},\bar{a}]$ and $B=[\underline{b},\bar{b}]$, then

(i) $A+B=[\underline{a}+\underline{b},\ \bar{a}+\bar{b}]$ (ii) $tA=[t\underline{a},t\bar{a}],\text{for}\ t\geq 0$ (iii) $-A=[-\bar{a},\ -\underline{a}]$ (iv) $A-B=[\underline{a}-\bar{b},\ \bar{a}-\underline{b}]$

(v) $A.B=[\min(\underline{a}.\bar{a},\underline{a}.\bar{b},\bar{a}.\underline{b},\ \bar{a}.\bar{b}),\max(\underline{a}.\bar{a},\underline{a}.\bar{b},\bar{a}.\underline{b},\ \bar{a}.\bar{b})]$ (vi) $\frac{1}{A}=\left[\frac{1}{\bar{a}},\frac{1}{\underline{a}}\right]$ (vii) $\frac{B}{A}=\left[\frac{\underline{b}}{\bar{a}},\frac{\bar{b}}{\underline{a}}\right]$

Corollary 3.1. If $A=[\underline{a},\bar{a}]$ and $n>0$ then we get the following corollary

(i) $1+A=[1+\underline{a},1+\bar{a}]$ (ii) $1-A=[1-\bar{a},1-\underline{a}]$ (iii) $A^n=\left[\underline{a}^n,\bar{a}^n\right]$ (iv) $A^{-n}=\left[\frac{1}{\bar{a}^n},\frac{1}{\underline{a}^n}\right]$

(v) $(1+A)^n=\left[\left(1+\underline{a}\right)^n,\left(1+\bar{a}\right)^n\right]$ (vi) $(1+A)^{-n}=\left[\frac{1}{(1+\bar{a})^n},\frac{1}{(1+\underline{a})^n}\right]$

(vii) $1-(1+A)^{-n}=\left[\frac{(1+\underline{a})^n-1}{(1+\underline{a})^n},\frac{(1+\bar{a})^n-1}{(1+\bar{a})^n}\right]$

(viii) $\frac{1-(1+A)^n}{A}=\left[\min\left(\frac{1-(1+\bar{a})^n}{\bar{a}},\frac{1-(1+\bar{a})^n}{\underline{a}},\frac{1-(1+\underline{a})^n}{\bar{a}},\frac{1-(1+\underline{a})^n}{\underline{a}}\right),\max\left(\frac{1-(1+\bar{a})^n}{\bar{a}},\frac{1-(1+\bar{a})^n}{\underline{a}},\frac{1-(1+\underline{a})^n}{\bar{a}},\frac{1-(1+\underline{a})^n}{\underline{a}}\right)\right]$

(x) $\frac{(1+A)^n-1}{A}=\left[\left(1+\bar{a}\right)^n-1,\left(1+\underline{a}\right)^n-1\right].\left[\frac{1}{\bar{a}},\frac{1}{\underline{a}}\right]$ (xi) $\ln\left(1+A\right)=[\ln\left(1+\underline{a}\right),\ln\left(1+\bar{a}\right)]$

(xii) $e^{[\underline{a},\bar{a}]}=[e^{\underline{a}},e^{\bar{a}}]$ (xiii) $e^{-[\underline{a},\bar{a}]}=e^{[-\bar{a},-\underline{a}]}=[e^{-\bar{a}},e^{-\underline{a}}]$

We have proved corollary 3.1 (i),(iii),(xi) and(xii):

Proof (i) $1+A=[1+\underline{a},1+\bar{a}]$

Let $x\in 1+A\Rightarrow x=1+a,$ where $\underline{a}\leq a\leq\bar{a}$

$\Rightarrow x=1+a,$ where $1+\underline{a}\leq 1+a\leq 1+\bar{a}$

$\Rightarrow x=1+a,$ where $1+\underline{a}\leq x\leq 1+\bar{a}$

$\Rightarrow x \in [1+\underline{a}, 1+\overline{a}]$

On the other hand;

Let $x \in [1+\underline{a}, 1+\overline{a}]$ $\exists\ \lambda \in [0,1]$ such that $x = \lambda(1+\underline{a}) + (1-\lambda)(1+\overline{a})$

$\Rightarrow x = 1 + \lambda\underline{a} + (1-\lambda)\overline{a}$ since $\lambda\underline{a} + (1-\lambda)\overline{a} \in A$

$\Rightarrow x \in 1 + A$

Hence $1 + A = [1+\underline{a}, 1+\overline{a}]$

Proof (iii) $A^n = [\underline{a}, \overline{a}]^n = \left[\underline{a}^n, \overline{a}^n\right]$

Let $x \in A^n \Rightarrow x = a^n$, where $\underline{a} \le a \le \overline{a}$

$\Rightarrow x = a^n$, where $\underline{a}^n \le a^n \le \overline{a}^n$

$\Rightarrow x = a^n$, where $\underline{a}^n \le x \le \overline{a}^n$

$\Rightarrow x \in [\underline{a}^n, \overline{a}^n]$

On the other hand;

Let $x \in [\underline{a}^n, \overline{a}^n] \Rightarrow \underline{a}^n \le x \le \overline{a}^n$

$\Rightarrow \underline{a} \le \sqrt[n]{x} \le \overline{a} \Rightarrow \sqrt[n]{x} \in [\underline{a}, \overline{a}] = A$

$\Rightarrow x \in A^n$ Hence $A^n = \left[\underline{a}^n, \overline{a}^n\right]$

Proof (xi) $\ln(1+A) = [\ln(1+\underline{a}), \ln(1+\overline{a})]$

Let $x \in \ln(1+A) \Rightarrow x = \ln(1+a)$ where $\underline{a} \le a \le \overline{a}$

$\Rightarrow x = \ln(1+a)$ where $1+\underline{a} \le 1+a \le 1+\overline{a}$

$\Rightarrow x = \ln(1+a)$ where $\ln(1+\underline{a}) \le \ln(1+a) \le \ln(1+\overline{a})$

$\Rightarrow x = \ln(1+a)$ where $\ln(1+\underline{a}) \le x \le \ln(1+\overline{a})$

$\Rightarrow x \in [\ln(1+\underline{a}), \ln(1+\overline{a})]$

On the other hand;

Let $x \in [\ln(1+\underline{a}), \ln(1+\overline{a})] \Rightarrow \ln(1+\underline{a}) \le x \le \ln(1+\overline{a})$

$\Rightarrow 1+\underline{a} \le e^x \le 1+\overline{a} \Rightarrow e^x \in [1+\underline{a}, 1+\overline{a}] = 1 + A$

$\Rightarrow x \in \ln(1+A)$ Hence $\ln(1+A) = [\ln(1+\underline{a}), \ln(1+\overline{a})]$

Proof (xii) $e^{[\underline{a},\overline{a}]} = [e^{\underline{a}}, e^{\overline{a}}]$

Let $x \in e^{[\underline{a},\overline{a}]} \Rightarrow \ln x = [\underline{a}, \overline{a}] \Rightarrow \underline{a} \le \ln x \le \overline{a} \Rightarrow e^{\underline{a}} \le x \le e^{\overline{a}} \Rightarrow x \in [e^{\underline{a}}, e^{\overline{a}}]$

On the other hand;

Let $x \in [e^{\underline{a}}, e^{\overline{a}}] \Rightarrow e^{\underline{a}} \le x \le e^{\overline{a}} \Rightarrow \underline{a} \le \ln x \le \overline{a}] \Rightarrow \ln x \in [\underline{a}, \overline{a}] \Rightarrow x \in e^{[\underline{a},\overline{a}]}$

Hence $e^{[\underline{a},\overline{a}]} = [e^{\underline{a}}, e^{\overline{a}}]$

Corollary 3.2. If $A = [\underline{a}, \overline{a}]$ and $n \in \mathrm{N}$, i is index then we get the following corollary

(i) $\sum_{i=1}^{n} (1+A)^i = \left[\sum_{i=1}^{n} (1+\underline{a})^i, \sum_{i=1}^{n} (1+\overline{a})^i\right]$ (ii) $\sum_{i=0}^{n} (1+A)^{-i} = \left[\sum_{i=0}^{n} \frac{1}{(1+\overline{a})^i}, \sum_{i=0}^{n} \frac{1}{(1+\underline{a})^i}\right]$

(iii) $\sum_{i=1}^{n} C_{t_i}(1+A)^{-i} = \left[\sum_{i=0}^{n} \frac{C_{t_i}}{(1+\bar{a})^i}, \sum_{i=0}^{n} \frac{C_{t_i}}{(1+\underline{a})^i}\right]$

4. Fuzzy interest rate

It is supposed that the risk discount rate or interest rate i may not be known exactly over the n periods. Because of political and economical and other factors interest rate may vary [10] and [16]. Countries Central Bank settles the interest rate in respect of inflation and other economic index.

Assumption 4.1. It is assume that the interest rate is fixed around some percent over the years. Let this rate is around i over the n periods of a financial contract. The interest rate $\tilde{i}$ is taken where $\tilde{i}$ a triangular fuzzy number defined by $\tilde{i} = (a,b,c)$, where $a < b < c$ and $a,b,c \in \mathbb{R}^+$. Any fuzzy number can be uniquely defined by a bounded interval [8][15]by taking it's $\alpha - cut$.

The alpha-cut of $\tilde{i} = (a,b,c)$ is $\tilde{i} = [a+(b-a)\alpha, c-(c-b)\alpha]$ where $\alpha \in [0,1]$.

The amount and duration of investment are assumed to have crisp value.

4.2. Mathematics of actuarial science and finance

The professionals of actuarial science and finance are responsible for financial studies of institutes and organizations. The inflow and out flow of money are recorded and are needed later for further investigations. Actuaries are the most responsible professionals, who use the financial record of the institutions/organizations to predict their future economical status with the help of established mathematical and statistical theorems [2],[3],[6],[7][11],[13],[17],[18],[20] and [21]. In this section the present value of annuities paid in advance and arrear, future values of annuities, equations of value and project or investment appraisal have been discussed.

Annuities 4.3.

Generally, an annuity is a series of payments in return for a deposit. Annuities are classified by annuity arrear and annuity advance.

4.4. Present values of annuities

The actuarial symbol $a_{\overline{n}|}(\ddot{a}_{\overline{n}|})$ represents the present value on an annuity paid in arrear (advance) consisting of n payments of 1 unit made at the end (start) of each of the next n time periods [11],[14]. It is supposed that interest rate (risk free rate of interest) is a fuzzy number (triangular fuzzy number). The formulae for the present values are

$$a_{\overline{n}|} = \frac{1-v^n}{i} \quad \text{and} \quad \ddot{a}_{\overline{n}|} = \frac{1-v^n}{d} = \frac{i}{d} a_{\overline{n}|} = a_{\overline{n-1}|} + 1$$

4.5. Present values of annuities in fuzzy form

The symbol $\tilde{a}_{\overline{n}|}(\tilde{\ddot{a}}_{\overline{n}|})$ represents the present value on an annuity paid in arrear (advance) consisting of n payments of 1 unit made at the end (start) of each of the next n time periods where the interest rate (risk free rate of interest) $\tilde{i} = (a,b,c)$ is a triangular fuzzy number. The formulae for the present values are:

$$\tilde{a}_{\overline{n}|}[\alpha]=\left[\frac{1-\left((1+a)+(b-a)\alpha\right)^{n}}{\left((1+a)+(b-a)\alpha\right)^{n}},\frac{1-\left((1+c)-(c-b)\alpha\right)^{n}}{\left((1+c)-(c-b)\alpha\right)^{n}}\right]\cdot\left[\frac{1}{c-(c-b)\alpha},\frac{1}{a+(b-a)\alpha}\right]$$

and applying the theorem 3.1 the following equation is obtained

$$A(\alpha)=\left[\frac{1-\left((1+a)+(b-a)\alpha\right)^{n}}{\left((1+a)+(b-a)\alpha\right)^{n}.\left(c-(c-b)\alpha\right)},\frac{1-\left((1+c)-(c-b)\alpha\right)^{n}}{\left((1+c)-(c-b)\alpha\right)^{n}.\left(c-(c-b)\alpha\right)},\frac{1-\left((1+a)+(b-a)\alpha\right)^{n}}{\left((1+a)+(b-a)\alpha\right)^{n}.\left(a+(b-a)\alpha\right)},\frac{1-\left((1+c)-(c-b)\alpha\right)^{n}}{\left((1+c)-(c-b)\alpha\right)^{n}.\left(a+(b-a)\alpha\right)}\right]$$

Therefore the present value of annuities is obtained in the fuzzy form as $\tilde{a}_{\overline{n}|}[\alpha]=\left[\min\left(\mathrm{A}(\alpha)\right),\max\left(\mathrm{A}(\alpha)\right)\right]$.

Again the actuarial symbol $\ddot{a}_{\overline{n}|}$ can be written in fuzzy expression as $\tilde{\ddot{a}}_{\overline{n}|}$ where $\tilde{\ddot{a}}_{\overline{n}|}=\tilde{a}_{\overline{n-1}|}+1$ and $\tilde{a}_{\overline{n-1}|}$ can be easily calculated from the expression $\tilde{a}_{\overline{n}|}$.

Thus the present value on an annuity paid in advance is obtained as the fuzzy form as

$$\tilde{\ddot{a}}_{\overline{n}|}[\alpha]=\left[\min\left\{\tilde{a}_{\overline{n-1}|}(\alpha)+1\right\},\max\left\{\tilde{a}_{\overline{n-1}|}(\alpha)+1\right\}\right]$$

where

$$\tilde{a}_{\overline{n-1}|}(\alpha)=\left[\frac{1-\left((1+a)+(b-a)\alpha\right)^{(n-1)}}{\left((1+a)+(b-a)\alpha\right)^{n}.\left(c-(c-b)\alpha\right)},\frac{1-\left((1+c)-(c-b)\alpha\right)^{(n-1)}}{\left((1+c)-(c-b)\alpha\right)^{n}.\left(c-(c-b)\alpha\right)},\frac{1-\left((1+a)+(b-a)\alpha\right)^{(n-1)}}{\left((1+a)+(b-a)\alpha\right)^{n}.\left(a+(b-a)\alpha\right)},\frac{1-\left((1+c)-(c-b)\alpha\right)^{(n-1)}}{\left((1+c)-(c-b)\alpha\right)^{n}.\left(a+(b-a)\alpha\right)}\right]$$

4.6. Future values of annuities

The future accumulated value of a series on n payment arrear (advance) each of amount 1 made at unit time over the period are formulated as

$$s_{\overline{n}|}=\frac{(1+i)^{n}-1}{i}\text{ and }\ddot{s}_{\overline{n}|}=\frac{(1+i)^{n}-1}{d}=\frac{i}{d}s_{\overline{n}|}=s_{\overline{n+1}|}-1.$$

4.7. Future values of annuities in fuzzy form

The accumulated future value of annuities arrear (advance) for fuzzy interest rate is denoted by $\tilde{s}_{\overline{n}|}$ $(\tilde{\ddot{s}}_{\overline{n}|})$ and are defined by

$$\tilde{s}_{\overline{n}|}[\alpha]=\frac{(1+i[\alpha])^{n}-1}{i[\alpha]}=\left[\left(a+(b-a)\alpha\right)^{n}-1,\left(c-(c-b)\alpha\right)^{n}-1\right]\cdot\left[\frac{1}{\left(c-(c-b)\alpha\right)},\frac{1}{\left(a+(b-a)\alpha\right)}\right]$$

Using multiplication rule of interval arithmetic's the future value of annuities is obtained in fuzzy form as $\tilde{s}_{\overline{n}|}[\alpha]=\left[\min(S(\alpha)),\max(S(\alpha))\right]$

where $S(\alpha)=\left[\frac{\left(\left(a+(b-a)\alpha\right)^{n}-1\right)}{\left(c-(c-b)\alpha\right)},\frac{\left(\left(a+(b-a)\alpha\right)^{n}-1\right)}{\left(a+(b-a)\alpha\right)},\frac{\left(\left(c-(c-b)\alpha\right)^{n}-1\right)}{\left(c-(c-b)\alpha\right)},\frac{\left(\left(c-(c-b)\alpha\right)^{n}-1\right)}{\left(a+(b-a)\alpha\right)}\right]$

From the above relation $\tilde{s}_{\overline{n}|}$ the values of $\tilde{\ddot{s}}_{\overline{n}|}$ can easily be found from the relation $\tilde{\ddot{s}}_{\overline{n}|}=\tilde{s}_{\overline{n-1}|}-1$.

Thus the accumulated future value of annuities paid in advance is obtained as

$$\tilde{\ddot{s}}_{\overline{n}|}=\left[\min\left(S^{(n-1)}(\alpha)-1\right),\max\left(S^{(n-1)}(\alpha)-1\right)\right]$$

where $s^{(n-1)}(\alpha)=\left[\frac{\left((a+(b-a)\alpha)^{(n-1)}-1\right)}{(c-(c-b)\alpha)},\frac{\left((a+(b-a)\alpha)^{(n-1)}-1\right)}{(a+(b-a)\alpha)},\frac{\left((c-(c-b)\alpha)^{(n-1)}-1\right)}{(c-(c-b)\alpha)},\frac{\left((c-(c-b)\alpha)^{(n-1)}-1\right)}{(a+(b-a)\alpha)}\right]$

4.8. Continuous annuities

The value at time 0 of an annuity payable continuously between 0 and time n, where the rate of payment per unit time is constant and equal to 1, is denoted by $\bar{a}_{\overline{n}|}$. The formulae for the present value and accumulated values of continuous payments are present value

$\bar{a}_{\overline{n}|}=\frac{1-v^n}{\delta}=\frac{i}{\delta}a_{\overline{n}|}$ and accumulated amount

$\bar{s}_{\overline{n}|}=\frac{(1+i)^n}{\delta}=\frac{i}{\delta}s_{\overline{n}|}$ where $\delta=\log(1+i)$.

4.9. Continuous annuities in fuzzy form

Present value and future accumulated value of continuous annuities for fuzzy interest are given as present value

$\tilde{\bar{a}}_{\overline{n}|}=\frac{1-\tilde{v}^n}{\tilde{\delta}}=\frac{\tilde{i}}{\tilde{\delta}}\tilde{a}_{\overline{n}|}$ which implies $\tilde{\bar{a}}_{\overline{n}|}[\alpha]=\left[\min\left\{\frac{\tilde{i}[\alpha]}{\tilde{\delta}[\alpha]}\tilde{a}_{\overline{n}|}[\alpha]\right\},\max\left\{\frac{\tilde{i}[\alpha]}{\tilde{\delta}[\alpha]}\tilde{a}_{\overline{n}|}[\alpha]\right\}\right]$

and accumulated value of continuous annuity $\tilde{\bar{s}}_{\overline{n}|}=\frac{(1+\tilde{i})^n}{\tilde{\delta}}=\frac{\tilde{i}}{\tilde{\delta}}\tilde{s}_{\overline{n}|}$ which

implies $\tilde{\bar{s}}_{\overline{n}|}[\alpha]=\left[\min\left\{\frac{\tilde{i}[\alpha]}{\tilde{\delta}[\alpha]}\tilde{s}_{\overline{n}|}[\alpha]\right\},\max\left\{\frac{\tilde{i}[\alpha]}{\tilde{\delta}[\alpha]}\tilde{s}_{\overline{n}|}[\alpha]\right\}\right]$.

4.10. Perpetuity annuities

An annuity that is payable forever is perpetuity annuity. The annuity perpetuity is defined for arrear (advance) by $a_{\overline{\infty}|}=\frac{1}{i}$ and $\ddot{a}_{\overline{\infty}|}=\frac{1}{d}$, where $d=\frac{i}{1+i}$.

4.11. Perpetuity annuities in fuzzy form

The present value of perpetuity annuity arrear (advance) when interest rate is fuzzy number defined by $\tilde{a}_{\overline{\infty}|}=\frac{1}{\tilde{i}}\Rightarrow\tilde{a}_{\overline{\infty}|}[\alpha]=\left[\min\left\{\frac{1}{\tilde{i}[\alpha]}\right\},\max\left\{\frac{1}{\tilde{i}[\alpha]}\right\}\right]$ and

$\tilde{\ddot{a}}_{\overline{\infty}|}=\frac{1}{\tilde{d}}\Rightarrow\tilde{\ddot{a}}_{\overline{\infty}|}[\alpha]=\left[\min\left\{\frac{\tilde{i}[\alpha]}{1+\tilde{i}[\alpha]}\right\},\max\left\{\frac{\tilde{i}[\alpha]}{1+\tilde{i}[\alpha]}\right\}\right]$

4.12. Deferred annuities

A deferred annuity means that one makes the deposit today, but the payments won't start until sometime in the future, say when one retires.

4.13. Present value of deferred annuity

The value at time 0 of a series of n payments, each of amounts 1, due at times $(m+1),(m+2),\ldots,(m+n)$ is denoted by ${}_{m|}a_{\overline{n}|}$.

Present value of deferred annuity payable in arrear is defined by ${}_{m|}a_{\overline{n}|}=a_{\overline{m+n}|}-a_{\overline{m}|}=v^m a_{\overline{n}|}$

and that of which payable in advance by ${}_{m|}\ddot{a}_{\overline{n}|} = \ddot{a}_{\overline{m+n}|} - \ddot{a}_{\overline{m}|} = v^m \ddot{a}_{\overline{n}|}$. The Present value of deferred annuity payable in continuous is defined by ${}_{m|}\bar{a}_{\overline{n}|} = \bar{a}_{\overline{m+n}|} - \bar{a}_{\overline{m}|} = v^m \bar{a}_{\overline{n}|}$.

4.14. Deferred annuities in fuzzy form

The deferred annuity of ${}_{m|}a_{\overline{n}|}$, ${}_{m|}\ddot{a}_{\overline{n}|}$ and ${}_{m|}\bar{a}_{\overline{n}|}$ for triangular fuzzy interest rate $\tilde{i} = (a,b,c)$ are denoted by ${}_{m|}\tilde{a}_{\overline{n}|}$, ${}_{m|}\tilde{\ddot{a}}_{\overline{n}|}$ and ${}_{m|}\tilde{\bar{a}}_{\overline{n}|}$. These are defined by ${}_{m|}\tilde{a}_{\overline{n}|} = \tilde{v}^m \tilde{a}_{\overline{n}|}$, ${}_{m|}\tilde{\ddot{a}}_{\overline{n}|} = \tilde{v}^m \tilde{\ddot{a}}_{\overline{n}|}$, ${}_{m|}\tilde{\bar{a}}_{\overline{n}|} = \tilde{v}^m \tilde{\bar{a}}_{\overline{n}|}$ where $\tilde{a}_{\overline{n}|}$, $\tilde{\ddot{a}}_{\overline{n}|}$ and $\tilde{\bar{a}}_{\overline{n}|}$ are fuzzy annuity, and $\tilde{v}^m = \left(1+\tilde{i}\right)^{-m}$, ${}_{m|}\tilde{a}_{\overline{n}|} = \tilde{v}^m \tilde{a}_{\overline{n}|} \Rightarrow {}_{m|}\tilde{a}_{\overline{n}|}[\alpha] = \tilde{v}^m[\alpha].\tilde{a}_{\overline{n}|}[\alpha]$ this implies

$${}_{m|}\tilde{a}_{\overline{n}|}[\alpha] = \left[\min\left\{\frac{\tilde{a}_{\overline{n}|}[\alpha]}{\left((1+c)-(c-b)\alpha\right)^m}, \frac{\tilde{a}_{\overline{n}|}[\alpha]}{\left((1+a)+(b-a)\alpha\right)^m}\right\}, \max\left\{\frac{\tilde{a}_{\overline{n}|}[\alpha]}{\left((1+c)-(c-b)\alpha\right)^m}, \frac{\tilde{a}_{\overline{n}|}[\alpha]}{\left((1+a)+(b-a)\alpha\right)^m}\right\}\right].$$

5. Price of forward contract

A forward contract is an agreement made between two parties under which one agrees to buy from the other a specified amount of asset (security) at a specified price, called the forward price, on a specified future date. In this section fuzziness are studied to evaluate the forward contract.

5.1. Forward price of a security with no income

The forward price of a security is found with no income as $K = S_0 e^{\delta \mathrm{T}}$ where K, S_0, δ and T are respectively forward price, present value, force (continuous) rate of interest and duration of the contract. The fuzzy form of this price is $\tilde{K} = S_0 e^{\tilde{\delta}\mathrm{T}}$ and the alpha-cuts is $\tilde{K}[\alpha] = \left[\min\left(S_0 e^{(\tilde{\delta}[\alpha].\mathrm{T})}\right), \max\left(S_0 e^{(\tilde{\delta}[\alpha].\mathrm{T})}\right)\right]$.

5.2. Forward price of a security with fixed cash income

The forward price of a security with fixed cash income I is found as $K = (S_0 - \mathrm{I}) e^{\delta \mathrm{T}}$ where $K, S_0, \delta, \mathrm{T}$ are defined earlier. The fuzzy form of this price is expressed by $\tilde{K} = (S_0 - \mathrm{I}) e^{\tilde{\delta}\mathrm{T}}$ and the alpha-cuts is found to be $\tilde{K}[\alpha] = \left[\min\left((S_0 - \mathrm{I}) e^{(\tilde{\delta}[\alpha].\mathrm{T})}\right), \max\left((S_0 - \mathrm{I}) e^{(\tilde{\delta}[\alpha].\mathrm{T})}\right)\right]$.

5.3. Forward price of a security with known dividend yield

The forward price of a security with known dividend yield D, when the dividends are payable continuously is found to be $K = S_0 e^{(\delta - \mathrm{D})\mathrm{T}}$ and this can be written in fuzzy form as $\tilde{K} = S_0 e^{(\tilde{\delta} - \mathrm{D})\mathrm{T}}$ and the alpha-cuts of which is found to be

$$\tilde{K}[\alpha] = \left[\min\left(S_0 e^{((\tilde{\delta}[\alpha] - \mathrm{D}).\mathrm{T})}\right), \max\left(S_0 e^{((\tilde{\delta}[\alpha] - \mathrm{D}).\mathrm{T})}\right)\right].$$

If the dividend yield is a fuzzy number then the forward price will be

$$\tilde{K}[\alpha] = \left[\min\left(S_0 e^{((\tilde{\delta}[\alpha] - \tilde{\mathrm{D}}[\alpha]).\mathrm{T})}\right), \max\left(S_0 e^{((\tilde{\delta}[\alpha] - \tilde{\mathrm{D}}[\alpha]).\mathrm{T})}\right)\right].$$

Example 5.1. (Forward Contract) The dividend yield of a portfolio of shares with current price of $673K is 2.8%pa payable continuously. Forward price of one year forward contract can be calculated, based on the portfolio, if assuming that the dividends are received continuously and the risk free effective rate of interest is around 4.6028%pa.

Solution. Suppose the forward price is $K = S_0 e^{(\delta - \mathrm{D})\mathrm{T}}$. It is provided that dividend D= 2.8% pa payable continuously, T=1 year, $S_0 = \$673\mathrm{K}$ and $\delta = \log(1+i) = \log(1.04628)$.

Therefore k=684539. The risk free effective rate of interest is around 4.6028% pa, it can be considered as a triangular fuzzy number which is $\tilde{i}$ = (0.04, 0.046028, 0.048).

The result $\tilde{K}$ is obtained with MATLAB script $\tilde{K} = (680594.07, 684538.89, 685829.40)$.

6. Fuzzy term structure of interest rate

The term structure of interest rate is referred to the term wise variation of interest rate. Assuming the bonds are non-convertible and non-callable, the price of the k^{th} bond, P_k is then can be expressed as the sum of the discounted cash flow $P_k = \sum_{i=1}^{n(k)} C_i^k f_{t_i^k}$ where f_t is the discounted value of one dollar at the end of maturity of t years: i.e $f_t = (1+i_t)^{-t}$, and i_t is the spot rate (also call the internal rate of return IRR).

The fuzzy term structure of interest rate comes when the rate is considered as a fuzzy number. The spot rate $\tilde{i}_t$ is considered to be a triangular fuzzy number. Hence, the price of bonds with term structure interest rate can be given as $\tilde{P}_k = \sum_{i=1}^{n(k)} C_i^k \tilde{f}_{t_i^k}$ which is also a fuzzy number and the alpha-cuts is found: $\tilde{P}_k[\alpha] = \left[\min\left(\sum_{i=1}^{n(k)} C_i^k \tilde{f}_{t_i^k}[\alpha] \right), \max\left(\sum_{i=1}^{n(k)} C_i^k \tilde{f}_{t_i^k}[\alpha] \right) \right]$.

Example 6.1. (Fuzzy term structure of interest rate) In this example we are calculating the price of a five year fixed security, redeemable at par, with 6% annual coupons if the annual term structure of interest rate is (7%, 7.25%, 7.5%, 7.75%, 8%,…) or around.

Solution The price per 100 nominal coupons is obtained from the equation of value, applying the interest rate to (7%, 7.25%, 7.5%, 7.75%, 8 %,…) respectively

$\mathrm{P} = 6*(v_{7\%} + v_{7.25\%}^2 + v_{7.5\%}^3 + v_{7.75\%}^4 + v_{8\%}^5) + 100 v_{8\%}^5$ => P=$92.25.

Applying the given interest as a triangular fuzzy number for each year respectively (6.75%, 7%, 7.25%), (7%, 7.25%, 7.5%), (7.25%, 7.5%, 7.75%), (7.5%, 7.75%, 8%), (7.75%, 8%, 8.25%).

The fuzzy form of the given security price can be expressed

$\tilde{\mathrm{P}} = 6*(\tilde{v}_{7\%} + \tilde{v}_{7.25\%}^2 + \tilde{v}_{7.5\%}^3 + \tilde{v}_{7.75\%}^4 + \tilde{v}_{8\%}^5) + 100 \tilde{v}_{8\%}^5$ and the final price is obtained

$\tilde{\mathrm{P}} = (91.31, 92.25, 93.20)$ with MATLAB script.

7. Application

In this section some applications of fuzzy interest rate are demonstrated which are equation of value, project appraisal and interest rate of return.

7.1. Equation of value

Many problems in actuarial work can be reduced to solving an equation of value [11] for an unknown quantity.

The examples are based on a financial security which operates in the following way:

A price p is paid (by the investor) in return for a series of interest payments of I payable at the end of the next n years, and a final redemption payment of R payable at the end of the n years. The equation of value for this investment is $p = \mathrm{I}a_{\overline{n}|} + \mathrm{R}v^n ;@ i\%$.

If the interest rate is a triangular fuzzy number $\tilde{i}$ defined by $\tilde{i} = (a,b,c)$ then the above equation can be given by a fuzzy expression $\tilde{p} = \mathrm{I}\tilde{a}_{\overline{n}|} + \mathrm{R}\tilde{v}^n ;@ \tilde{i}\%$, where I, R and n are crisp number. Any P can be found if I, R and $\tilde{i}$ are given. The interest rate $\tilde{i}$ can be found if P, I and R are given.

7.2. Procedure of finding $\tilde{i}$

The fuzzy form of the equation of value is

$\tilde{p} = \mathrm{I}\tilde{a}_{\overline{n}|} + \mathrm{R}\tilde{v}^n ;@ \tilde{i} = (a,b,c)$ this implies $\tilde{p} = \mathrm{I}\left(\frac{1-(1+\tilde{i})^n}{\tilde{i}}\right) + \mathrm{R}\left(1+\tilde{i}\right)^{-n} ;@ \tilde{i} = (a,b,c)$

$$\tilde{p}[\alpha] = \left[\min\left\{\mathrm{I}\left(\frac{1-(1+\tilde{i}[\alpha])^n}{\tilde{i}[\alpha]}\right) + \mathrm{R}\left(1+\tilde{i}[\alpha]\right)^{-n}\right\}, \max\left\{\mathrm{I}\left(\frac{1-(1+\tilde{i}[\alpha])^n}{\tilde{i}[\alpha]}\right) + \mathrm{R}\left(1+\tilde{i}[\alpha]\right)^{-n}\right\}\right] \text{ where } \alpha \in [0,1].$$

Example 7.1. The interest rate can be found if provided p=78.92, I=5, R=125 and n=10.

Solution. The equation of value is $\tilde{p} = \mathrm{I}\tilde{a}_{\overline{n}|} + \mathrm{R}\tilde{v}^n ;@ \tilde{i} = (a,b,c)$

After putting the given values it becomes $78.92 = 5\tilde{a}_{\overline{n}|} + 125\tilde{v}^n ;@ \tilde{i} = (a,b,c)$

Next step is to find $\tilde{i} = (a,b,c)$. At the end of the computation it is obtained that $\tilde{i} = (0.12, 0.123, 0.125)$ which is a triangular fuzzy number.

7.3. Project appraisal

This refers to the process of assessing, in a structured way, the case for proceeding with a project proposal. In short, project appraisal is the effort of assessing a project's feasibility. It often involves comparing various options, using economic appraisal or some other decision analysis technique [4].

A project is studied only on the basis of its financial value that is, on basis of the present value of the money. The higher the present value of an investment or project the more viable it would be [5] and [20].

Example 7.2. Two hypothetical cash flows of Project A and Project B is considered.

Project A: Delegates all the development works to outside companies. The estimated cash flows are

Year	Cash flows	Remarks
Beginning of year 1	($150,000)	Contractors' fees
Beginning of year 2	($250,000)	Contractors' fees
Beginning of year 3	($250,000)	Contractors' fees
End of year 3	$1,000,000	Sales

Project B: Carries out all the development work in-house by purchasing the necessary equipment and using the company's own staff. The staff cost is assumed to be paid throughout the year. The estimated cash flows are

Year	Cash flows	Remarks
Beginning of year 1	($325,000)	New equipment
Throughout of year 1	($75,000)	Staff costs
Throughout of year 2	($90,000)	Staff costs
Throughout of year 2	($120,000)	Staff costs
End of year 3	$1,000,000	Sales

We are evaluating the Project A and Project B using a risk discount rate is around 20% per annum.

Solution. Given the risk discount rate is around 20% per annum. Taking $\tilde{i} = \text{around}\, 20\% = (.19,.20,.21)$ as a triangular fuzzy number. The net present values of Project A and B can respectively be written as

$NPV_A = -150 - 250\tilde{v} - 250\tilde{v}^2 + 100\tilde{v}^3 \; @ \; \tilde{i} = \text{around}\, 20\% = (.19,.20,.21)$ and

$NPV_B = -325 - 75\tilde{\bar{a}}_{\overline{1|}} - 90\tilde{v}\tilde{\bar{a}}_{\overline{1|}} - 120\tilde{v}^2\tilde{\bar{a}}_{\overline{1|}} + 100\tilde{v}^3 \; @ \; \tilde{i} = \text{around}\, 20\% = (.19,.20,.21)$

After calculating the net present value of the cash flow of project A it is obtained that $\text{NPV}_\text{A} = (37.1090, 46.7593, 56.7906)$. Similarly calculating project B it is obtained that $\text{NPV}_\text{B} = (25.2872,\ 41.3704,\ 57.8948)$.

From the consideration of computed present value it can be concluded that the project A is more feasible/profitable than the project B.

8. Result

The applications of fuzzy interest rates have been presented with example through out the paper. Since state of economy is uncertain, deterministic system of interest rate model is not the satisfactory tool to use in calculating the worth of future financial decision. Now a days varying interest rate is a common technique that are used to improve the performance. Taking interest rate as a fuzzy number that fits to the deterministic and varying model at the same time, the results obtained from the fuzzy system is a fuzzy number. Taking different alpha level of the interest rate around 0.09 the resultant interval has been tabulated in (Table 1). Taking Interest rates are around 0.08, 0.14, 0.20 and 0.25 the results have been shown in (Table 2). It has been observed from the tabulated values that when the alpha level is 1, the obtained results are equal that of the exact interest rate. A fuzzy number can be uniquely defined as a bounded interval. The results of deterministic and varying interest rate lie in this bounded interval.

9. Conclusion

In this paper fuzzy interest rate is applied with example to the future and present value of money, annuity arrear and advance, price of forward contract, fuzzy term structure rate of interest, equation of value and evaluation of project. For all the tools the interest rate is a triangular fuzzy number of the form $\tilde{i} = (a, b, c)$ and the calculation was performed by taking $\alpha - cuts$ and interval arithmetic. The messages of this study are that fuzzy system is a very advanced and suitable tool that can be applied in actuarial and financial mathematics.

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Table 1: Taking interest rate is around 9%. The triangular fuzzy number is taken as $\tilde{i} = (0.08, 0.09, 0.10)$ corresponding α-level is $\tilde{i}[\alpha] = (0.08 + 0.01\alpha, 0.10 - 0.01\alpha)$, n=10 year. The results of α-levels have been shown in vertically.

α-level	$(1+\tilde{i})^n$	$(1+\tilde{i})^{-n}$	$\tilde{a}_{\overline{n}\vert}$	$\tilde{\ddot{a}}_{\overline{n}\vert}$	$\tilde{\bar{a}}_{\overline{n}\vert}$	$s_{\overline{n}\vert}$	$\tilde{\ddot{s}}_{\overline{n}\vert}$	$\tilde{\bar{s}}_{\overline{n}\vert}$
0.1	2.1790	0.3891	6.1711	5.5489	5.2950	14.5555	13.0881	12.4893
	2.5703	0.4589	6.6799	8.8257	8.4907	15.8612	20.9562	20.1609
0.2	2.1992	0.3926	6.1977	5.6941	5.4360	14.6249	13.4364	12.8274
	2.5470	0.4547	6.6500	8.5992	8.2691	15.7854	20.4124	19.6288
0.3	2.2197	0.3962	6.2246	5.8428	5.5805	14.6946	13.7934	13.1741
	2.5239	0.4505	6.6202	8.3791	8.0537	15.7100	19.8837	19.1116
0.4	2.2402	0.3998	6.2516	5.9953	5.7287	14.7647	14.1593	13.5297
	2.5010	0.4464	6.5907	8.1649	7.8443	15.6349	19.3694	18.6089
0.5	2.2610	0.4035	6.2788	6.1516	5.8807	14.8351	14.5345	13.8945
	2.4782	0.4423	6.5613	7.9566	7.6407	15.5603	18.8691	18.1200
0.6	2.2819	0.4072	6.3062	6.3118	6.0366	14.9059	14.9192	14.2687
	2.4557	0.4382	6.5322	7.7539	7.4427	15.4860	18.3823	17.6444
0.7	2.3030	0.4110	6.3338	6.4762	6.1966	14.9771	15.3138	14.6527
	2.4333	0.4342	6.5033	7.5566	7.2500	15.4122	17.9084	17.1818
0.8	2.3243	0.4147	6.3615	6.6448	6.3608	15.0487	15.7187	15.0468
	2.4112	0.4302	6.4745	7.3645	7.0625	15.3387	17.4471	16.7316
0.9	2.3457	0.4186	6.3895	6.8177	6.5293	15.1206	16.1340	15.4514
	2.3892	0.4263	6.4460	7.1774	6.8800	15.2656	16.9979	16.2934
1	2.3674	0.4224	6.4177	6.9952	6.7023	15.1929	16.5603	15.8668
	2.3674	0.4224	6.4177	6.9952	6.7023	15.1929	16.5603	15.8668

Table 2: Taking Interest rates are around 0.08, 0.14, 0.20 and 0.25 and the results have been shown vertically in the table:

$\tilde{i}$ and n=5.	$(1+\tilde{i})^n$	$(1+\tilde{i})^{-n}$	$\tilde{a}_{\overline{n}\vert}$	$\tilde{\ddot{a}}_{\overline{n}\vert}$	$\tilde{\bar{a}}_{\overline{n}\vert}$	$s_{\overline{n}\vert}$	$\tilde{\ddot{s}}_{\overline{n}\vert}$	$\tilde{\bar{s}}_{\overline{n}\vert}$
(0.06,0.08,0.10)	1.34	0.62	3.79	2.50	2.39	5.64	3.72	3.55
	1.47	0.68	3.99	4.31	4.15	5.87	6.34	6.10
	1.61	0.75	4.21	7.44	7.23	6.11	10.79	10.48
(0.12,0.14,0.15)	1.76	0.50	3.35	3.08	2.88	6.35	5.84	5.45
	1.93	0.52	3.43	3.91	3.67	6.61	7.54	7.06
	2.01	0.57	3.60	5.05	4.77	6.74	9.44	8.92
(0.18,0.20,0.22)	2.29	0.37	2.86	2.86	2.59	7.15	7.14	6.48
	2.49	0.40	2.99	3.59	3.28	7.44	8.93	8.16
	2.70	0.44	3.13	4.51	4.16	7.74	11.16	10.29
(0.24,0.25,0.26)	2.93	0.31	2.64	3.06	2.74	8.05	9.36	8.36
	3.05	0.33	2.69	3.36	3.01	8.21	10.26	9.19

3.18	0.34	2.75	3.69	3.32	8.37	11.24	10.11

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