Journal of Physical Sciences, Vol. 16, 2012, 125-132
ISSN: 0972-8791, www.vidyasagar.ac.in/journal
Published on 31 December 2012

Reliability Estimations of Components in Series System Using Type-II Censored and Masked Life Data

Xiao-lin Shi

School of Electronic Engineering
Xi'an University of Posts and Telecommunications
Xi'an –710121, Xi'an, China. E-mail: linda20016@20016.com

Received September 30, 2012; accepted December 18, 2012

ABSTRACT

This paper introduces estimations of reliabilities index for Rayleigh components in a series system using Type-II censored and masked system life data. We derive the Maximum likelihood estimation (MLE) of the unknown parameters for Rayleigh components and reliabilities index. In view of the limitation of MLE under complete masked case, we employ the Bayes approach. Bayes estimations of the Rayleigh parameters and reliabilities index are obtained under the Square Error loss function. Also, a numerical example is given by means of the Monte-Carlo simulation for comparing the effects of the maximum likelihood estimation and the Bayes estimation, which shows that the Bayes estimation is better than MLE.

Keywords: Type II censored; the masked data; series system; the Rayleigh component; reliability estimations

AMS Mathematics Subject Classification (2010):

1. Introduction

In reliability analysis, the reliability estimations of components in a system are often obtained using system life test data. These estimations are very useful since they can reflect the actual operational capacity of individual components in system environment. In addition, we can predict the further reliabilities of components in some new systems, see Usher and Hodgson (1988). System life test data generally include system failure time and information on the exact component causing the system failure. In practice, however, the true component responsible for the failure of system is sometimes unknown due to various reasons such as the constraints of cost and time. Therefore, the cause of system failure is masked.

Recently, estimating component reliability using the masked data has been considered by several authors. Usher and Hodgson (1988) firstly proposed the concept of masked data and obtained likelihood function of the life test sample. Literature[2-4] derived the maximum likelihood and Bayes estimates of components' parameters and reliabilities in the series system when the lifetime of components obey Weibull, Pareto and Geometric distribution, respectively. Sarhan (2003) extended the reliability analysis of masked data in the parallel case, while the reliability estimating of verse Weibull component was discussed in a parallel system by Zhang (2009). Lin & Guess (1994) presented a proportional probability model for a dependent masked probability in the series system. Based on this model, reliability estimations of Geometric and Pareto components from series system are studied by Sarhan (2007) and Xu (2009), respectively. Fan and Wang (2011) studied Statistical inference in accelerated life tests for Weibull series systems with masked data. Literature (2012) considers the parametric estimation in a wide class of parametric distribution families based on right-censored competing risks data and with masked failure cause.

Rayleigh distribution has wide applications in reliability analysis. In this paper, we mainly consider the estimations of parameters and reliabilities index for Rayleigh components in series system. Both MLE and Bayes approaches are utilized, and their results are compared in the simulation.

2. Maximum Likelihood Estimation

Suppose that N identical systems are put on the life test, the test is stopped when the n-system failure (n is a pre-given constant $n < N$). Each system consists of k independent but non-identical components connected in series, and each component life distribution independent of each other. The random variables $T_{ij}(i=1,2,\ldots,N,\ j=1,2,\ldots,k)$ denote the lifetime of the j -th component in the i -th system, so the lifetime of the i -th system is $T_i = \min(T_{i1}, T_{i2}, \cdots,\ T_{ik}),\ (i=1,2,\cdots,N)$. T_{ij} obey Rayleigh distribution, and its probability density as following

$$f_j(t) = 2t/\lambda_j \exp(-t^2/\lambda_j), \quad t > 0\,. \tag{1}$$

Reliability function and failure rate function of the j -th component in the i -th system are $R_j(t) = \exp(-t^2/\lambda_j),\ \ H_j(t) = 2t/\lambda_j,\ t > 0,\ j = 1,2,\ldots,k$. After the life test is stopped, we can obtain the observed data $(t_1, S_1),\ \ (t_2, S_2), \cdots, (t_n, S_n)$, where $S_i(i=1,\cdots,n)$ express the set of possible failure cause in i -th system. For non-failure system, we only observed censored lifetime $t_i = t_n$, $i = n+1, n+2, \cdots, N$ .When masking is independent of failure cause, similar to the literature[7], we can derive the likelihood function

$$L(\lambda_1, \lambda_2) = \prod_{i=1}^{n} (\sum_{j \in S_i} f_j(t_i) \prod_{l \in M_j} R_l(t_i))(\prod_{k=n+1}^{N} \prod_{l=1}^{k} R_l(t_k) \qquad (M_j = 1,2,\cdots, j-1, j+1, \cdots, K) \quad . \tag{2}$$

In this paper, we consider the simple case of $k=2$. Let n_1 and n_2 be the numbers of system failures for which the component 1 and 2 cause the failure, respectively. While n_{12} denotes the number of failed systems that the cause is not directly observed. That is, $n_j\,(j=1,2)$ is the number of observations when $S_i=\{j\}$ and n_{12} is the number of observations when $S_i=\{1,2\}$. Note that $n=n_1+n_2+n_{12}$, the likelihood function becomes

$$L(\lambda_1,\lambda_2)=2^n(\prod_{i=1}^n t_i)(1/\lambda_1)^{n_1}(1/\lambda_2)^{n_2}(1/\lambda_1+1/\lambda_2)^{n_{12}}\exp\{-(1/\lambda_1+1/\lambda_2)\sum_{i=1}^N t_i^2\}. \qquad (3)$$

The maximum likelihood estimation of λ_1,λ_2 can be obtained by maximizing the likelihood function given by formula (3). One can derive the log-likelihood function as following

$$\ln L(\lambda_1,\lambda_2))=n\ln 2+\sum_{i=1}^n \ln t_i-(n_{12}+n_1)\ln\lambda_1-(n_{12}+n_2)\ln\lambda_2-(1/\lambda_1+1/\lambda_2)\sum_{i=1}^N t_i^2+n_{12}\ln(\lambda_1+\lambda_2).$$

Thus, by setting the derivative zero, the likelihood equations can be obtained as

$$\partial\ln L(\lambda)/\partial\lambda_1=-(n_{12}+n_1)/\lambda_1+\sum_{i=1}^N t_i^2/\lambda_1^2+n_{12}/(\lambda_1+\lambda_2)=0$$
$$\partial\ln L(\lambda)/\partial\lambda_2=-(n_{12}+n_2)/\lambda_2+\sum_{i=1}^N t_i^2/\lambda_2^2+n_{12}/(\lambda_1+\lambda_2)=0$$

Solving the equations above with respect to λ_1,λ_2, we can directly get

$$\hat\lambda_j=\sum_{i=1}^N t_i^2\Big/[n_j+n_{12}\cdot n_j/(n_1+n_2)],\quad (j=1,2). \qquad (4)$$

According to the invariance property of maximum likelihood estimation, by replacing the parameters λ_1,λ_2 with their MLE $\hat\lambda_1,\hat\lambda_2$ in $R_j(t)$ and $H_j(t)$, we can get the MLE of the reliability function and failure rate function as following

$$\hat R_j(t)=\exp\{-t^2/\hat\lambda_j\}\ ,\ H_j(t)=2t/\lambda_j,\quad j=1,2.$$

3. Bayes Estimation

From the above discussion we can conclude that when $n_1=n_2=0$, we can not get the maximum likelihood estimation of parameters from (4). Thus Bayes approach is considered in this section. In order to derive the Bayesian estimation, the prior distributions of the parameters λ_1,λ_2 should be considered initially. In this paper, the prior probability density function (pdf) is taken as following form

$$\pi_j(\lambda_j)=\frac{\beta_j^{\alpha_j}}{\Gamma(\alpha_j)}(\frac{1}{\lambda_j})^{\alpha_j+1}\exp(-\beta_j/\lambda_j),\ (\lambda_j>0,\alpha_j>0,\beta_j>0,\ j=1,2). \qquad (5)$$

where $\alpha_j, \beta_j (j=1,2)$ called hyper-parameters are given by historical data or experience of experts.

Assume that the parameters λ_1 and λ_2 is independent of each other. Then the joint prior pdf of λ_1, λ_2 are $\pi(\lambda_1,\lambda_2)=\pi_1(\lambda_1)\pi_2(\lambda_2)$. By using binomial expansion, we have

$$(1/\lambda_1+1/\lambda_2)^{n_{12}}=\sum_{k=0}^{n_{12}}\binom{n_{12}}{k}(\frac{1}{\lambda_1})^k(\frac{1}{\lambda_2})^{n_{12}-k}.$$

Hence the likelihood function becomes

$$L(data\,|\,\lambda_1,\lambda_2)=2^n\prod_{i=1}^n t_i\exp\{-(1/\lambda_1+1/\lambda_2)\sum_{i=1}^N t_i^{\,2}\}\sum_{k=0}^{n_{12}}\binom{n_{12}}{k}(\frac{1}{\lambda_1})^{n_1+k}(\frac{1}{\lambda_2})^{n_2+n_{12}-k}. \qquad (6)$$

Once the likelihood function, see (6), and the joint prior pdf are constructed, we can derive the joint posterior pdf of λ_1, λ_2 as following

$$\pi(\lambda_1,\lambda_2|data)=\frac{\pi(\lambda_1,\lambda_2)L(data\,|\,\lambda_1,\lambda_2)}{\int_0^{+\infty}\int_0^{+\infty}\pi(\lambda_1,\lambda_2)L(data\,|\,\lambda_1,\lambda_2)d\lambda_1 d\lambda_2}$$

$$=\frac{e^{-(\beta_1+T)/\lambda_1}e^{-(\beta_2+T)/\lambda_2}\sum_{k=0}^{n_{12}}\binom{n_{12}}{k}(\frac{1}{\lambda_1})^{A+1}(\frac{1}{\lambda_2})^{B+1}}{\sum_{k=0}^{n_{12}}\binom{n_{12}}{k}\int_0^{+\infty}\int_0^{+\infty}(\frac{1}{\lambda_1})^{A+1}(\frac{1}{\lambda_2})^{B+1}e^{-(\beta_1+T)/\lambda_1}e^{-(\beta_2+T)/\lambda_2}d\lambda_1 d\lambda_2}$$

$$\underline{\underline{\mu_j=1/\lambda_j}}\ \frac{e^{-(\beta_1+T)/\lambda_1}e^{-(\beta_2+T)/\lambda_2}\sum_{k=0}^{n_{12}}\binom{n_{12}}{k}(\frac{1}{\lambda_1})^{A+1}(\frac{1}{\lambda_2})^{B+1}}{\sum_{k=0}^{n_{12}}\binom{n_{12}}{k}\int_0^{+\infty}\mu_1^{A-1}e^{-(\beta_1+T)\mu_1}d\mu_1\int_0^{+\infty}\mu_2^{B-1}e^{-(\beta_2+T)\mu_2}d\mu_2}$$

$$=\frac{e^{-(\beta_1+T)/\lambda_1}e^{-(\beta_2+T)/\lambda_2}\sum_{k=0}^{n_{12}}\binom{n_{12}}{k}(\frac{1}{\lambda_1})^{A+1}(\frac{1}{\lambda_2})^{B+1}}{\sum_{k=0}^{n_{12}}\binom{n_{12}}{k}(\beta_1+T)^{-A}\Gamma(A)(\beta_2+T)^{-B}\Gamma(B)}$$

$$=\frac{1}{I_0}e^{-(\beta_1+T)/\lambda_1}e^{-(\beta_2+T)/\lambda_2}\sum_{k=0}^{n_{12}}\binom{n_{12}}{k}(\frac{1}{\lambda_1})^{A+1}(\frac{1}{\lambda_2})^{B+1}.$$

where $I_0=\sum_{k=0}^{n_{12}}\binom{n_{12}}{k}(T+\beta_1)^{-A}\Gamma(A)(T+\beta_2)^{-B}\Gamma(B)$ and

$T=\sum_{i=1}^N t_i^{\,2}, A=n_1+k+\alpha_1, B=n_2+n_{12}-k+\alpha_2$.

Thus, the marginal posterior pdf of λ_1, λ_2 can be formulated as in the following respective relations

$$\pi_1(\lambda_1 \mid data) = \int_0^{+\infty} \pi(\lambda_1, \lambda_2 \mid data) d\lambda_2$$
$$= \frac{1}{I_0} e^{-(\beta_1+T)/\lambda_1} \sum_{k=0}^{n_{12}} \binom{n_{12}}{k} (\frac{1}{\lambda_1})^{A+1} (T+\beta_2)^{-B} \Gamma(B),$$
$$\pi_2(\lambda_2 \mid data) = \int_0^{+\infty} \pi(\lambda_1, \lambda_2 \mid data) d\lambda_1$$
$$= \frac{1}{I_0} e^{-(\beta_2+T)/\lambda_2} \sum_{k=0}^{n_{12}} \binom{n_{12}}{k} (\frac{1}{\lambda_2})^{B+1} (T+\beta_1)^{-A} \Gamma(A).$$

Under the square error loss function $L(\theta,d) = (d-\theta)^2$, the Bayes estimators of λ_1 and λ_2 are their posterior mean, that is $\hat{\lambda}_j = E(\lambda_j \mid data)$ $j = 1,2$. Hence the Bayes estimators of λ_1 and λ_2 are respectively

$$\hat{\lambda}_1 = E(\lambda_1 \mid data) = \int_0^{+\infty} \lambda_1 \pi_1(\lambda_1 \mid data) d\lambda_1$$
$$= J_1^{(1)} / \sum_{k=0}^{n_{12}} \binom{n_{12}}{k} (T+\beta_1)^{-A} \Gamma(A) (T+\beta_2)^{-B} \Gamma(B) = J_1^{(1)} / I_0 .$$
$$\hat{\lambda}_2 = E(\lambda_2 \mid data) = \int_0^{+\infty} \lambda_2 \pi_2(\lambda_2 \mid data) d\lambda_2$$
$$= J_2^{(1)} / \sum_{k=0}^{n_{12}} \binom{n_{12}}{k} (T+\beta_1)^{-A} \Gamma(A) (T+\beta_2)^{-B} \Gamma(B) = J_2^{(1)} / I_0 .$$

where

$$J_1^{(1)} = \sum_{k=0}^{n_{12}} \binom{n_{12}}{k} (T+\beta_2)^{-B} \Gamma(B) (T+\beta_1)^{-(A-1)} \Gamma(A-1), J_2^{(1)} = \sum_{k=0}^{n_{12}} \binom{n_{12}}{k} (T+\beta_1)^{-A} \Gamma(A) (T+\beta_2)^{-(B-1)} \Gamma(B-1) .$$

Similarly, we can obtain the Bayes estimators for the reliability function R_1, R_2 below

$$\hat{R}_1(t) = \int_0^{+\infty} e^{-t^2/\lambda_1} \pi_1(\lambda_1 \mid data) d\lambda_1 = \frac{1}{I_0} \sum_{k=0}^{n_{12}} \binom{n_{12}}{k} (T+\beta_2)^{-B} \Gamma(B) \int_0^{+\infty} (\frac{1}{\lambda_1})^{A+1} e^{-(T+\beta_1+t^2)/\lambda_1} d\lambda_1$$
$$= \frac{1}{I_0} \sum_{k=0}^{n_{12}} \binom{n_{12}}{k} (T+\beta_2)^{-B} \Gamma(B) (T+\beta_1+t^2)^{-A} \Gamma(A) = \sum_{k=0}^{n_{12}} \binom{n_{12}}{k} \ I_1 / I_0.$$
$$\hat{R}_2 = \int_0^{+\infty} e^{-t^2/\lambda_2} \pi_2(\lambda_2 \mid data) d\lambda_2$$
$$= \frac{1}{I_0} \sum_{k=0}^{n_{12}} \binom{n_{12}}{k} (T+\beta_1)^{-A} \Gamma(A) \int_0^{+\infty} (\frac{1}{\lambda_2})^{B+1} e^{-(T+\beta_2+t^2)/\lambda_2} d\lambda_2$$
$$= \frac{1}{I_0} \sum_{k=0}^{n_{12}} \binom{n_{12}}{k} (T+\beta_1)^{-A} \Gamma(A) (T+\beta_2+t^2)^{-B} \Gamma(B) = \sum_{k=0}^{n_{12}} \binom{n_{12}}{k} \ I_2 / I_0.$$

In the reliability analysis, Component failure rate function is calculated by the formula $H(t) = -R'(t)/R(t)$. Hence the Bayes estimators of Component failure rate function H_1 and H_2 are respectively

$$\hat{H}_1(t)=\int_0^{+\infty}\frac{2t}{\lambda_1}\pi_1(\lambda_1\left|data\right.)d\lambda_1=\frac{2t}{I_0}\sum_{k=0}^{n_{12}}\binom{n_{12}}{k}(T+\beta_2)^{-B}\Gamma(B)(T+\beta_1)^{-(A+1)}\Gamma(A+1)$$

$$=2t\sum_{k=0}^{n_{12}}\binom{n_{12}}{k}\ M_1/I_0,$$

$$\hat{H}_2(t)=\int_0^{+\infty}\frac{2t}{\lambda_2}\pi_2(\lambda_2\left|data\right.)d\lambda_2$$

$$=\frac{2t}{I_0}\sum_{k=0}^{n_{12}}\binom{n_{12}}{k}(T+\beta_1)^{-A}\Gamma(A)(T+\beta_2)^{-(B+1)}\Gamma(B+1)=2t\sum_{k=0}^{n_{12}}\binom{n_{12}}{k}\ M_2/I_0$$

4. Numerical Study

In this section, we assume that there are 50 same systems are put on the life test at the same time. Each system is consisted of two independent components connected in series. The lifetimes of the components obey Rayleigh distribution with parameters λ_1,λ_2 .The test is stopped when the n system fail ($n<N$). The hyper-parameters in prior distribution are taken as $\alpha_1=3,\ \ \alpha_2=4,\beta_1=\beta_2=2$, and we Let $\lambda_1=1,\lambda_2=0.9$, $t_0=1.5$ .According to the given masking level l and censoring number n, we obtain the observed data $(t_1,s_1),\ (t_2,s_2),\cdots(t_r,s_n)\ \ (t_{n+1},*),\cdots,(t_N,*)$ by using Monte-Carlo method. Where $t_{n+1}=t_{n+2}=\cdots=t_N=t_n$. Then we can get the value of n_1,n_2,n_{12} . After taking these datas to the theoretical results in section 2 and section 3, the MLE and Bayes estimations of $\lambda_j,R_j,H_j(j=1,2)$ are calculated. Repeat the above steps above 1000 times, and then the mean squared errors (MSE) of these estimations are computed, and the results are presented in the table 1 and the table 2. (In the table 1, N means for the mean square error can not be given.)

Table 1. The mean squared error of the parameter estimations

n	estimations	l =0	l =20%	l =40%	l =60%	l =100%
		MLE Bayes	MLE Bayes	MLE Bayes	MLE Bayes	MLE Bayes
10	$\hat{\lambda}_1$	0.2589 0.0574	0.3060 0.0585	0.3159 0.0651	0.3832 0.0682	N 0.1702
	$\hat{\lambda}_2$	0.2483 0.0571	0.2583 0.0580	0.2701 0.0610	0.3002 0.0661	N 0.1375
20	$\hat{\lambda}_1$	0.1180 0.0462	0.1219 0.0478	0.1278 0.0552	0.1390 0.0653	N 0.1138
	$\hat{\lambda}_2$	0.1038 0.0391	0.1048 0.0438	0.1145 0.0473	0.1284 0.0522	N 0.0709
30	$\hat{\lambda}_1$	0.0708 0.0382	0.0823 0.0396	0.0850 0.0479	0.0930 0.0597	N 0.0661
	$\hat{\lambda}_2$	0.0676 0.0335	0.0775 0.0346	0.0798 0.0371	0.0864 0.0465	N 0.0629
40	$\hat{\lambda}_1$	0.0506 0.0322	0.0589 0.0342	0.0621 0.0376	0.0743 0.0427	N 0.0525
	$\hat{\lambda}_2$	0.0459 0.0278	0.0523 0.0301	0.0528 0.0311	0.0767 0.0324	N 0.0368

Table 2. The mean squared error of the Bayes estimations for the parameter and reliabilities index

l	n=30, $t_0 = 1.5$					
	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{R}_1$	$\hat{R}_2$	$\hat{H}_1$	$\hat{H}_2$
20%	0.0396	0.0346	0.0306	0.0351	0.0331	0.0389
40%	0.0479	0.0371	0.0332	0.0434	0.0368	0.0425
60%	0.0597	0.0465	0.0457	0.0498	0.0471	0.0512
100%	0.0661	0.0629	0.0662	0.0687	0.0676	0.0681

From the results shown in table 1, one can conclude that:

(1) When the masking level l and censoring number n are given, the mean square error of Bayes estimation for the parameters is always less than the mean square error of the MLE. Therefore, the effect of the Bayes estimation is better than the effect of maximum likelihood estimation.

(2) For a given Censoring number n, the mean square error of the MLE and Bayes estimation for the parameters decrease with increasing masking level l. However, for the cases in which masking level is 100% , the mean square error of the MLE for the parameters can not be obtained. But based on the discussion of Section 3, the mean square error of the Bayes estimation can still be calculated, and the results are good.

The results presented in Table 2 shows, for a given Censoring number n, the mean squared error of the Bayes estimations for the parameter and components reliabilities index decrease with increasing masking level l. However, the mean squared errors of the Bayes estimations are smaller.

5. **Conclusion**

In this paper, we discuss the estimations of Rayleigh components reliabilities index in a series system using Type-II censored and masked system life data. MLE and Bayes approach are exploited for estimating. The Maximum likelihood estimation and Bayes estimations of the Rayleigh parameters and reliabilities index are obtained. Also, a numerical example is given by means of the Monte-Carlo simulation for comparing the effects of the maximum likelihood estimation and the Bayes estimation, which shows that the mean squared errors of the Bayes estimations are smaller than MLE.

Acknowledgements. This work is supported by the National Natural Science Foundation of China (No.71171164) and the Natural Science Foundation of Education Department of Shaanxi Province (No. 2010JK838).

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