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Balanced Interval-Valued Fuzzy Graphs

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ABSTRACT

In this paper, we discuss notion of ring sum of product interval-valued fuzzy graphs. We define tensor product of two interval-valued fuzzy graphs and shown that the tensor product of two product interval-valued fuzzy graphs is a product interval-valued fuzzy graph. Likewise, given three independent theorems based on ring sum, join and isomorphism of product interval-valued fuzzy graphs. Finally, we define balanced and strictly balanced interval-valued fuzzy graphs and investigated several properties.

Keywords: Interval-valued fuzzy graph, isomorphism, ring sum, product interval-valued fuzzy graph, density, balanced interval-valued fuzzy graph.

1. Introduction

Presently, science and technology is featured with complex processes and phenomena for which complete information is not always available. For such cases, mathematical models are developed to handle various types of systems containing elements of uncertainty. A large number of these models is based on an extension of the ordinary set theory, namely, fuzzy sets. Graph theory has numerous applications to problems in computer science, electrical engineering, system analysis, operations research, economics, networking routing, transportation, etc. In many cases, some aspects of a graph-theoretic problem may be uncertain. For example, the vehicle travel time or vehicle capacity on a road network are not crisp number. In such cases, it is natural to deal with the uncertainty using the methods of fuzzy sets and fuzzy logic. But, using hypergraphs as the models of various systems (social, economic systems, communication networks and others) leads to difficulties. In 1965, Zadeh [24] introduced the notion of a fuzzy subset of a set as a method for representing uncertainty. The theory of fuzzy sets has become a vigorous area of research in different disciplines including medical and life sciences, engineering, statics, graph theory, computer networks, decision making and automata theory.

In 1975, Rosenfeld [14] introduced the concept of fuzzy graphs, and proposed another elaborated definition, including fuzzy vertex and fuzzy edges, and several fuzzy analogs of graph theoretic concepts such as paths, cycles, connectedness, etc. Zadeh [25] introduced the notion of interval-valued fuzzy sets as an extension of fuzzy set [24] in which the values of the membership degrees are interval of number instead of the number. Akram and Dudek [2] defined interval-valued fuzzy graph. Then Akram and

Karunambigai [5] defined length, distance, eccentricity, radius and diameter of a bipolar fuzzy graph and introduced the concept of self centered bipolar fuzzy graphs. Akram and Davvaz [4] discussed the properties of strong intuitionistic fuzzy graphs and also they introduced the concept of intuitionistic fuzzy line graphs. Akram introduced the concept of bipolar fuzzy graphs and studied some properties on it [3]. Akram et al. [6] defined Certain types of vague graphs. Talebi and Rashmanlou [20] studied properties of isomorphism and complement on interval-valued fuzzy graphs. Likewise, they defined isomorphism on vague graphs [21]. Bhattacharya [7] gave some remarks on fuzzy graphs. Ramaswamy and Poornima introduced product fuzzy graphs. Hawary in [1] defined complete fuzzy graphs and gave three new operations on it. Nagoorgani and Malarvizhi [10, 11] investigated isomorphism properties on fuzzy graphs. Also they defined the self complementary fuzzy graphs. The complement of fuzzy graphs was studied by Sunitha and Vijayakumar [15]. It consists to define bijective correspondence existence which preserve adjacent relation between vertex sets of two graphs.

Samanta and Pal introduced fuzzy tolerance graph [16], irregular bipolar fuzzy graphs [18], fuzzy k-competition graphs and p-Competition fuzzy graphs [19], bipolar fuzzy hypergraphs [17] and investigated several properties. Pal and Rashmanlou [12] studied lots of properties of irregular interval-valued fuzzy graphs.

In this paper, we have discussed notion of ring sum of product interval-valued fuzzy graphs. We further provided three independent theorems based on ring sum, join and isomorphism of product interval-valued fuzzy graphs. The concept of density and balance of interval-valued fuzzy graphs are introduced and investigated several useful properties.

2. Preliminaries

Let V be a Universe of discourse. It may be taken as the set of vertices of a graph G. If the membership value of $u \in V$ is non-zero, then u is consider as a vertex of G.

Definition 2.1. *A fuzzy graph is a pair* $G = (\sigma, \mu)$, *where* σ *is a fuzzy subset of* V *and* μ *is fuzzy relation on* V *such that,* $\mu(u,v) \le \sigma(u) \wedge \sigma(v)$ *for all* $u, v \in V$, *where* $x \wedge y$ *represents the minimum among* x *and* y.

A very special type of fuzzy graph called complete fuzzy graph is defined below.

Definition 2.2. *A fuzzy graph* $G = (\sigma, \mu)$ *is complete if* $\mu(u,v) = \sigma(u) \wedge \sigma(v)$ *for all* $u, v \in V$.

The main objective of this paper is to study of interval-valued fuzzy graph and this graph is based on the interval-valued fuzzy set defined below.

Definition 2.3. *An interval-valued fuzzy set* A *in* V *is defined as*

$$A = \{(x, [\mu_{A^-}(x), \mu_{A^+}(x)]) : x \in V\},$$

where $\mu_{A^-}(x)$ and $\mu_{A^+}(x)$ are fuzzy subsets of V such that $\mu_{A^-}(x) \le \mu_{A^+}(x)$ for all $x \in V$.

For any two interval-valued fuzzy sets $A = \{(x, [\mu_{A^-}(x), \mu_{A^+}(x)]) \mid x \in V\}$ and

$B = \{(x, [\mu_{B^-}(x), \mu_{B^+}(x)]) \mid x \in V\}$ in V we define:

$$A \cup B = \{(x, [\max(\mu_{A^-}(x), \mu_{B^-}(x)), \max(\mu_{A^+}(x), \mu_{B^+}(x))]) \mid x \in V\},$$
$$A \cap B = \{(x, [\min(\mu_{A^-}(x), \mu_{B^-}(x)), \min(\mu_{A^+}(x), \mu_{B^+}(x))]) \mid x \in V\}.$$

Definition 2.4. *By an interval-valued fuzzy graph of a crisp graph* $G^* = (V, E)$ *we mean a pair* $G = (A, B)$*, where* $A = [\mu_{A^-}, \mu_{A^+}]$ *is an interval-valued fuzzy set on* V *and* $B = [\mu_{B^-}, \mu_{B^+}]$ *is an interval-valued fuzzy set defined on* E*, such that*
$\mu_{B^-}(xy) \le \min(\mu_{A^-}(x), \mu_{A^-}(y)),\quad \mu_{B^+}(xy) \le \min(\mu_{A^+}(x), \mu_{A^+}(y))$ *for all* $xy \in E.$

Definition 2.5. *Let* $H_1 = (A_1, B_1)$ *and* $G = (A, B)$ *be two interval valued fuzzy graphs whose underline graphs be* $H_1^* = (V_1, E_1)$ *and* $G^* = (V, E)$*. Then* H_1 *is said to be a subgraph of* G *if*
$(i)\, V_1 \subseteq V$, where $\mu_{A_1^-(u_i)} = \mu_{A_2^-(u_i)}$, $\mu_{A_1^+(u_i)} = \mu_{A_2^+(u_i)}$ for all $u_i \in V_1$, $i = 1,2,3,\cdots,n$.
$(ii)\, E_1 \subseteq E$, where $\mu_{B_1^-(v_iv_j)} = \mu_{B_2^-(v_iv_j)}$, $\mu_{B_1^+(v_iv_j)} = \mu_{B_2^+(v_iv_j)}$ for all $v_iv_j \in E_1$, $i = 1,2,\cdots,n$.

Definition 2.6. *An interval-valued fuzzy graph* $G = (A, B)$ *of a graph* $G^* = (V, E)$ *is said to be complete interval-valued fuzzy graph if*
$\mu_{B^-}(xy) = \min(\mu_{A^-}(x), \mu_{A^-}(y))$ and $\mu_{B^+}(xy) = \min(\mu_{A^+}(x), \mu_{A^+}(y))$
for all $xy \in E$.

Definition 2.7. *The complement of an interval-valued strong fuzzy graph* $G = (A, B)$ *of a graph* $G^* = (V, E)$ *is an interval-valued fuzzy graph* $\overline{G} = (\overline{A}, \overline{B})$ *of* $\overline{G^*} = (V, V \times V)$*, where* $\overline{A} = A = [\mu_{A^-}, \mu_{A^+}]$ *and* $\overline{B} = [\overline{\mu_{B^-}}, \overline{\mu_{B^+}}]$. $\overline{\mu_{B^-}}$ *and* $\overline{\mu_{B^+}}$ *are defined as*

$$\overline{\mu_{B^-}}(xy) = \min(\mu_{A^-}(x), \mu_{A^-}(y)) - \mu_{B^-}(xy),$$
$$\overline{\mu_{B^+}}(xy) = \min(\mu_{A^+}(x), \mu_{A^+}(y)) - \mu_{B^+}(xy) \text{ for all } xy \in E.$$

Definition 2.8. *Let* G *be an interval-valued fuzzy graph. The neighbourhood of a vertex* x *in* G *is defined by* $N(x) = [N^-(x), N^+(x)]$*, where*

$N^-(x) = \{y \in V : \mu_{B^-}(xy) \le \min(\mu_{A^-}(x), \mu_{A^-}(y))\}$ and

$N^+(x) = \{y \in V : \mu_{B^+}(xy) \le \min(\mu_{A^+}(x), \mu_{A^+}(y))\}$. Also, the neighbourhood degree of a vertex x in G is defined by $deg(x) = [deg^-(x), deg^+(x)]$ where $deg^-(x) = \sum_{y \in N(x)} \mu_{A^-}(y)$ and $deg^+(x) = \sum_{y \in N(x)} \mu_{A^+}(y)$.

Definition 2.9. *Let G be an interval-valued fuzzy graph. The closed neighbourhood degree of the vertex x is defined by $deg[x] = [deg^-[x], deg^+[x]]$, where $deg^-[x] = deg^-(x) + \mu_{A^-}(x)$ and $deg^+[x] = deg^+(x) + \mu_{A^+}(x)$.*

Definition 2.10. *An interval-valued fuzzy graph $G = (A, B)$ is said to be regular interval-valued fuzzy graph if all the vertices have the same closed neighborhood degree.*

Definition 2.11. *Let $G = (A, B)$ be an interval-valued fuzzy graph of a graph $G^* = (V, E)$. If $\mu_{B^-}(xy) \le \mu_{A^-}(x) \times \mu_{A^-}(y)$ and $\mu_{B^+}(xy) \le \mu_{A^+}(x) \times \mu_{A^+}(y)$* for all $x, y \in V$, then the interval-valued fuzzy graph G is called product interval-valued fuzzy graph of G^*, where $\times$ represent ordinary multiplication.

Remark 2.12. *If $G = (A, B)$ is a product interval-valued fuzzy graph, then since $\mu_{A^-}(x)$ and $\mu_{A^+}(x)$ are less than or equal to 1, it follows that:*

$$\mu_{B^-}(xy) \le \mu_{A^-}(x) \times \mu_{A^-}(y) \le \mu_{A^-}(x) \wedge \mu_{A^-}(y) \text{ and}$$
$$\mu_{B^+}(xy) \le \mu_{A^+}(x) \times \mu_{A^+}(y) \le \mu_{A^+}(x) \wedge \mu_{A^+}(y) \text{ for all } x, y \in V.$$

Thus every product interval-valued fuzzy graph is an interval-valued fuzzy graph.

Remark 2.13. *If $G = (A, B)$ is a product interval-valued fuzzy sub graph of G^* whose vertex set is V, we assume that $\mu_{A^-}(v) \ne 0, \mu_{A^+}(v) \ne 0$ for all $v \in V$ and μ_{B^-}, μ_{B^+}* are symmetric.

Definition 2.14. *A product interval-valued fuzzy graph $G = (A, B)$ is said to be complete if $\mu_{B^-}(xy) = \mu_{A^-}(x) \times \mu_{A^-}(y)$* and $\mu_{B^+}(xy) = \mu_{A^+}(x) \times \mu_{A^+}(y)$ for all $x, y \in V$.

Definition 2.15. *The complement of a product interval-valued fuzzy graph G is denoted by $\overline{G} = (\overline{A}, \overline{B})$ where $\overline{A} = A = [\mu_{A^-}(x), \mu_{A^+}(x)]$ and $\overline{B} = [\overline{\mu_{B^-}}(y), \overline{\mu_{B^+}}(y)]$ is defined by*

$$\overline{\mu_{B^-}}(xy) = \mu_{A^-}(x) \times \mu_{A^-}(y) - \mu_{B^-}(xy), \overline{\mu_{B^+}}(xy) = \mu_{A^+}(x) \times \mu_{A^+}(y) - \mu_{B^+}(xy)$$

for all $x, y \in V$. It follows that $\overline{G}$ is a product interval-valued fuzzy graph.

Lemma 2.16. *Consider the product interval-valued fuzzy graphs $G_1 = (A_1, B_1)$ and*

$G_2=(A_2,B_2)$, *the isomorphism between two product interval-valued fuzzy graphs* G_1 *and* G_2 *is a bijective mapping* $h:V_1\to V_2$ *such that*

$$\begin{cases}\mu_{A_1^-}(u)=\mu_{A_2^-}(h(u))\\ \mu_{A_1^+}(u)=\mu_{A_2^+}(h(u))\end{cases} \textit{for all } u\in V_1,$$

and

$$\begin{cases}\mu_{B_1^-}(uv)=\mu_{B_2^-}(h(u)h(v))\\ \mu_{B_1^+}(uv)=\mu_{B_2^+}(h(u)h(v))\end{cases} \textit{for all } u,v\in V_1.$$

If G_1 and G_2 are isomorphic, then we write $G_1\cong G_2$.

An automorphism of G is isomorphism of G with itself.

Definition 2.17. *Let* $G_1=(A_1,B_1)$ *and* $G_2=(A_2,B_2)$ *be interval-valued fuzzy graphs with underlying set* V_1 *and* V_2, *respectively. Then we denote the intersection* G_1 *and* G_2 *by* $G_1\cap G_2=(A_1\cap A_2,B_1\cap B_2)$ *and defined as follows:*

$(\mu_{A_1^-}\cap\mu_{A_2^-})(u)=min(\mu_{A_1^-}(u),\mu_{A_2^-}(u))$,

$(\mu_{A_1^+}\cap\mu_{A_2^+})(u)=min(\mu_{A_1^+}(u),\mu_{A_2^+}(u))$, for all $u\in V_1\cap V_2$.

$(\mu_{B_1^-}\cap\mu_{B_2^-})(uv)=min(\mu_{B_1^-}(uv),\mu_{B_2^-}(uv))$,

$(\mu_{B_1^+}\cap\mu_{B_2^+})(uv)=min(\mu_{B_1^+}(uv),\mu_{B_2^+}(uv))$, for all $uv\in E_1\cap E_2$.

The following result can be easily verified.

Proposition 2.18. *Let* $G_1=(A_1,B_1)$ *and* $G_2=(A_2,B_2)$ *be two product interval-valued fuzzy graphs of* G_1 *and* G_2 *respectively. Then* $(A_1\cap A_2,B_1\cap B_2)$ *is a product interval-valued fuzzy graph of* G.

Lemma 2.19. *The union of two product interval-valued fuzzy graphs* $G_1=(A_1,B_1)$ and $G_2=(A_2,B_2)$ is defined as $G=G_1\cup G_2$ whose vertex and edge sets are $(A_1\cup A_2)$ and $(B_1\cup B_2)$. Also,

$$(\mu_{A_1^-}\cup\mu_{A_2^-})(u)=\begin{cases}\mu_{A_1^-}(u) & \textit{if } u\in V_1-V_2,\\ \mu_{A_2^-}(u) & \textit{if } u\in V_2-V_1,\\ \mu_{A_1^-}(u)\cup\mu_{A_2^-}(u) & \textit{if } u\in V_1\cap V_2.\end{cases}$$

$$(\mu_{A_1^+} \cup \mu_{A_2^+})(u) = \begin{cases} \mu_{A_1^+}(u) & if\ u \in V_1 - V_2, \\ \mu_{A_2^+}(u) & if\ u \in V_2 - V_1, \\ \mu_{A_1^+}(u) \cup \mu_{A_2^+}(u) & if\ u \in V_1 \cap V_2. \end{cases}$$

and

$$(\mu_{B_1^-} \cup \mu_{B_2^-})(uv) = \begin{cases} \mu_{B_1^-}(uv) & if\ uv \in E_1 - E_2, \\ \mu_{B_2^-}(uv) & if\ uv \in E_2 - E_1, \\ \mu_{B_1^-}(uv) \cup \mu_{B_2^-}(uv) & if\ uv \in E_1 \cap E_2. \end{cases}$$

$$(\mu_{B_1^+} \cup \mu_{B_2^+})(uv) = \begin{cases} \mu_{B_1^+}(uv) & if\ uv \in E_1 - E_2, \\ \mu_{B_2^+}(uv) & if\ uv \in E_2 - E_1, \\ \mu_{B_1^+}(uv) \cup \mu_{B_2^+}(uv) & if\ uv \in E_1 \cap E_2. \end{cases}$$

Definition 2.20. *The join* $G_1 + G_2 = (A_1 + A_2, B_1 + B_2)$ *of two product interval-valued fuzzy graphs* G_1 *and* G_2 *is defined as follows:* $(\mu_{A_1^-} + \mu_{A_2^-})(u) = (\mu_{A_1^-} \cup \mu_{A_2^-})(u)$,
$(\mu_{A_1^+} + \mu_{A_2^+})(u) = (\mu_{A_1^+} \cup \mu_{A_2^+})(u)$, if $u \in V_1 \cup V_2$.
$(\mu_{B_1^-} + \mu_{B_2^-})(uv) = (\mu_{B_1^-} \cup \mu_{B_2^-})(uv)$, $(\mu_{B_1^+} + \mu_{B_2^+})(uv) = (\mu_{B_1^+} \cup \mu_{B_2^+})(uv)$, if $uv \in E_1 \cap E_2$.
$(\mu_{B_1^-} + \mu_{B_2^-})(uv) = min(\mu_{A_1^-}(u), \mu_{A_2^-}(v))$, $(\mu_{B_1^+} + \mu_{B_2^+})(uv) = min(\mu_{A_1^+}(u), \mu_{A_2^+}(v))$,
if $uv \in E'$, where E' is the set of all edges joining the nodes of V_1 and V_2.

Theorem 2.21. *Let* $G_1 = (A_1, B_1)$ *and* $G_2 = (A_2, B_2)$ *be the product interval-valued fuzzy graphs then,* (i) $(\overline{G_1 \cup G_2}) \cong \overline{G_1} + \overline{G_2}$ (ii) $(\overline{G_1 + G_2}) \cong \overline{G_1} \cup \overline{G_2}$.

3. Product Interval-valued Fuzzy Graphs

In this section, different types of product on interval-valued fuzzy graphs are defined and investigated whether the resultant graphs are interval-valued fuzzy graphs.

Definition 3.1. *The tensor product* $G_1 \otimes G_2$ *of two interval-valued fuzzy graphs* $G_1 = (A_1, B_1)$ *and* $G_2 = (A_2, B_2)$ *of* $G_1^* = (V_1, E_1)$ *and* $G_2^* = (V_2, E_2)$ *respectively is defined as a pair* (A, B)*, where* $A = [\mu_{A^-}, \mu_{A^+}]$ *and* $B = [\mu_{B^-}, \mu_{B^+}]$ *are interval-valued fuzzy sets on* $V = V_1 \times V_2$ *and*

$E = \{((u_1,u_2),(v_1,v_2)) \mid (u_1,v_1) \in E_1, (u_2,v_2) \in E_2\}$, respectively which satisfies the followings

$$(i)\begin{cases}(\mu^-_{A_1} \otimes \mu_{A_2^-})(u_1,u_2) = \min(\mu_{A_1^-}(u_1), \mu_{A_2^-}(u_2)) \\ (\mu_{A_1^+} \otimes \mu_{A_2^+})(u_1,u_2) = \min(\mu_{A_1^+}(u_1), \mu_{A_2^+}(u_2))\end{cases} \quad \textit{for all } (u_1,u_2) \in V_1 \times V_2,$$

$$(ii)\begin{cases}(\mu^-_{B_1} \otimes \mu_{B_2^-})((u_1,u_2)(v_1,v_2)) = \min(\mu_{B_1^-}(u_1v_1), \mu_{B_2^-}(u_2v_2)) \\ (\mu_{B_1^+} \otimes \mu_{B_2^+})((u_1,u_2)(v_1,v_2)) = \min(\mu_{B_1^+}(u_1v_1), \mu_{B_2^+}(u_2v_2))\end{cases}$$

for all $u_1v_1 \in E_1$ *and* $u_2v_2 \in E_2$.

Proposition 3.2. *The tensor product of two product interval-valued fuzzy graphs is a product interval-valued fuzzy graphs.*

Proof. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be product interval-valued fuzzy graphs. We prove that $G_1 \otimes G_2$ is a product interval-valued fuzzy graph.

Let $u_1v_1 \in E_1$ and $u_2v_2 \in E_2$. Then

$$\begin{aligned}(\mu_{B_1^-} \otimes \mu_{B_2^-})((u_1,u_2)(v_1,v_2)) &= min(\mu_{B_1^-}(u_1v_1), \mu_{B_2^-}(u_2v_2)) \\ &\le min(\mu_{A_1^-}(u_1) \times \mu_{A_1^-}(v_1), \mu_{A_2^-}(u_2) \times \mu_{A_2^-}(v_2)) \\ &\le min(\mu_{A_1^-}(u_1), \mu_{A_2^-}(u_2)) \times min(\mu_{A_1^-}(v_1), \mu_{A_2^-}(v_2)) \\ &= (\mu_{A_1^-} \otimes \mu_{A_2^-})(u_1,u_2) \times (\mu_{A_1^-} \otimes \mu_{A_2^-})(v_1,v_2),\end{aligned}$$

Similarly, $(\mu_{B_1^+} \otimes \mu_{B_2^+})((u_1,u_2)(v_1,v_2)) \le (\mu_{A_1^+} \otimes \mu_{A_2^+})(u_1,u_2) \times (\mu_{A_1^+} \otimes \mu_{A_2^+})(v_1,v_2)$.

Hence, $G_1 \otimes G_2$ is a product interval-valued fuzzy graphs.

Proposition 3.3. *Let* $G_1 = (A_1, B_1)$ *and* $G_2 = (A_2, B_2)$ *be product interval-valued fuzzy graphs of* G_1^* *and* G_2^*, *respectively. If* G_1 *and* G_2 *be complete, then* $G_1 \otimes G_2$ *is not necessarily complete.*

Definition 3.4. *Let* $G_1 = (A_1, B_1)$ *and* $G_2 = (A_2, B_2)$ *be the product interval-valued fuzzy graphs with* $G_1^* = (V_1, E_1)$ *and* $G_2^* = (V_2, E_2)$ *respectively. Then the ring sum of two product interval-valued fuzzy graphs* G_1 *and* G_2 is denoted by $G = G_1 \oplus G_2 = (A_1 \oplus A_2, B_1 \oplus B_2)$

$$\begin{cases}(\mu_{A_1^-} \oplus \mu_{A_2^-})(u) = (\mu_{A_1^-} \cup \mu_{A_2^-})(u) \\ (\mu_{A_1^+} \oplus \mu_{A_2^+})(u) = (\mu_{A_1^+} \cup \mu_{A_2^+})(u)\end{cases} \textit{for all } u \in V_1 \cup V_2,$$

and

$$(\mu_{B_1^-} \oplus \mu_{B_2^-})(uv) = \begin{cases} \mu_{B_1^-}(uv) & if \ \ uv \in E_1 - E_2, \\ \mu_{B_2^-}(uv) & if \ \ uv \in E_2 - E_1, \\ 0 & otherwise \end{cases}$$

$$(\mu_{B_1^+} \oplus \mu_{B_2^+})(uv) = \begin{cases} \mu_{B_1^+}(uv) & if \ \ uv \in E_1 - E_2, \\ \mu_{B_2^+}(uv) & if \ \ uv \in E_2 - E_1, \\ 0 & otherwise. \end{cases}$$

Proposition 3.5. *Let* $G_1 = (A_1, B_1)$ *and* $G_2 = (A_2, B_2)$ *be the product interval-valued fuzzy graphs whose underlying graphs are* $G_1 = (V_1, E_1)$ *and* $G_2 = (V_2, E_2)$ *respectively. Then the ring sum of* G_1 *and* G_2 *is denoted by* $G = G_1 \oplus G_2 = (A_1 \oplus A_2, B_1 \oplus B_2)$ *which is a product interval-valued fuzzy graph.*

Theorem 3.6. *Let* $G_1 = (A_1, B_1)$ *and* $G_2 = (A_2, B_2)$ *be two product interval-valued fuzzy graphs with* $E_1 \cap E_2 = \varnothing$ *then,* $\overline{G_1 \oplus G_2} \cong \overline{G_1} + \overline{G_2}$.

Theorem 3.7. *Let* $G_1 = (A_1, B_1)$ *and* $G_2 = (A_2, B_2)$ *be two product interval-valued fuzzy graphs with* $E_1 \cap E_2 = \varnothing$ *, then* $\overline{G_1 + G_2} \cong \overline{G_1} \oplus \overline{G_2}$.

Theorem 3.8. *If* G *is the ring sum of two subgraphs* G_1 *and* G_2 *with* $E_1 \cap E_2 = \varnothing$ *, then every complete product interval-valued fuzzy subgraph* (A, B) *of* G *is a ring sum of complete product interval-valued fuzzy subgraph of* G_1 *and complete product interval-valued fuzzy subgraph of* G_2 *.*

Proof. We define the interval-valued fuzzy subsets A_1, A_2, B_1 and B_2 of V_1, V_2, E_1 and E_2 as follows

$$\begin{cases} \mu_{A_1^-}(u) = \mu_{A^-}(u) & if \ \ u \in V_1 - V_2, \\ \mu_{A_1^+}(u) = \mu_{A^+}(u) & if \ \ u \in V_1 - V_2, \\ \mu_{A_2^-}(u) = \mu_{A^-}(u) & if \ \ u \in V_2 - V_1, \\ \mu_{A_2^+}(u) = \mu_{A^+}(u) & if \ \ u \in V_2 - V_1. \end{cases} \qquad \begin{cases} \mu_{B_1^-}(uv) = \mu_{B^-}(uv) & if \ \ uv \in E_1 - E_2, \\ \mu_{B_1^+}(uv) = \mu_{B^+}(uv) & if \ \ uv \in E_1 - E_2, \\ \mu_{B_2^-}(uv) = \mu_{B^-}(uv) & if \ \ uv \in E_2 - E_1, \\ \mu_{B_2^+}(uv) = \mu_{B^+}(uv) & if \ \ uv \in E_2 - E_1. \end{cases}$$

So (A_1, B_1) is product interval-valued fuzzy graph of G_1 and (A_2, B_2) is product interval-valued fuzzy graph of G_2 and $A = (A_1 \oplus A_2)$ by definition of ring sum of product interval-valued fuzzy graphs G_1 and G_2 . If $uv \in E_1 \cup E_2$ then

$\mu_{B^-}(uv) = (\mu_{B_1^-} \oplus \mu_{B_2^-})(uv), \mu_{B^+}(uv) = (\mu_{B_1^+} \oplus \mu_{B_2^+})(uv).$

If $uv \in E_1 - E_2$ then $\mu_{B^-}(uv) = (\mu_{B_1^-} \oplus \mu_{B_2^-})(uv), \mu_{B^+}(uv) = (\mu_{B_1^+} \oplus \mu_{B_2^+})(uv)$

(by definition of ring sum of product interval-valued fuzzy graph).

If $uv \in E_2 - E_1$ then $\mu_{B^-}(uv) = (\mu_{B_1^-} \oplus \mu_{B_2^-})(uv), \mu_{B^+}(uv) = (\mu_{B_1^+} \oplus \mu_{B_2^+})(uv).$

If $uv \in E'$ i.e. $u \in V_1$ and $v \in V_2$ then

$$(\mu_{B_1^-} \oplus \mu_{B_2^-})(uv) = 0 = \mu_{B^-}(uv), (\mu_{B_1^+} \oplus \mu_{B_2^+})(uv) = 0 = \mu_{B^+}(uv).$$

Other equalities are also holds because (A, B) is a complete product interval-valued fuzzy graph.

4. Balanced Interval-Valued Fuzzy Graphs

The density of a crisp graph $G^* = (V, E)$ is defined by

$$D(G^*) = \frac{2\sum |E|}{|V|(|V|-1)}.$$

This gives the number of edges per unit vertex. $D(G^*)$ is non-negative for any graph G^* and its maximum value is 1, when G^* is complete. Thus, $0 \le D(G^*) \le 1$. Higher value of $D(G^*)$ represent more edges in G^*. If G^* has no edges then $D(G^*)$ is 0.

However, for a fuzzy graph $G = (\sigma, \mu)$ the density is defined as

$$D(G) = \frac{2\sum \mu(uv)}{\sum \sigma(u) \wedge \sigma(v)}.$$

Like crisp graph, the lower bound of $D(G)$ is 0 when $\mu(uv) = 0$ for all edges, but, for the complete fuzzy graph the upper bound is 2. That is, for the fuzzy graph $0 \le D(G) \le 2$.

Motivated from this definition for fuzzy graph, the density of interval-valued fuzzy graph is defined below.

Definition 4.1. *The density of an interval-valued fuzzy graph* $G = (A, B)$ *is* $D(G) = [D^-(G), D^+(G)]$, *where* $D^-(G)$ *is defined by*

$$D^-(G) = \frac{2\sum_{u,v \in V} (\mu_{B^-}(uv))}{\sum_{(u,v) \in E} (\mu_{A^-}(u) \wedge \mu_{A^-}(v))}, \quad \textit{for all } u, v \in V$$

and $D^+(G)$ is defined by

$$D^+(G) = \frac{2\sum_{u,v \in V} (\mu_{B^+}(uv))}{\sum_{(u,v) \in E} (\mu_{A^+}(u) \wedge \mu_{A^+}(v))}, \quad \textit{for all } u, v \in V.$$

Definition 4.2. *An interval-valued fuzzy graph* $G=(A,B)$ *is balanced if* $D(H)\leq D(G)$ *, that is,* $D^{-}(H)\leq D^{-}(G)$, $D^{+}(H)\leq D^{+}(G)$ *for all subgraphs* H *of* G.

Example 4.3. *Consider a graph* $G^{*}=(V,E)$ *such that* $V=\{v_1,v_2,v_3,v_4\}$, $E=\{v_1v_2,v_2v_3,v_3v_4,v_4v_1,v_2v_4\}$. Let A be an interval-valued fuzzy set of V and B be an interval-valued fuzzy set of $E\subseteq V\times V$ defined by

$$A=\langle(\frac{v_1}{0.4},\frac{v_2}{0.3},\frac{v_3}{0.3},\frac{v_4}{0.2}),(\frac{v_1}{0.6},\frac{v_2}{0.5},\frac{v_3}{0.7},\frac{v_4}{0.7})\rangle,$$

$$B=\langle(\frac{v_1v_2}{0.24},\frac{v_2v_3}{0.24},\frac{v_3v_4}{0.16},\frac{v_1v_4}{0.16},\frac{v_2v_4}{0.16}),(\frac{v_1v_2}{0.425},\frac{v_2v_3}{0.425},\frac{v_3v_4}{0.595},\frac{v_1v_4}{0.51},\frac{v_2v_4}{0.425})\rangle.$$

For this graph

$$D^{-}(G)=2(\frac{0.24+0.24+0.16+0.16+0.16}{0.3+0.3+0.2+0.2+0.2})=1.6$$

$$D^{+}(G)=2(\frac{0.425+0.425+0.595+0.51+0.425}{0.5+0.5+0.7+0.6+0.5})=1.7$$

$D(G)=[D^{-}(G),D^{+}(G)]=[1.6,1.7].$

Let $H_1=\{v_1,v_2\}$, $H_2=\{v_1,v_3\}$, $H_3=\{v_1,v_4\}$, $H_4=\{v_2,v_3\}$, $H_5=\{v_2,v_4\}$, $H_6=\{v_3,v_4\}$, $H_7=\{v_1,v_2,v_3\}$, $H_8=\{v_1,v_3,v_4\}$, $H_9=\{v_1,v_2,v_4\}$, $H_{10}=\{v_2,v_3,v_4\}$, $H_{11}=\{v_1,v_2,v_3,v_4\}$ be the non-empty subgraphs of G. Densities of these subgraphs are

$D[H_1]=[1.6,1.7]$, $D[H_2]=[0,0]$, $D[H_3]=[1.6,1.7]$,
$D[H_4]=[1.6,1.7]$, $D[H_5]=[1.6,1.7]$, $D[H_6]=[1.6,1.7]$, $D[H_7]=[1.6,1.7]$, $D[H_8]=[1.6,1.7]$, $D[H_9]=[1.6,1.7]$, $D[H_{10}]=[1.6,1.7]$, $D[H_{11}]=[1.6,1.7]$.

Thus, it is verified that $D(H)\leq D(G)$ for all subgraphs H of G.
Hence, G is a balanced interval-valued fuzzy graph.

Definition 4.4. *An interval-valued fuzzy graph* $G=(A,B)$ *is strictly balanced if for every* $u,v\in V$, $D(H)=D(G)$ *for all non-empty subgraphs* H *of* G.

Theorem 4.5. *Every complete interval-valued fuzzy graph is balanced.*
Proof. Let $G=(A,B)$ be a complete interval-valued fuzzy graph, then by the definition of complete interval-valued fuzzy graph G, we have
$\mu_{B^-}(uv)=\mu_{A^-}(u)\wedge\mu_{A^-}(v)$ and $\mu_{B^+}(uv)=\mu_{A^-}(u)\wedge\mu_{A^-}(v)$ for every $u,v\in V$

$$\sum_{u,v\in V}\mu_{B^-}(uv)=\sum_{(u,v)\in E}\mu_{A^-}(u)\wedge\mu_{A^-}(v) \text{ and}$$

$$\sum_{u,v\in V}\mu_{B^+}(uv)=\sum_{(u,v)\in E}\mu_{A^+}(u)\wedge\mu_{A^+}(v) \text{ for every } u,v\in V.$$

Now, $D(G)=\left[\frac{2\sum_{u,v\in V}\mu_{B^-}(uv)}{\sum_{(u,v)\in E}(\mu_{A^-}(u)\wedge\mu_{A^-}(v))},\frac{2\sum_{u,v\in V}\mu_{B^+}(uv)}{\sum_{(u,v)\in E}(\mu_{A^+}(u)\wedge\mu_{A^+}(v))}\right]$

$$=\left[\frac{2\sum_{(u,v)\in E}(\mu_{A^-}(u)\wedge\mu_{A^-}(v))}{\sum_{(u,v)\in E}(\mu_{A^-}(u))\wedge\mu_{A^-}(v)},\frac{2\sum_{(u,v)\in E}(\mu_{A^+}(u)\wedge\mu_{A^+}(v))}{\sum_{(u,v)\in E}(\mu_{A^+}(u))\wedge\mu_{A^+}(v)}\right]=[2,2].$$

Also, every subgraph of a complete interval-valued fuzzy graph is complete. therefore, it is easy to verify that $D(H)=[2,2]$ for every $H\subseteq G$. Thus, G is balanced.

Note 4.6. *The converse of Theorem 4.6 is need not be true, that is every balanced interval-valued fuzzy graph is not necessarily complete.*

Definition 4.7. *An interval-valued fuzzy graph* $G=(A,B)$ *of a given graph* $G^*=(V,E)$ *is called strong interval-valued fuzzy graph if*
$\mu_{B^-}(xy)=\min(\mu_{A^-}(x),\mu_{A^-}(y))$ and $\mu_{B^+}(xy)=\min(\mu_{A^+}(x),\mu_{A^+}(y))$ for all $xy\in E$.

Corollary 4.8. *Every strong interval-valued fuzzy graph is balanced.*

Theorem 4.9. *Let* $G=(A,B)$ *be a strictly balanced interval-valued fuzzy graph and* $\overline{G}=(\overline{A},\overline{B})$ *be its complement, then* $D(G)+D(\overline{G})=[2,2]$.

Proof. Let $G=(A,B)$ be a strictly balanced interval-valued fuzzy graph and $\overline{G}=(\overline{A},\overline{B})$ be its complement. Let H be a non-empty subgraph of G. Since G is strictly balanced $D(G)=D(H)$ for every $H\subseteq G$ and $u,v\in V$.

In $\overline{G}$,

$$\overline{\mu_{B^-}(uv)}=\mu_{A^-}(u)\wedge\mu_{A^-}(v)-\mu_{B^-}(uv) \qquad (1)$$

and $$\overline{\mu_{B^+}(uv)}=\mu_{A^+}(u)\wedge\mu_{A^+}(v)-\mu_{B^+}(uv) \qquad (2)$$

for every $u,v\in V$. Dividing (1) by $\mu_{A^-}(u)\wedge\mu_{A^-}(v)$

$$\frac{\overline{\mu_{B^-}(uv)}}{\mu_{A^-}(u)\wedge\mu_{A^-}(v)}=1-\frac{\mu_{B^-}(uv)}{\mu_{A^-}(u)\wedge\mu_{A^-}(v)},$$

for every $u,v\in V$ and dividing (2) by $\mu_{A^+}(u)\wedge\mu_{A^+}(v)$,

$\dfrac{\overline{\mu_{B^+}(uv)}}{\mu_{A^+}(u)\wedge\mu_{A^+}(v)}=1-\dfrac{\mu_{B^+}(uv)}{\mu_{A^+}(u)\wedge\mu_{A^+}(v)}$, for every $u,v\in V$.

Therefore, $\sum_{u,v\in V}\dfrac{\overline{\mu_{B^-}(uv)}}{\mu_{A^+}(u)\wedge\mu_{A^+}(v)}=1-\sum_{u,v\in V}\dfrac{\mu_{B^-}(uv)}{\mu_{A^-}(u)\wedge\mu_{A^-}(v)}$,

where $u,v\in V$ and

$$\sum_{u,v\in V}\frac{\overline{\mu_{B^+}(uv)}}{\mu_{A^+}(u)\wedge\mu_{A^+}(v)}=1-\sum_{u,v\in V}\frac{\mu_{B^+}(uv)}{\mu_{A^+}(u)\wedge\mu_{A^+}(v)},\ \text{where } u,v\in V.$$

$$2\sum_{u,v\in V}\frac{\overline{\mu_{B^-}(uv)}}{\overline{\mu_{A^-}(u)}\wedge\overline{\mu_{A^-}(v)}}=2-2\sum_{u,v\in V}\frac{\mu_{B^-}(uv)}{\mu_{A^-}(u)\wedge\mu_{A^-}(v)},\ \text{where } u,v\in V \text{ and}$$

$$2\sum_{u,v\in V}\frac{\overline{\mu_{B^+}(uv)}}{\overline{\mu_{A^+}(u)}\wedge\overline{\mu_{A^+}(v)}}=2-2\sum_{u,v\in V}\frac{\mu_{B^+}(uv)}{\mu_{A^+}(u)\wedge\mu_{A^+}(v)},$$

where $u,v\in V$

$D^-(\overline{G})=2-D^-(G)$ and $D^+(\overline{G})=2-D^+(G)$.

Now,

$D(G)+D(\overline{G})=[D^-(G),D^+(G)]+[D^-(\overline{G}),D^+(\overline{G})]$

$=[D^-(G)+D^-\overline{G},D^+(G)+D^+(\overline{G})]$

Hence, $D(G)+D(\overline{G})=[2,2]$.

Theorem 4.10. *The complement of strictly balanced interval-valued fuzzy graph is strictly balanced.*

Definition 4.11. *Let $G_1=(A_1,B_1)$ and $G_2=(A_2,B_2)$ be two interval-valued fuzzy graphs whose underlying crisp graphs are $G_1^*=(V_1,E_1)$ and $G_2^*=(V_2,E_2)$. Assume that $V_1\cap V_2=\varnothing$. The direct product of G_1 and G_2 is defined as $G_1\sqcap G_2=(\sigma_1\sqcap\sigma_2,\mu_1\sqcap\mu_2)$ with the underlying crisp graph $G^*=(V_1\times V_2,E)$ where $E=\{(u_1,v_1),(u_2,v_2):(u_1,u_2)\in E_1,(v_1,v_2)\in E_2\}$,*

- $(\mu_{A_1^-}\sqcap\mu_{A_2^-})(u_1,v_1)=\mu_{A_1^-}(u_1)\wedge\mu_{A_2^-}(v_1)$ for all $(u_1,v_1)\in V_1\times V_2$ and $(\mu_{A_1^+}\sqcap\mu_{A_2^+})(u_1,v_1)=\mu_{A_1^+}(u_1)\wedge\mu_{A_2^+}(v_1)$ for all $(u_1,v_1)\in V_1\times V_2$.
- $(\mu_{B_1^-}\sqcap\mu_{B_2^-})(u_1v_1,u_2v_2)=\mu_{B_1^-}(u_1u_2)\wedge\mu_{B_2^-}(v_1v_2)$ for all $u_1u_2\in E_1,v_1v_2\in E_2$ and $(\mu_{B_1^+}\sqcap\mu_{B_2^+})(u_1v_1,u_2v_2)=\mu_{B_1^+}(u_1u_2)\wedge\mu_{B_2^+}(v_1v_2)$ for all $u_1u_2\in E_1,v_1v_2\in E_2$.

Theorem 4.12. *The direct product of two interval-valued fuzzy graphs is also an interval-valued fuzzy graph.*

Proof. Let $u_1v_1 \in E_1$ and $u_2v_2 \in E_2$ so we have

$$(\mu_{B_1^-} \sqcap \mu_{B_2^-})(u_1v_1, u_2v_2) = min(\mu_{B_1^-}(u_1u_2), \mu_{B_2^-}(v_1v_2))$$
$$\leq min(min(\mu_{A_1^-}(u_1), \mu_{A_1^-}(u_2)), min(\mu_{A_2^-}(v_1), \mu_{A_2^-}(v_2)))$$
$$= min(min(\mu_{A_1^-}(u_1), \mu_{A_2^-}(v_1)), min(\mu_{A_1^-}(u_2), \mu_{A_2^-}(v_2)))$$
$$= min((\mu_{A_1^-} \sqcap \mu_{A_2^-})(u_1, v_1), (\mu_{A_1^-} \sqcap \mu_{A_2^-})(u_2, v_2)).$$

Also

$$(\mu_{B_1^+} \sqcap \mu_{B_2^+})(u_1v_1, u_2v_2) = min(\mu_{B_1^+}(u_1u_2), \mu_{B_2^+}(v_1v_2))$$
$$\leq min(min(\mu_{A_1^+}(u_1), \mu_{A_1^+}(u_2)), min(\mu_{A_2^+}(v_1), \mu_{A_2^+}(v_2)))$$
$$= min(min(\mu_{A_1^+}(u_1), \mu_{A_2^+}(v_1)), min(\mu_{A_1^+}(u_2), \mu_{A_2^+}(v_2)))$$
$$= min((\mu_{A_1^+} \sqcap \mu_{A_2^+})(u_1, v_1), (\mu_{A_1^+} \sqcap \mu_{A_2^+})(u_2, v_2)).$$

Theorem 4.13. *Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two interval-valued fuzzy graphs such that $G_1 \sqcap G_2$ is complete, then either G_1 or G_2 must be complete.*

Theorem 4.14. *Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two interval-valued fuzzy graphs. Then $D(G_1) = D(G_2) = D(G_1 \sqcap G_2)$ if and only if $D(G_i) \leq D(G_1 \sqcap G_2)$ for i=1,2.*

Theorem 4.15. *Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two balanced interval-valued fuzzy graphs. Then $G_1 \sqcap G_2$ is balanced if and only if $D(G_1) = D(G_2) = D(G_1 \sqcap G_2)$.*

Proof. Let $G_1 \sqcap G_2$ be balanced interval-valued fuzzy graphs. Then by definition, $D(G_i) \leq D(G_1 \sqcap G_2)$ for $i = 1,2$. So by Theorem 4.18, $D(G_1) = D(G_2) = D(G_1 \sqcap G_2)$.

Conversely, suppose that $D(G_1) = D(G_2) = D(G_1 \sqcap G_2)$. We have to prove that $G_1 \sqcap G_2$ is balanced.

Let $[\frac{n_1}{r_1}, \frac{n_2}{r_2}]$ be the density of an interval-valued fuzzy graph G_1. Let $[\frac{a_1}{b_1}, \frac{a_2}{b_2}]$ and $[\frac{a_3}{b_3}, \frac{a_4}{b_4}]$ be the densities of the interval-valued fuzzy subgraphs H_1 and H_2 of G_1 and G_2 respectively. Since G_1 and G_2 are balanced and

$$D(G_1) = D(G_2) = [\frac{n_1}{r_1}, \frac{n_2}{r_2}], \text{ where } 0 \le [\frac{n_1}{r_1}, \frac{n_2}{r_2}] \le [2,2],$$

$$D(H_1) = [\frac{a_1}{b_1}, \frac{a_2}{b_2}] \le [\frac{n_1}{r_1}, \frac{n_2}{r_2}],\ D(H_2) = [\frac{a_3}{b_3}, \frac{a_4}{b_4}] \le [\frac{n_1}{r_1}, \frac{n_2}{r_2}].$$

Thus $a_1r_1 + a_3r_1 \le b_1n_1 + b_3n_1$ and $a_2r_2 + a_4r_2 < b_2n_2 + b_4n_2$. Hence,

$$D(H_1) + D(H_2) \le [\frac{a_1 + a_3}{b_1 + b_3}, \frac{a_2 + a_4}{b_2 + b_4}] \le [\frac{n_1}{r_1}, \frac{n_2}{r_2}] = D(G_1 \sqcap G_2).$$

Thus, $D(H) \le D(G_1 \sqcap G_2)$ for any subgraph H of $G_1 \sqcap G_2$. Therefore, $D(G_1 \sqcap G_2)$ is balanced.

Theorem 4.16. *Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be isomorphic interval-valued fuzzy graphs. If G_2 is balanced, then G_1 is balanced.*

5. Conclusions

The interval-valued fuzzy models give more precision, flexibility and compatibility to the system as compared to the classical and fuzzy models. In this paper, we discussed product interval-valued fuzzy graphs and it's complement. The notion of a ring sum and join of product interval-valued fuzzy graphs are discussed.

In the definition of density, the lower and upper limits of the interval are multiplied by 2 and its is proved that the upper bound of the density is [2,2]. As a result the density of any interval-valued fuzzy graph lies between [0,0] and [2,2]. If we omit this 2, the density of any interval-valued fuzzy graph lies between [0,0] and [1,1]. It is better to define the density of a fuzzy graph as $D(G) = \frac{2\sum \mu(uv)}{\sum \sigma(u) \wedge \sigma(v)}$. Finally, we defined balanced and strictly balanced interval-valued fuzzy graphs. In our future work, we will focus on energy and hyperenergetic of interval valued fuzzy graphs which are very useful in physics and chemistry.

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