Journal of Physical Sciences, Vol. 17, 2013, 1-9
ISSN: 0972-8791, www.vidyasagar.ac.in/journal
Published on 26 December 2013

Strong Common Coupled Fixed Point Result in Fuzzy Metric Spaces

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Received 1 January 2013; accepted 2 June 2013

ABSTRACT

Coupled fixed point problems have attracted large attention in recent times. In this paper we have proved a fuzzy coupled fixed point result. Precisely, we have defined the concept of strong coupled fixed point and have established the existence of a unique common strong coupled fixed point for two mappings on fuzzy metric spaces which satisfy certain contractive inequality. The main theorem has a corollary and is supported with an example. The example demonstrates that the theorem properly contains its corollary.

Keywords: fuzzy metric space, completeness, t-norm, strong coupled fixed point, common fixed point.

1. Introduction

The concept of fuzzy sets was introduced by Zadeh [24] in 1965. Afterwards, fuzzy concepts made headways in almost all branches of mathematics. In particular, fuzzy metric space was introduced by Kramosil and Michalek [10]. George and Veeramani modified the definition of Kramosil and Michalek in [7]. The topology in such spaces is a Hausdroff topology. There are several fixed point results for mappings defined on fuzzy metric spaces in the sense of George and Veeramani. We have noted some of these works in [2, 5, 13, 14, 16] and [22].

Definition 1.1. [20] A binary operation $*: [\mathbf{0},\mathbf{1}]^{\mathbf{2}} \rightarrow [\mathbf{0},\mathbf{1}]$ is called a *t*-norm if the following properties are satisfied:

(i) $*$ is associative and commutative,
(ii) $a * 1 = 1$ for all $a \in [\mathbf{0},\mathbf{1}]$,
(iii) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d,$ for each $a, b, c, d \in [\mathbf{0},\mathbf{1}]$.

Examples of continuous t--norms are $a *_1 b = \min\{a, b\}$, $a *_2 b = \frac{ab}{Max\{a,b,\lambda\}}$ for $a < \lambda < 1$ and $a *_3 b =$ ab.

Kramosil and Michalek defined fuzzy metric space in the following way.

Definition 1.2. [10] The 3-tuple $(X, M, *)$ is called a fuzzy metric space if $\boldsymbol{X}$ is an arbitrary non-empty set, $*$ is a t-norm and M is a fuzzy set on $X^2 \times [0, \infty)$ satisfying the following conditions:

(i) $M(x, y, 0) = 0$,

(ii) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$,

(iii) $M(x, y, t) = M(y, x, t)$,

(iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ and

(v) $M(x, y, .) : (0, \infty) \longrightarrow [0,1]$ is left-continuous , where $t, s > 0$ and $x, y, z \in X$.

George and Veeramani in their paper [7] introduced a modification of the above definition. The motivation was to make the corresponding induced topology into a Hausdroff topology. Following Mihet [13], we call such spaces GV-fuzzy metric space.

Definition 1.3.[7] The 3-tuple $(X, M, *)$ is called a GV-fuzzy metric space if $\boldsymbol{X}$ is an arbitrary non-empty set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times [0, \infty)$ satisfying the following conditions for each $x, y, z \in X$ and $t, s > 0$:

(i) $M(x, y, t) = 0$,

(ii) $M(x, y, t) = 1$ if and only if $x = y$,

(iii) $M(x, y, t) = M(y, x, t)$,

(iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ and

(v) $M(x, y, .) : (0, \infty) \longrightarrow [0,1]$ is continuous.

Let $(X, M, *)$ be a GV-fuzzy metric space. For $t > 0$, $0 < r < 1$, the open ball $B(x, t, r)$ with center $x \in X$ is defined by

$$B(x, t, r) = \{y \in X : M(x, y, t) > 1 - r\}.$$

A subset $A \subset X$ is called open if for each $x \in A$, there exist $t > 0$ and $0 < r < 1$ such that $B(x, t, r) \subset A$. Let τ denote the family of all open subsets of X. Then τ is called the topology on X induced by the fuzzy metric M. This topology is Hausdorff and first countable [7].

Example 1.4. [7] Let $\boldsymbol{X} = \mathbb{R}$. Let $\boldsymbol{a} * \boldsymbol{b} = \boldsymbol{a} \, . \, \boldsymbol{b}$ for all $\boldsymbol{a}, \boldsymbol{b} \in [\mathbf{0}, \mathbf{1}]$. For each $\boldsymbol{t} \in (\mathbf{0}, \infty)$, let

$$M(x, y, t) = \frac{t}{t + |x - y|}$$

for all $x, y \in X$. Then $(\mathbb{R}, M, *)$ is a GV-fuzzy metric space.

Example 1.5. Let (X, d) be a metric space and $\boldsymbol{\psi}$ be an increasing and a continuous function of $\mathbb{R}_+$ into $(0,1)$ such that $\lim_{t \to \infty} \psi(t) = 1$. Three generic examples of these functions are $(t) = \frac{t}{t+1}$, $\psi(t) = \sin\left(\frac{\pi t}{2t+1}\right)$ and $\psi(t) = 1 - e^{-t}$. Let $*$ be any continuous t-norm. For each $t \in (0, \infty)$, let $M(x, y, t) = \psi(t)^{d(x,y)}$ for all $x, y \in X$.Then $(X, M, *)$ is a fuzzy metric space.

Definition 1.6. [10] Let $(X, M, *)$ be a GV-fuzzy metric space.

(i) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if $\lim_{n \to \infty} M(x_n, x, t) = 1$ for all $t > 0$.

(ii) A sequence $\{x_n\}$ in X is called a Cauchy sequence if for each $0 < \varepsilon < 1$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for each $n, m \geq n_0$.

(iii) A GV-fuzzy metric space in which every Cauchy sequence is convergent, is said to be complete.

Lemma 1.7. [17] $\boldsymbol{M}$ is a continuous function on $\boldsymbol{X}^{\mathbf{2}} \times (\mathbf{0}, \infty)$.

Our purpose in this paper is to prove a strong coupled common fixed point theorem in complete GV-fuzzy metric spaces. Coupled fixed point problems belong to a category of problems in fixed point theory in which much interest has been generated recently after the publication of a coupled contraction mapping theorem by Bhaskar and Lakshmikantham [1]. The result proved by Bhaskar and Lakshmikantham in [1] was generalised to coupled coincidence point results in [3] and [11] under two separate sets of conditions. Some other works in this line of research are noted in [6, 12, 15, 19]. Coupled fixed point problems have also been studied in probabilistic metric spaces [23], in cone metric spaces [9, 18] and in *G*- metric spaces [4]. One of the reasons of this widespread interest in coupled fixed point problems is their potential applicability [1].

Definition 1.8. [1] An element $x, y \in X \times X$ is called a coupled fixed point of the mapping $F: X \times X \to X$ if $F(x, y) = x, F(y, x) = y$.

Further Lakshmikantham and Ciric have introduced the concept of coupled coincidence point.

Definition 1.9. [1] An element $x, y \in X \times X$ is called a coupled coincidence point of a mapping $F: X \times X \to X$ and $g : X \to X$ if $F(x, y) = g(x),\ F(y, x) = g(y)$.
If, in particular, $x = g(x) = F(x, y)$ and $y = g(y) = F(y, x)$, then (x, y) is a coupled common fixed point of g and F .

Here we define strong coupled fixed point in the following.

Definition 1.10. In particular, if $\boldsymbol{x} = \boldsymbol{y}$ in the coupled common fixed point of $\boldsymbol{g}$ and we call (x, x) or simply x a strong coupled common fixed point of g and F. If further $g = I$, the identity map, then we call (x, x) or simply x a strong coupled fixed point of F.

Coupled fixed point results were established in fuzzy metric spaces by Sedghi et al [21] in which they had established a fuzzy version of the result of Bhaskar et al [1]. After that, common coupled fixed point results in fuzzy metric spaces were established by Hu [8].

Definition 1.11. [3, 11] Let $\boldsymbol{X}$ be a non-empty set and $F: X \times X \to X$ and $g : X \to X$. We say that F and g commute if $g\big(F(x, y)\big) = F(g(x), g(y))$ for all $x, y \in X$.

In this paper we have established that for two mappings $F: X \times X \to X$ and $g : X \to X$, where $(X, M, *)$ is a GV-fuzzy metric space, under certain conditions, there exists a unique strongly coupled common fixed point for g and F. The theorem has a corollary and is supported with an example.

2. Main Results

Theorem 2.1. Let $(X, M, *)$ be a complete GV-fuzzy metric space where $*$ is any continuous t-norm satisfying $a * b \geq a.b$ for all $a, b \in [0,1]$. Let there be functions $F: X \times X \to X$ and $g : X \to X$ such that

$$M(F(x, y), F(u, v), t) \geq \gamma(M(g(x), g(u), t) * M(g(y), g(v), t)) \tag{2.1}$$

for all $x, y, u, v \in X,\ t > 0$ where $\gamma : [0,1] \to [0,1]$ is a monotone increasing continuous function such that $\gamma(a) > \sqrt{a}$ for each $a \in (0,1)$. Let g be continuous, commute with F and is such that $F(X \times X) \subseteq g(X)$. Then g and F have a unique strongly coupled common fixed point.

Proof. Let $\boldsymbol{x_0}, \boldsymbol{y_0} \in \boldsymbol{X}$. Since $F(X \times X) \subseteq g(X)$, we can choose $\boldsymbol{x_1}, \boldsymbol{y_1} \in \boldsymbol{X}$ such that $\boldsymbol{g}(\boldsymbol{x_1}) = \boldsymbol{F}(\boldsymbol{x_0}, \boldsymbol{y_0})$ and that $\boldsymbol{g}(\boldsymbol{y_1}) = \boldsymbol{F}(\boldsymbol{y_0}, \boldsymbol{x_0})$ for some $x_0, y_0 \in X$. Again we can choose $x_2, y_2 \in X$ such that $\boldsymbol{g}(\boldsymbol{x_2}) = \boldsymbol{F}(\boldsymbol{x_1}, \boldsymbol{y_1})$ and $\boldsymbol{g}(\boldsymbol{y_2}) = \boldsymbol{F}(\boldsymbol{y_1}, \boldsymbol{x_1})$ Continuing this process, we construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$g(x_{n+1}) = F(x_n, y_n)$ and $g(y_{n+1}) = F(y_n, y_n)$ for all $n \geq 0$. (2.2)

Let, for all $t > 0$, $n \geq 0$,

$$\delta_n(t) = M(g(x_n), (g(x_{n+1}), t) * M(g(y_n), (g(y_{n+1}), t)\ . \qquad (2.3)$$

From (2.1) and (2.2), for all $t > 0$, $n \geq 0$, we have

$$\begin{aligned} M(g(x_n),(g(x_{n+1}),t) &= M(F(x_{n-1},y_{n-1}), F(x_n,y_n),t) \\ &\geq \gamma(M(g(x_{n-1}),(g(x_n),t) * M(g(y_{n-1}),(g(y_n),t)) \\ &= \gamma(\delta_{n-1}(t)). \qquad (2.4) \end{aligned}$$

Similarly, from (2.1) and (2.2) for all $t > 0$, $n \geq 0$, we have

$$\begin{aligned} M(g(y_n),(g(y_{n+1}),t) &= M(F(y_{n-1},x_{n-1}), F(y_n,x_n),t) \\ &\geq \gamma(M(g(y_{n-1}),(g(y_n),t) * M(g(x_{n-1}),(g(x_n),t)) \\ &= \gamma(\delta_{n-1}(t)). \qquad (2.5) \end{aligned}$$

From (2.4) and (2.5), for all $t > 0$, $n \geq 0$, we have

$$\delta_n(t) \geq \gamma(\delta_{n-1}(t)) * \gamma(\delta_{n-1}(t)) \geq (\gamma(\delta_{n-1}(t)))^2 \qquad (\text{Since } a * b \geq ab). \qquad (2.6)$$

From (2.6), by a property of γ, we have $\delta_n(t) \geq \gamma(\delta_{n-1}(t))$ for all $t > 0$.

Thus for each $t > 0, \{\delta_n(t); n \geq 0\}$ is an increasing sequence in [0,1] and hence tends to a limit $a(t) \leq 1$. We claim that $a(t) = 1$ for all $t > 0$. If there exists $t_0 > 0$ such that $a(t_0) < 1$, then taking limit as $n \to \infty$ for $t = t_0$ in (2.6), and using the properties of γ, we get $a(t_0) \geq (\gamma(a(t_0)))^2 > a(t_0)$, which is a contradiction. Hence $a(t) = 1$ for every $t > 0$, that is, for all $t > 0$,

$$\lim_{n\to\infty} \delta_n(t) = \lim_{n\to\infty} M(g(x_n), (g(x_{n+1}), t) * M(g(y_n), (g(y_{n+1}), t) \qquad (2.7)$$

Now we prove that $\{g(x_n)\}$ and $\{g(y_n)\}$ are Cauchy sequences. Let, to the contrary, at least one of $\{g(x_n)\}$ and $\{g(y_n)\}$ be not a Cauchy sequence. Then there exist $\epsilon, \lambda \in (0,1)$ such that for each integer k, there are two integers $l(k)$ and $m(k)$ such that $m(k) > l(k) \geq k$ and

either $M(g(x_{l(k)}), g(x_{m(k)}), \epsilon) \leq 1 - \lambda$ for all $k \geq 1$

or $M(g(y_{l(k)}), g(y_{m(k)}), \epsilon) \leq 1 - \lambda$ for all $k \geq 1$

Let us write, for all $t > 0$, $k \geq 0$,

$$r_k(t) = M\big(g(x_{l(k)}), g(x_{m(k)}), t\big) * M\left(g\left(y_{l(k)}\right), g\left(y_{m(k)}\right), t\right).$$

Then, from the above, for all $k \geq 0$,

$$r_k(\epsilon) = M\big(g(x_{l(k)}), g(x_{m(k)}), \epsilon\big) * M\left(g\left(y_{l(k)}\right), g\left(y_{m(k)}\right), \epsilon\right) \leq 1 - \lambda \tag{2.8}$$

By choosing $m(k)$ to be the smallest integer exceeding *l(k)* for which (2.8) holds, for all $k > 0$, we have

$$M\big(g(x_{l(k)}), g(x_{m(k)-1}), \epsilon\big) * M\big(g(y_{l(k)}), g(y_{m(k)-1}), \epsilon\big) > 1 - \lambda.$$

From the above inequality, by the continuity of M, we can have some α with $0 < 2\alpha < \epsilon$ such that, for all $k \geq 1$,

$$M\big(g(x_{l(k)}), g(x_{m(k)-1}), \epsilon - 2\alpha\big) * M\big(g(y_{l(k)}), g(y_{m(k)-1}), \epsilon - 2\alpha\big) > 1 - \lambda \tag{2.9}$$

From (2.8) and (2.9), for all $k > 0$, we have

$$\begin{aligned} 1 - \lambda \geq r_k(\epsilon) &\geq M\big(g(x_{l(k)}), g(x_{m(k)-1}), \epsilon - \alpha\big) * M\big(g(x_{m(k)-1}), g(x_{m(k)}), \alpha\big) \\ &\quad * M\big(g(y_{l(k)}), g(y_{m(k)-1}), \epsilon - \alpha\big) * M\big(g(y_{m(k)-1}), g(y_{m(k)}), \alpha\big) \\ &= M\big(g(x_{l(k)}), g(x_{m(k)-1}), \epsilon - \alpha\big) * M\left(g\left(y_{l(k)}\right), g\left(y_{m(k)-1}\right), \epsilon - \alpha\right) * \delta_{m(k)-1}(\alpha) \\ &> (1 - \lambda) * \delta_{m(k)-1}(\alpha). \end{aligned}$$

Taking the limit as $k \to \infty$, and using (2.7), we get

$\lim_{k\to\infty} r_k(\epsilon) = 1 - \lambda.$

Since $M(x, y, t_1) \geq M(x, y, t_2)$ whenever $t_1 \geq t_2$, it follows that $M(x, y, \epsilon) \leq 1 - \lambda$ implies

$M(x, y, \epsilon_1) \leq 1 - \lambda$, for all $x, y \in X$ whenever $\epsilon_1 \leq \epsilon$.

Hence the above derivation is valid if ϵ is replaced by any smaller value. Thus we conclude that

$$\lim_{k\to\infty} r_k(\epsilon) = 1 - \lambda \quad \text{for all } 0 < \epsilon_1 \leq \epsilon. \tag{2.10}$$

Again, for all $k > 0$,

$$\begin{aligned} r_k(\epsilon) &= M\big(g(x_{l(k)}), g(x_{m(k)}), \epsilon\big) * M\big(g(y_{l(k)}), g(y_{m(k)}), \epsilon\big) \\ &\geq M\big(g(x_{l(k)}), g(x_{l(k)+1}), \alpha\big) * M\big(g(x_{l(k)+1}), g(x_{m(k)+1}), \epsilon - 2\alpha\big) \\ &\quad * M\big(g(x_{m(k)+1}), g(x_{m(k)}), \alpha\big) * M\left(g\left(y_{l(k)}\right), g\left(y_{l(k)+1}\right), \alpha\right) \\ &\quad * M\left(g\left(y_{l(k)+1}\right), g\left(y_{m(k)+1}\right), \epsilon - 2\alpha\right) * M\left(g\left(y_{m(k)+1}\right), g\left(y_{m(k)}\right), \alpha\right) \end{aligned}$$

Using the notation of (2.3), for all $k > 0$, we have

$$\begin{aligned} r_k(\epsilon) \geq \delta_{l(k)}(\alpha) * \delta_{m(k)}(\alpha) * M\big(g(x_{l(k)+1}), g(x_{m(k)+1}), \epsilon - 2\alpha\big) \\ * M\big(g(y_{l(k)+1}), g(y_{m(k)+1}), \epsilon - 2\alpha\big) \end{aligned} \tag{2.11}$$

From (2.1) and (2.2, for all $k > 0$, it follows that

$$\begin{aligned} M\big(g(x_{l(k)+1}), g(x_{m(k)+1}), \epsilon - 2\alpha\big) &= M\big(F(x_{l(k)}, y_{l(k)}), F(x_{m(k)}, y_{m(k)}), \epsilon - 2\alpha\big) \\ &\geq \gamma\big(M\big(g(x_{l(k)}), g(x_{m(k)}), \epsilon - 2\alpha\big) * M\big(g(y_{l(k)}), g(y_{m(k)}), \epsilon - 2\alpha\big)\big) \\ &= \gamma(r_k(\epsilon - 2\alpha) \end{aligned} \tag{2.12}$$

Also from (2.1) and (2.2) we have, for all $k > 0$,

$$M\big(g(y_{m(k)+1}), g(y_{m(k)+1}), \epsilon - 2\alpha\big) = M\big(F(y_{m(k)}, x_{m(k)}), F(y_{l(k)}, x_{l(k)}), \epsilon - 2\alpha\big)$$

$$M(g(y_{m(k)+1}), g(y_{m(k)+1}), \epsilon - 2\alpha) = M(F(y_{m(k)}, x_{m(k)}), F(y_{l(k)}, x_{l(k)}), \epsilon - 2\alpha)$$
$$\geq \gamma(M(g(x_{l(k)}), g(x_{m(k)}), \epsilon - 2\alpha) * M(g(y_{l(k)}), g(y_{m(k)}), \epsilon - 2\alpha))$$
$$= \gamma(r_k(\epsilon - 2\alpha) \qquad (2.13)$$

Inserting (2.12) and (2.13) in (2.11) we obtain, for all $k > 0$,

$$r_k(\epsilon) \geq \delta_{l(k)}(\alpha) * \delta_{m(k)}(\alpha) * \gamma(r_k(\epsilon - 2\alpha)) * \gamma(r_k(\epsilon - 2\alpha)$$
$$\geq \delta_{l(k)}(\alpha) * \delta_{m(k)}(\alpha) * (\gamma(r_k(\epsilon - 2\alpha)))^2$$

(Since $a * b \geq a \,.\, b$).

$$1 - \lambda \geq (\gamma(1 - \lambda))^2 > (1 - \lambda), \qquad (2.14)$$

which is a contradiction. Therefore, $\{g(x_n)\}$ and $\{g(y_n)\}$ are Cauchy sequences.

Since X complete, there exist $x, y \in X$ such that

$$\lim_{n\to\infty} g(x_n) = x \text{ and } \lim_{n\to\infty} g(y_n) = y\,. \qquad (2.15)$$

From (2.15), and the continuity of g, we obtain

$$\lim_{n\to\infty} g(g(x_n)) = g(x) \quad \text{and} \quad \lim_{n\to\infty} g(g(y_n)) = g(y) \qquad (2.16)$$

From (2.2), and the commutativity of g and F , for all $n \geq 0$, we have

$$g(g(x_{n+1})) = g(F(x_n, y_n)) = F(g(x_n), g(y_n)), \qquad (2.17)$$

and

$$g(g(y_{n+1})) = g(F(y_n, x_n)) = F(g(y_n), g(x_n)), \qquad (2.18)$$

We now show that $g(x) = F(x, y)$ and $g(y) = F(y, x)$

From (2.1) and (2.17), for all $t > 0$, $n \geq 0$, we have

$$M(g(g(x_{n+1})), F(x, y), t) = M(g(F(x_n, y_n)), F(x, y), t)$$
$$= M(F(g(x_n), g(y_n)), F(x, y), t)$$
$$\geq \gamma(M(g(g(x_n)), g(x), t) * (M(g(g(y_n)), g(y), t).$$

Letting $n \to \infty$ in the above inequality, by (2.16), the continuity of γ and lemma 1.7, for all $t > 0$, we have

$$M(g(x), F(x, y), t) \geq \gamma(1) = 1,$$

that is, $g(x) = F(x, y)$ (2.19)

Again from (2.1) and (2.18) we get, for all $t > 0$, $n \geq 0$,

$$M(g(g(y_{n+1})), F(y, x), t) = M(g(F(y_n, x_n)), F(y, x), t)$$
$$\geq \gamma(M(g(g(y_n)), g(y), t) * (M(g(g(x_n)), g(x), t)$$

Letting $n \to \infty$ in the above inequality, by (2:16), the continuity of γ and lemma 1.7, for all $t > 0$, we have $M(g(y), F(y, x), t) \geq \gamma(1) = 1$ for all *t>0,* that is

$G(y) = F(y,x).$ (2.20)

Thus (x, y) is a coupled coincidence point of g and F.
Now we prove that $x = y$. Let, to the contrary, $x \neq y$. Then, $M(x, y, s) < 1$ for $s > 0$. Then, from (2.1) and (2.2), we get

$$M(g(x_{n+1}), g(y_{n+1}), s) = M(F(x_n, y_n), F(x_n, y_n), s)$$

$$\geq \gamma(M\big(g(x_n)\big), g(y_n), s) * M\big(g(y_n)\big), g(x_n), s)) .$$

Letting $n \to \infty$ in the above inequality, by (2:15), lemma 1.7, the continuity of * and the properties of γ, we have

$$M(x, y, s) \geq \gamma(M(x, y, s) * M(y, x, s) \geq \lambda(M(x, y, s)^2) > M(x, y, s),$$

which is a contraction.

Thus we have proved that $x = y$. From (2.19) and (2.20) we have that

$$g(x) = F(x, y). \tag{2.21}$$

Next we prove that $g(x) = x$. If otherwise, then $0 < M(x, y, s) < 1$ for $s > 0$.

From (2.1), (2.2) and (2.21), we have that

$$M\big(g(y_n)\big), g(x), s) = M(F(y_n, x_n), F(x, x), s)$$
$$\geq \gamma(M\big(g(x_n)\big), g(x), s) * M\big(g(x_n)\big), g(x), s)).$$

Letting $n \to \infty$ in the above inequality, by (2.15), lemma 1.7, the properties γ, and the fact that $x = y$, we have

$$\begin{aligned} M(x, g(x), s) &\geq \gamma(M(x, g(x), s) * M(y, g(y), s)) \\ &= \gamma(M(x, g(x), s) * M(x, g(x), s)) \\ &= \gamma(M(x, g(x), s)^2) \\ &> M(x, g(x), s, \end{aligned}$$

which is contraction. Hence we have established that $x = g(x) = F(x, x)$, that is, g and F have a strongly coupled common fixed point. Next we prove the uniqueness of the fixed point.

If x and u are such that, $x = g(x) = F(x, x)$ and $u = g(u) = F(u, u)$,
then, by (2.1) and a property of γ, for all $t > 0$, we have

$$\begin{aligned} M(g(x), g(u), t) &= \gamma(M(F(x, x), F(u, u), t) \\ &\geq \gamma(M(g(x), g(u), t) * M(g(x), g(u), t)) \quad \text{(by (2.1))} \\ &> M(g(x), g(u), t), \end{aligned}$$

which is a contradiction.
Hence the strongly coupled x common fixed point is unique.
This completes the proof of the theorem.

Corollary 2.2. Let (*X, M,* $*$) be a complete GV-fuzzy metric space such that $a * b \geq \boldsymbol{a.b}$ for all *a, b* ∈ [0,1]. Let there be a function $F: X \times X \to X$ such that

$$M(F(x, y), F(u, v), t) \geq \gamma(M(x,u,t) * M(y,v,t)) \quad (2.22)$$

for all *x, y, u, v* ∈ *X*, where $\gamma: [0,1] \to [0,1]$ is a continuous function such that $\gamma(a) > \sqrt{a}$ for each $a \in (0,1)$. Then there exists a unique strongly coupled fixed point of F, that is, a unique $x \in X$ such that $x = F(x, x)$.

Proof. If we take *g* = *I*, the identity map, in theorem 2.1, then there exists a unique $\boldsymbol{x} \in \boldsymbol{X}$ such that $x = F(x, x)$.

Example 2.3. Let (*X, M,* *) be a fuzzy metric space, where *X* = [-2,2], $a*b = a.b$ for all $a, b \in [\mathbf{0}, \mathbf{1}]$ and, for *x, y*∈ *X, t* > 0,

$$M(x, y, t) = \varphi(t)^{|x-y|}$$

where φ is an increasing and continuous function of R_+ into (0, 1) given by $\varphi(t) = \frac{t}{t+1}$ for each $t \in (0, \infty)$. We define the map $F: X \times X \to X$ as:

$$F(x, y) = \frac{x^2}{24} + \frac{y^2}{24} - 2, \text{ for all } x, y \in X.$$

Let $g: X \to X$ be given by $g(x) = -|x|$ for all $x \in X$.

Then

$$F(X \times X) = [-2, \frac{-5}{3}] \subset g(X) = [-2,0].$$

Let $\gamma: [0,1] \to [0,1]$ be defined as $\gamma(a) = a^{\frac{1}{6}}$ for each $a \in (0,1), \gamma(0) = 0$ *and* $\gamma(1) = 1$. Then, for all x, y, u, v ∈ *X* , t > 0, we have

$$M(F(x,y),F(u,v),t) = \left(\frac{t}{t+1}\right)^{\frac{|x^2-u^2+y^2-v^2|}{24}} \geq \left(\frac{t}{t+1}\right)^{\frac{|x-u|+|y-v|}{6}}$$

$$= \left(\frac{t}{t+1}\right)^{\frac{|x-u|}{6}} \left(\frac{t}{t+1}\right)^{\frac{|y-v|}{6}} = M(g(x),g(y),t)^{\frac{1}{6}}. M(g(y),g(v),t)^{\frac{1}{6}}$$

$$= \gamma(M(g(x), g(u), t) * M(g(y), g(v), t)).$$

Thus all the conditions of theorem 2.1 hold. Here (6-2$\sqrt{15}$, 6-2$\sqrt{15}$) is the unique strong common coupled fixed point of F.

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