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Bipolar Anti Fuzzy HX Group and its Lower Level Sub HX Groups

***R. Muthuraj* [1] *and M.Sridharan* [2]**

[1]Department of Mathematics, H.H.The Rajah's College,Pudukkottai – 622 001, Tamilnadu, India. E-mail : rmr1973@yahoo.com
[2] Department of Mathematics, PSNA College of Engineering and Technology, Dindigul-624 622, Tamilnadu , India.
E-mail : msridharan1972@gmail.com

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ABSTRACT

In this paper, we define a new algebraic structure of a bipolar fuzzy HX group and a bipolar anti fuzzy sub HX group of a HX group and lower level sub HX group of a HX group and some related properties are investigated. The purpose of this study is to implement the fuzzy set theory and group theory in bipolar anti fuzzy sub HX group of a HX group. Characterizations of lower level subsets of a bipolar anti fuzzy sub HX group of a HX group are given. We also discussed the relation between a given a bipolar anti fuzzy sub HX group of a HX group and its lower level sub HX groups and investigate the conditions under which a given HX group has a properly inclusive chain of sub HX groups. In particular, we formulate how to structure a bipolar anti fuzzy sub HX group of a HX group by a given chain of sub HX groups. We also establish the relation between bipolar fuzzy HX group and bipolar anti fuzzy HX group of a HX group.

***Keywords*: ** HX group, fuzzy subgroup, fuzzy HX group, anti fuzzy HX group, bipolar fuzzy HX group, bipolar anti fuzzy HX group

1. Introduction

The concept of fuzzy sets was initiated by Zadeh [13]. Then it has become a vigorous area of research in engineering, medical science, social science, graph theory etc. Rosenfeld [9] gave the idea of fuzzy subgroups. Li Hongxing [4] introduce the concept of HX group and the authors Luo Chengzhong , Mi Honghai , Li Hongxing [5] introduce the concept of fuzzy HX group. The author W.R.Zhang[10],[11] commenced the concept of bipolar fuzzy sets as a generalization of fuzzy sets in 1994. In fuzzy sets the membership degree of elements range over the interval [0,1]. The membership degree expresses the degree of belongingness of elements to a fuzzy set. The membership degree 1 indicates that an element completely belongs to its corresponding fuzzy set and membership degree 0 indicates that an element does not belong to fuzzy set. The membership degrees on the interval (0, 1) indicate the partial membership to the fuzzy set. Sometimes, the membership degree means the satisfaction degree of elements to some property or constraint corresponding to a fuzzy set .In case of Bipolar-valued fuzzy sets membership degree range is enlarged from the interval [0, 1] to [-1, 1]. In a bipolar-valued fuzzy set, the membership degree 0 means that the elements are irrelevant to the corresponding property, the membership degree (0,1] indicates that elements somewhat satisfy the property and the membership degree [-1,0) indicates that elements somewhat satisfy the implicit counter-property.

2. Preliminaries

In this section, we site the fundamental definitions that will be used in the sequel. Throughout this paper, G = (G , *) is a group, e is the identity element of G, and xy, we mean x * y.

Definition 2.1. Let G be a finite group. In $2^G - \{\phi\}$, a nonempty set $\vartheta \subset 2^G-\{\phi\}$ is called a HX group on G, if ϑ is a group with respect to the algebraic operation defined by AB = { ab /a $\in$ A and b $\in$ B}, which its unit element is denoted by E.

Definition 2.2. Let G be any non-empty set. A fuzzy subset μ of G is a function $\mu : G \to [0,1]$.

Definition 2.3. Let μ be a fuzzy subset defined on G. Let $\vartheta \subset 2^G-\{\phi\}$ be a HX group on G. A fuzzy set λ_μ defined on ϑ is said to be a fuzzy subgroup induced by μ on ϑ or a fuzzy HX subgroup on ϑ if for any A, B $\in\vartheta$,

i. $\lambda_\mu(AB) \geq \min\{\lambda_\mu(A), \lambda_\mu(B)\}$
ii. $\lambda_\mu(A^{-1}) = \lambda_\mu(A)$

Definition 2.4. Let μ be a fuzzy subset defined on G. Let $\vartheta \subset 2^G-\{\phi\}$ be a HX group on G.A fuzzy set λ_μ defined on ϑ is said to be an anti fuzzy subgroup induced by μ on ϑ or an anti fuzzy HX subgroup on ϑ if for any A,B $\in\vartheta$,

i. $\lambda_\mu(AB) \leq \max\{\lambda_\mu(A), \lambda_\mu(B)\}$
ii. $\lambda_\mu(A^{-1}) = \lambda_\mu(A)$

Definition 2.5. Let G be a non-empty set. A bipolar-valued fuzzy set or bipolar fuzzy set μ in G is an object having the form $\mu=\{\langle x, \mu^+(x), \mu^-(x)\rangle$ / for all $x\in G\}$, where $\mu^+ : G \to [0,1]$ and $\mu^- : G \to [-1,0]$ are mappings. The positive membership degree $\mu^+(x)$ denotes the satisfaction degree of an element x to the property corresponding to a bipolar-valued fuzzy set $\mu = \{\langle x, \mu^+(x), \mu^-(x)\rangle$ / for all $x\in G\}$ and the negative membership degree $\mu^-(x)$ denotes the satisfaction degree of an element x to some implicit counter property corresponding to a bipolar-valued fuzzy set $\mu = \{\langle x, \mu^+(x), \mu^-(x)\rangle$ / for all $x\in G\}$. If $\mu^+(x) \neq 0$ and $\mu^-(x) = 0$, it is the situation that x is regarded as having only positive satisfaction for $\mu = \{\langle x, \mu^+(x), \mu^-(x)\rangle$ / for all

$x \in G\}$. If $\mu^+(x) = 0$ and $\mu^-(x) \neq 0$, it is the situation that x does not satisfy the property of $\mu = \{\langle x, \mu^+(x), \mu^-(x)\rangle$ / for all $x \in G\}$, but somewhat satisfies the counter property of $\mu = \{\langle x, \mu^+(x), \mu^-(x)\rangle$ / for all $x \in G\}$. It is possible for an element x to be such that $\mu^+(x) \neq 0$ and $\mu^-(x) \neq 0$ when the membership function of property overlaps that its counter property over some portion of G. For the sake of simplicity, we shall use the symbol $\mu = (\mu^+, \mu^-)$ for the bipolar-valued fuzzy set $\mu = \{\langle x, \mu^+(x), \mu^-(x)\rangle$ / for all $x \in G\}$.

Definition 2.6. Let G be a group. A bipolar-valued fuzzy set or bipolar fuzzy set μ in G is a bipolar fuzzy subgroup of G if for all x, y $\in$G,

i. $\mu^+(xy) \geq \min \{\mu^+(x), \mu^+(y)\}$,
ii. $\mu^-(xy) \leq \max \{\mu^-(x), \mu^-(y)\}$
iii. $\mu^+(x^{-1}) = \mu^+(x)$, $\mu^-(x^{-1}) = \mu^-(x)$

Example 2. Let G = { 1,-1, i, -i } be a group. Let $\mu = \{ \langle x, \mu^+(x), \mu^-(x) \rangle$ / for all $x \in G \}$ be a bipolar fuzzy subset and $\mu^+ : G \to [0,1]$ and $\mu^- : G \to [-1,0]$ are defined by $\mu^+(x) = 0.8$ for x = 1,$\mu^+(x) = 0.6$ for x = -1, $\mu^+(x) = 0.5$ for x = i,-i. and $\mu^-(x) = -0.6$ for x =1, $\mu^-(x) = -0.4$ for x = -1, $\mu^-(x) = -0.3$ for x = i, -i.

By routine Calculations, Clearly μ is a bipolar fuzzy subgroup of G.

Definition 2.7. Let G be a group. A bipolar-valued fuzzy set or bipolar fuzzy set μ in G is a bipolar anti fuzzy subgroup of G if for all x, y $\in$G,

i. $\mu^+(xy) \geq \min \{\mu^+(x), \mu^+(y)\}$
ii. $\mu^-(xy) \leq \max \{\mu^-(x), \mu^-(y)\}$
iii. $\mu^+(x^{-1}) = \mu^+(x)$, $\mu^-(x^{-1}) = \mu^-(x)$

Example 2.2. Let G = { 1,-1, i, -i } be a group. Let $\mu = \{ \langle x, \mu^+(x), \mu^-(x) \rangle$ / for all $x \in G \}$ be a bipolar fuzzy subset and $\mu^+ : G \to [0,1]$ and $\mu^- : G \to [-1,0]$ are defined by $\mu^+(x) = 0.4$ for x =1, $\mu^+(x) = 0.5$ for x = -1, $\mu^+(x) = 0.7$ for x = i,-i. and $\mu^-(x) = -0.3$ for x =1, $\mu^-(x) = -0.6$ for x = -1,$\mu^-(x) = -0.8$ for x = i, -i
By routine Calculations, Clearly μ is a bipolar anti fuzzy subgroup of G.

Definition 2.8. Let ϑ be a non-empty set. A bipolar-valued fuzzy set or bipolar fuzzy set λ_μ in ϑ is an object having the form $\lambda_\mu = \{\langle A, \lambda_\mu^+(A), \lambda_\mu^-(A)\rangle$ / for all $x \in A \subset G\}$, where $\lambda_\mu^+ : \vartheta \to [0,1]$ and $\lambda_\mu^- : \vartheta \to [-1,0]$ are mappings. The positive membership degree $\lambda_\mu^+(A)$ denotes the satisfaction degree of an element A to the property corresponding to a bipolar-valued fuzzy set $\lambda_\mu = \{\langle A, \lambda_\mu^+(A), \lambda_\mu^-(A)\rangle$ / for all $x \in A \subset G$ } and the negative membership degree $\lambda_\mu^-(A)$ denotes the satisfaction degree of an element A to some implicit counter property corresponding to a bipolar-valued fuzzy set $\lambda_\mu =\{\langle A, \lambda_\mu^+(A), \lambda_\mu^-(A)\rangle$ / for all $x \in A \subset G\}$. If $\lambda_\mu^+(A) \neq 0$ and $\lambda_\mu^-(A) = 0$, it is the situation that A is regarded as having only positive satisfaction for $\lambda_\mu = \{\langle A, \lambda_\mu^+(A), \lambda_\mu^-(A)\rangle$ / for all $x \in A \subset G\}$. If $\lambda_\mu^+(A) = 0$ and $\lambda_\mu^-(A) \neq 0$, it is the situation that A does not satisfy the property of $\lambda_\mu = \{\langle A, \lambda_\mu^+(A),$

$\lambda_\mu^-(A)\rangle$ / for all $x \in A \subset G\}$, but somewhat satisfies the counter property of $\lambda_\mu = \{\langle A, \lambda_\mu^+(A), \lambda_\mu^-(A)\rangle$ / for all $x \in A \subset G\}$. It is possible for an element A to be such that $\lambda_\mu^+(A) \neq 0$ and $\lambda_\mu^-(A) \neq 0$ when the membership function of property overlaps that its counter property over some portion of ϑ. For the sake of simplicity, we shall use the symbol $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$. For the bipolar-valued fuzzy set $\lambda_\mu = \{\langle A, \lambda_\mu^+(A), \lambda_\mu^-(A)\rangle$ / for all $x \in A \subset G\}$.

Definition 2.9. Let μ be a bipolar fuzzy subset defined on G .Let $\vartheta \subset 2^G-\{\phi\}$ be a HX group on G.A bipolar fuzzy set λ_μ defined on ϑ is said to be a fuzzy subgroup induced by μ on ϑ or a fuzzy HX subgroup on ϑ , if for any for $A , B \in \vartheta$,

i. $\lambda_\mu^+(AB) \geq \min\{\lambda_\mu^+(A), \lambda_\mu^+(B)\}$,

ii. $\lambda_\mu^-(AB) \leq \max\{\lambda_\mu^-(A), \lambda_\mu^-(B)\}$

iii. $\lambda_\mu^+(A^{-1}) = \lambda_\mu^+(A)$, $\lambda_\mu^-(A^{-1}) = \lambda_\mu^-(A)$

where $\lambda_\mu^+(A) = \max\{\mu^+(x)$ / for all $x \in A \subset G\}$ and

$\lambda_\mu^-(A) = \min\{\mu^-(x)$ / for all $x \in A \subset G\}$

Example 2.3. Let $G = \{\{1,2,3,4,5,6\}, \bullet_7\}$ be a group. Let $\mu = \{\langle x, \mu^+(x), \mu^-(x)\rangle$ / for all $x\in G\}$ be a bipolar fuzzy subset in G and $\mu^+ : G \to [0,1]$ and $\mu^- : G \to [-1,0]$ are defined by $\mu^+(x) = 0.8$ for $x =1$, $\mu^+(x) = 0.6$ for $x = 2,4$, $\mu^+(x) = 0.5$ for $x = 3,5,6$ and $\mu^-(x) = - 0.7$ for $x =1$, $\mu^-(x) = - 0.6$ for $x = 2,4$, $\mu^-(x) = - 0.4$ for $x = 3,5,6$.
By routine Calculations, Clearly μ is a bipolar fuzzy subgroup of G.
Let $\vartheta = \{ \{1,6\},\{2,5\},\{3,4\}\}$be a HX subgroup of G.
Let $\lambda_\mu = \{\langle A, \lambda_\mu^+(A), \lambda_\mu^-(A)\rangle$ / for all $x \in A \subset G$ } be a bipolar fuzzy set in ϑ and $\lambda_\mu^+ : \vartheta \to [0,1]$ and $\lambda_\mu^- : \vartheta \to [-1,0]$
In ϑ, Let us consider A = {1,6}, B = {2,5},C = {3,4}

$\bullet_7$	A	B	C
A	A	B	C
B	B	C	A
C	C	A	B

where $\lambda_\mu^+(A) = \max\{\mu^+(x)$ / for all $x \in A \subset G\}$ and $\lambda_\mu^-(A) = \min\{\mu^-(x)$ / for all $x \in A \subset G\}$

Now $\lambda_\mu^+(A) = \lambda_\mu^+(\{1,6\}) = \max\{\mu^+(1), \mu^+(6)\} = \max\{0.8,0.5\} = 0.8$
$\lambda_\mu^+(B) = \lambda_\mu^+(\{2,5\}) = \max\{\mu^+(2), \mu^+(5)\} = \max\{0.6,0.5\} = 0.6$
$\lambda_\mu^+(C) = \lambda_\mu^+(\{3,4\}) = \max\{\mu^+(3), \mu^+(4)\} = \max\{0.5,0.6\} = 0.6$
$\lambda_\mu^-(A) = \lambda_\mu^-(\{1,6\}) = \min\{\mu^-(1), \mu^-(6)\} = \min\{-0.7,-0.4\} = - 0.7$
$\lambda_\mu^-(B) = \lambda_\mu^-(\{2,5\}) = \min\{\mu^-(2), \mu^-(5)\} = \min\{-0.6,-0.4\} = - 0.6$
$\lambda_\mu^-(C) = \lambda_\mu^-(\{3,4\}) = \min\{\mu^-(3), \mu^-(4)\} = \min\{-0.4,-0.6\} = - 0.6$

By routine Calculations, Clearly λ_μ is a bipolar fuzzy HX subgroup of ϑ.

Definition 2.10. Let μ be a bipolar fuzzy subset defined on G. Let $\vartheta \subset 2^G-\{\phi\}$be a HX group on G.A bipolar anti fuzzy set λ_μ defined on ϑ is said to be an anti fuzzy subgroup induced by μ on ϑ or an anti fuzzy HX subgroup on ϑ , if for any for A , B $\in\vartheta$,

i. $\lambda_\mu^+(AB) \leq \max \{ \lambda_\mu^+(A), \lambda_\mu^+(B)\}$,
ii. $\lambda_\mu^-(AB) \geq \min \{ \lambda_\mu^-(A), \lambda_\mu^-(B)\}$
iii. $\lambda_\mu^+(A^{-1}) = \lambda_\mu^+(A)$, $\lambda_\mu^-(A^{-1}) = \lambda_\mu^-(A)$

where $\lambda_\mu^+(A) = \min \{\mu^+(x) / \text{ for all } x \in A\subset G\}$ and
$\lambda_\mu^-(A) = \max\{\mu^-(x) / \text{ for all } x \in A\subset G\}$

Example 2.4. Let G = $\{\{1,2,3,4,5,6\}, \bullet_7\}$ be a group. Let $\mu = \{\langle x, \mu^+(x), \mu^-(x)\rangle / \text{ for all } x\in G\}$ be a bipolar fuzzy subset in G and $\mu^+ : G \to [0,1]$ and $\mu^- : G \to [-1,0]$ are defined by $\mu^+(x) = 0.3$ for x = 1, $\mu^+(x) = 0.7$ for x = 2,4 , $\mu^+(x) = 0.8$ for x = 3,5,6 and $\mu^-(x) = -0.2$ for x = 1, $\mu^-(x) = -0.5$ for x = 2,4, $\mu^-(x) = -0.6$, for x = 3,5,6.
By routine Calculations, Clearly μ is a bipolar anti fuzzy subgroup of G.

$\bullet_7$	A	B	C
A	A	B	C
B	B	C	A
C	C	A	B

Let ϑ = { {1,6},{2,5},{3,4}}be a HX subgroup of G.
Let $\lambda_\mu = \{\langle A, \lambda_\mu^+(A), \lambda_\mu^-(A)\rangle / \text{ for all } x \in A \subset G \}$ be a bipolar fuzzy set in ϑ and $\lambda_\mu^+ : \vartheta \to [0,1]$ and $\lambda_\mu^- : \vartheta \to [-1,0]$
In ϑ, Let us consider A = {1,6}, B = {2,5},C = {3,4}
where $\lambda_\mu^+(A) = \min\{\mu^+(x) / \text{ for all } x \in A\subset G\}$ and
$\lambda_\mu^-(A) = \max\{\mu^-(x) / \text{ for all } x \in A\subset G\}$
Now $\lambda_\mu^+(A) = \lambda_\mu^+(\{1,6\}) = \min\{\mu^+(1), \mu^+(6)\} = \min \{0.3,0.8\} = 0.3$
$\lambda_\mu^+(B) = \lambda_\mu^+(\{2,5\}) = \min\{\mu^+(2), \mu^+(5)\} = \min \{0.7,0.8\} = 0.7$
$\lambda_\mu^+(C) = \lambda_\mu^+(\{3,4\}) = \min\{\mu^+(3), \mu^+(4)\} = \min \{0.8,0.7\} = 0.7$
$\lambda_\mu^-(A) = \lambda_\mu^-(\{1,6\}) = \max \{\mu^-(1), \mu^-(6)\} = \max \{-0.2,-0.6\} = -0.2$
$\lambda_\mu^-(B) = \lambda_\mu^-(\{2,5\}) = \max \{\mu^-(2), \mu^-(5)\} = \max \{-0.5,-0.6\} = -0.5$
$\lambda_\mu^-(C) = \lambda_\mu^-(\{3,4\}) = \max \{\mu^-(3), \mu^-(4)\} = \max \{-0.6,-0.5\} = -0.5$
By routine Calculations, Clearly λ_μ is a bipolar anti fuzzy HX subgroup of ϑ.

3. Properties of bipolar anti fuzzy HX subgroup

In this section, we discuss some of the properties of bipolar anti fuzzy HX group.

Theorem 3.1. Let G be a group. If μ is a bipolar anti fuzzy subgroup of G then the bipolar fuzzy set λ_μ is a bipolar anti fuzzy HX subgroup of ϑ.

Proof. Let μ be a bipolar anti fuzzy subgroup on G, and λ_μ be a bipolar fuzzy subset on G for any A, B $\in \vartheta \subset$ G

i. $\max\{ \lambda_\mu^+(A), \lambda_\mu^+(B) \}$

$= \max \{ \min \{\mu^+(x) \,/\, \text{for all } x\in A\subset G\}, \min \{\mu^+(y) \,/\, \text{for all } y\in B\subset G\}\}$

$= \max \{\mu^+(x_0), \mu^+(y_0)\}$, some $x_0\in A$, $y_0\in B$ and $A,B\subset G$

$\geq \mu^+(x_0y_0)$, μ is a bipolar anti fuzzy subgroup on G

$= \min \{\mu^+(xy) \,/\, \text{for all } x\in A , y\in B \text{ and } A,B \subset G\}$

$= \lambda_\mu^+(AB)$

So, $\lambda_\mu^+(AB) \leq \max\{ \lambda_\mu^+(A), \lambda_\mu^+(B) \}$

ii. $\min\{ \lambda_\mu^-(A), \lambda_\mu^-(B)\}$

$= \min\{ \max\{\mu^-(x) \,/\, \text{for all } x \in A\subset G\}, \max\{\mu^-(y) \,/\, \text{for all } y \in B\subset G\}\}$

$= \min\{ \mu^-(x_0), \mu^-(y_0)\}$, for some $x_0 \in A$, $y_0 \in B$ and $A,B \subset G$

$\leq \mu^-(x_0y_0)$, Since μ is a bipolar anti fuzzy subgroup on G

$= \max\{\mu^-(xy) \,/\, \text{for all } x \in A, y \in B \text{ and } A,B \subset G\}$

$= \lambda_\mu^-(AB)$

So, $\lambda_\mu^-(AB) \geq \min\{ \lambda_\mu^-(A), \lambda_\mu^-(B)\}$

iii. $\lambda_\mu^+(A) = \min\{\mu^+(x) \,/\, \text{for all } x\in A \subset G\}$

$= \min\{\mu^+(x^{-1}) \,/\, \text{for all } x^{-1}\in A \subset G\}$

$= \min\{\mu^+(x^{-1}) \,/\, \text{for all } x^{-1}\in A^{-1} \subset G\}$

$= \lambda_\mu^+(A^{-1})$

$\lambda_\mu^-(A) = \max\{\mu^-(x) \,/\, \text{for all } x \in A\subset G\}$

$= \max\{\mu^-(x^{-1}) \,/\, \text{for all } x^{-1} \in A\subset G\}$

$= \max\{\mu^-(x^{-1}) \,/\, \text{for all } x^{-1}\in A^{-1}\subset G\}$

$= \lambda_\mu^-(A^{-1})$

Hence λ_μ is a bipolar anti fuzzy HX subgroup of ϑ. □

Remark 3.2. If μ is a bipolar fuzzy subset of a group G and λ_μ be a bipolar anti fuzzy HX subgroup on ϑ, such that $\lambda_\mu^+(A) = \min\{\mu^+(x) \,/\, \text{for all } x\in A\subset G\}$ and $\lambda_\mu^-(A) = \max\{\mu^-(x) \,/\, \text{for all } x \in A\subset G\}$, then μ need not be a bipolar anti fuzzy subgroup of G, which can be illustrated by the following Example,

Let G = {{1,2,3,4,5,6}, $\bullet_7$} be a group. Let $\mu = \{\langle x, \mu^+(x), \mu^-(x)\rangle \,/\, \text{for all } x\in G\}$ be a bipolar fuzzy subset in G and $\mu^+ : G \to [0,1]$ and $\mu^- : G \to [-1,0]$ are defined by $\mu^+(x) = 0.3$ for x =1, $\mu^+(x) = 0.7$ for x = 2,4, $\mu^+(x) = 0.8$ for x = 3,5 $\mu^+(x) = 0.9$ for x = 6 and $\mu^-(x) = -0.2$ for x = 1 , $\mu^-(x) = -0.5$ for x = 2,4 $\mu^-(x) = -0.6$ for x = 3,5 , $\mu^-(x) = -0.6$ for x = 6.

By routine Calculations, Clearly μ is a bipolar anti fuzzy subgroup of G.

Let ϑ = {{1,6},{2,5},{3,4}}be a HX subgroup of G.

Let $\lambda_\mu = \{\langle A, \lambda_\mu^+(A), \lambda_\mu^-(A) \rangle \,/\, \text{for all } x\in A\subset G \}$ bipolar fuzzy set in ϑ

and $\lambda_\mu^+ : \vartheta \to [0,1]$ and $\lambda_\mu^- : \vartheta \to [-1,0]$

In ϑ, let us consider A = {1,6}, B = {2,5}, C = {3,4} where $\lambda_\mu^+(A) = \min\{\mu^+(x) \,/\, \text{for all } x \in A \subset G\}$ and

$\lambda_\mu^-(A) = \max\{\mu^-(x) \,/\, \text{for all } x \in A \subset G\}$

Now $\lambda_\mu^+(A) = \lambda_\mu^+(\{1,6\}) = \min\{\mu^+(1), \mu^+(6)\} = \min\{0.3,0.9\} = 0.3$

$\lambda_\mu^+(B) = \lambda_\mu^+(\{2,5\}) = \min\{\mu^+(2), \mu^+(5)\} = \min\{0.7,0.8\} = 0.7$

$\lambda_\mu^+(C) = \lambda_\mu^+(\{3,4\}) = \min\{\mu^+(3), \mu^+(4)\} = \min\{0.8,0.7\} = 0.7$

$\lambda_\mu^-(A) = \lambda_\mu^-(\{1,6\}) = \max\{\mu^-(1), \mu^-(6)\} = \max\{-0.2,-0.8\} = -0.2$

$\lambda_\mu^-(B) = \lambda_\mu^-(\{2,5\}) = \max\{\mu^-(2), \mu^-(5)\} = \max\{-0.5,-0.6\} = -0.5$

$\lambda_\mu^-(C) = \lambda_\mu^-(\{3,4\}) = \max\{\mu^-(3), \mu^-(4)\} = \max\{-0.6,-0.5\} = -0.5$

By routine Calculations, Clearly λ_μ is a bipolar anti fuzzy HX subgroup of ϑ.

But by Calculation,

$\mu^+(2 \cdot 3) \le \max\{\mu^+(2), \mu^+(3)\}$ and $\mu^-(2 \cdot 3) \ge \min\{\mu^-(2), \mu^-(3)\}$

$\mu^+(6) \le \max\{0.7, 0.8\}$ $\quad\quad$ $\mu^-(6) \ge \min\{-0.5, -0.6\}$

$0.9 \le 0.8$ is not true . $\quad\quad$ $-0.8 \ge -0.6$ is not true.

So μ is not a bipolar anti fuzzy subgroup of G.

Theorem 3.3. Let λ_μ be a bipolar anti fuzzy HX subgroup of a HX group ϑ then

i. $\lambda_\mu^+(A) \ge \lambda_\mu^+(E)$ and $\lambda_\mu^-(A) \le \lambda_\mu^-(E)$ for all $A \in \vartheta$ and E is the identity element of ϑ

ii. The subset $H = \{A \in \vartheta \,/\, \lambda_\mu^+(A) = \lambda_\mu^+(E) \text{ and } \lambda_\mu^-(A) = \lambda_\mu^-(E)\}$ is a sub HX group of ϑ

Proof. i. Let $A \in \vartheta$,

$$\begin{aligned} \lambda_\mu^+(E) &= \lambda_\mu^+(AA^{-1}) \\ &\le \max\{\lambda_\mu^+(A), \lambda_\mu^+(A^{-1})\} \\ &= \max\{\lambda_\mu^+(A), \lambda_\mu^+(A)\} \\ &= \lambda_\mu^+(A) \end{aligned}$$

Therefore, $\lambda_\mu^+(A) \ge \lambda_\mu^+(E)$, for all $A \in \vartheta$.

Similarly, for all $A \in \vartheta$,

$$\begin{aligned} \lambda_\mu^-(E) &= \lambda_\mu^-(AA^{-1}) \\ &\ge \min\{\lambda_\mu^-(A), \lambda_\mu^-(A^{-1})\} \\ &= \min\{\lambda_\mu^-(A), \lambda_\mu^-(A)\} \\ &= \lambda_\mu^-(A) \end{aligned}$$

Therefore, $\lambda_\mu^-(A) \ge \lambda_\mu^-(E)$, for all $A \in \vartheta$.

$\bullet_7$	A	B	C
A	A	B	C
B	B	C	A
C	C	A	B

ii. Let $H = \{A \in \vartheta \,/\, \lambda_\mu^+(A) = \lambda_\mu^+(E) \text{ and } \lambda_\mu^-(A) = \lambda_\mu^-(E)\}$.

Clearly H is non-empty as $E \in H$. Let $A,B \in H$.

Then, $\lambda_\mu^+(A) = \lambda_\mu^+(B) = \lambda_\mu^+(E)$, $\lambda_\mu^-(A) = \lambda_\mu^-(B) = \lambda_\mu^-(E)$

$$\begin{aligned} \lambda_\mu^+(AB^{-1}) &\le \max\{\lambda_\mu^+(A), \lambda_\mu^+ B^{-1})\} \\ &= \max\{\lambda_\mu^+(A), \lambda_\mu^+(B)\} \end{aligned}$$

$$= \max \{ \lambda_\mu^+(E), \lambda_\mu^+(E)\}$$
$$= \lambda_\mu^+(E).$$

That is, $\lambda_\mu^+(AB^{-1}) \leq \lambda_\mu^+(E)$ and obviously $\lambda_\mu^+(AB^{-1}) \geq \lambda_\mu^+(E)$

Hence, $\lambda_\mu^+(AB^{-1}) = \lambda_\mu^+(E)$.

$$\begin{aligned}\lambda_\mu^-(AB^{-1}) &\geq \min \{ \lambda_\mu^-(A), \lambda_\mu^-(B^{-1})\}\\ &= \min \{ \lambda_\mu^-(A), \lambda_\mu^-(B)\}\\ &= \min \{ \lambda_\mu^-(E), \lambda_\mu^-(E)\}\\ &= \lambda_\mu^-(E).\end{aligned}$$

That is $\lambda_\mu^-(AB^{-1}) \geq \lambda_\mu^-(E)$ and Obviously $\lambda_\mu^-(AB^{-1}) \leq \lambda_\mu^-(E)$.

Hence $\lambda_\mu^-(AB^{-1}) = \lambda_\mu^-(E)$

Since, $\lambda_\mu^+(AB^{-1}) = \lambda_\mu^+(E)$ and $\lambda_\mu^-(AB^{-1}) = \lambda_\mu^-(E)$ and hence, $AB^{-1} \in H$

Then H is a sub HX group of ϑ. □

Theorem 3.4. Let λ_μ be a bipolar fuzzy HX subgroup of a HX group ϑ if and only if λ_μ^c is a bipolar anti fuzzy HX subgroup of HX group ϑ.

Proof. A bipolar fuzzy set λ_μ is called bipolar fuzzy HX subgroup of a HX group ϑ if for any $A , B \in \vartheta$,

i. $\lambda_\mu^+(AB) \geq \min\{ \lambda_\mu^+(A), \lambda_\mu^+(B)\}$,

ii. $\lambda_\mu^-(AB) \leq \max \{ \lambda_\mu^-(A), \lambda_\mu^-(B)\}$.

iii. $\lambda_\mu^+(A^{-1}) = \lambda_\mu^+(A) , \lambda_\mu^-(A^{-1}) = \lambda_\mu^-(A)$

Now

$$\begin{aligned}
&\lambda_\mu^+(AB) \geq \min\{ \lambda_\mu^+(A), \lambda_\mu^+(B)\}\\
\Leftrightarrow\ & 1- \lambda_\mu^{+c}(AB) \geq \min \{ 1- \lambda_\mu^{+c}(A) , 1- \lambda_\mu^{+c}(B)\}\\
\Leftrightarrow\ & \lambda_\mu^{+c}(AB) \leq 1- \min \{ 1- \lambda_\mu^{+c}(A),1- \lambda_\mu^{+c}(B) \}\\
\Leftrightarrow\ & \lambda_\mu^{+c}(AB) \leq \max \{ \lambda_\mu^{+c}(A) , \lambda_\mu^{+c}(B) \}\\
&\lambda_\mu^-(AB) \leq \max \{ \lambda_\mu^-(A) , \lambda_\mu^-(B) \}\\
\Leftrightarrow\ & -1 - \lambda_\mu^{-c}(AB) \leq \max \{ (-1 - \lambda_\mu^{-c}(A) , -1- \lambda_\mu^{-c}(B)\}\\
\Leftrightarrow\ & \lambda_\mu^{-c}(AB) \geq -1 - \max\{(-1 - \lambda_\mu^{-c}(A) , (-1- \lambda_\mu^{-c}(B)\}\\
\Leftrightarrow\ & \lambda_\mu^{-c}(AB) \geq \min \{ \lambda_\mu^{-c}(A) , \lambda_\mu^{-c}(B)\}
\end{aligned}$$

Therefore,

$$\lambda_\mu^{+c}(AB) \leq \max \{ \lambda_\mu^{+c}(A) , \lambda_\mu^{+c}(B) \} \text{ and}$$
$$\lambda_\mu^{-c}(AB) \geq \min \{ \lambda_\mu^{-c}(A) , \lambda_\mu^{-c}(B) \}$$

We have $\lambda_\mu^+(A^{-1}) = \lambda_\mu^+(A)$

$$\begin{aligned}
\Leftrightarrow\ & 1- \lambda_\mu^{+c}(A^{-1}) = 1 - \lambda_\mu^{+c}(A)\\
\Leftrightarrow\ & \lambda_\mu^{+c}(A^{-1}) = \lambda_\mu^{+c}(A)\\
& \lambda_\mu^-(A^{-1}) = \lambda_\mu^-(A)\\
\Leftrightarrow\ & -1 - \lambda_\mu^{-c}(A^{-1}) = -1 - \lambda_\mu^{-c}(A)\\
\Leftrightarrow\ & \lambda_\mu^{-c}(A^{-1}) = \lambda_\mu^{-c}(A)
\end{aligned}$$

Therefore, $\lambda_\mu^{+c}(A^{-1}) = \lambda_\mu^{+c}(A)$, $\lambda_\mu^{-c}(A^{-1}) = \lambda_\mu^{-c}(A)$

Hence λ_μ^c is a bipolar anti fuzzy HX subgroup of HX group ϑ. □

Theorem 3.5. Let λ_μ be a bipolar anti fuzzy HX subgroup of a HX group ϑ with identity E, then

i. $\lambda_\mu^+(AB^{-1}) = \lambda_\mu^+(E) \Rightarrow \lambda_\mu^+(A) = \lambda_\mu^+(B)$ and

ii. $\lambda_\mu^-(AB^{-1}) = \lambda_\mu^-(E) \Rightarrow \lambda_\mu^-(A) = \lambda_\mu^-(B)$ for all $A,B\in\vartheta$

Proof. Let λ_μ is a bipolar anti fuzzy HX subgroup of HX group ϑ with identity E and $\lambda_\mu^+(AB^{-1}) = \lambda_\mu^+(E)$, $\lambda_\mu^-(AB^{-1}) = \lambda_\mu^-(E)$ then for all $A,B\in\vartheta$.

i.
$$\begin{aligned}\lambda_\mu^+(A) &= \lambda_\mu^+(AE)\\ &= \lambda_\mu^+(A(B^{-1}B))\\ &\le \max\{\lambda_\mu^+(AB^{-1}), \lambda_\mu^+(B)\}\\ &= \max\{\lambda_\mu^+(E), \lambda_\mu^+(B)\}\\ &= \lambda_\mu^+(B).\end{aligned}$$
$$\lambda_\mu^+(A) \le \lambda_\mu^+(B)$$

Now, $\lambda_\mu^+(B) = \lambda_\mu^+(B^{-1})$, Since λ_μ is a bipolar anti fuzzy HX subgroup of HX group ϑ.
$$\begin{aligned}\lambda_\mu^+(B) &= \lambda_\mu^+(EB^{-1})\\ &= \lambda_\mu^+((A^{-1}A)B^{-1})\\ &= \lambda_\mu^+(A^{-1}(AB^{-1}))\\ &\le \max\{\lambda_\mu^+(A^{-1}), \lambda_\mu^+(AB^{-1})\}\\ &= \max\{\lambda_\mu^+(A), \lambda_\mu^+(E)\}\\ &= \lambda_\mu^+(A).\end{aligned}$$
$$\lambda_\mu^+(B) \le \lambda_\mu^+(A)$$

Hence $\lambda_\mu^+(A) = \lambda_\mu^+(B)$

Therefore $\lambda_\mu^+(AB^{-1}) = \lambda_\mu^+(E) \Rightarrow \lambda_\mu^+(A) = \lambda_\mu^+(B)$

ii.
$$\begin{aligned}\lambda_\mu^-(A) &= \lambda_\mu^-(AE)\\ &= \lambda_\mu^-(A(B^{-1}B))\\ &= \lambda_\mu^-((AB^{-1})B)\\ &\ge \min\{\lambda_\mu^-(AB^{-1}), \lambda_\mu^-(B)\}\\ &= \min\{\lambda_\mu^-(E), \lambda_\mu^-(B)\}\\ &= \lambda_\mu^-(B).\end{aligned}$$
$$\lambda_\mu^-(A) \ge \lambda_\mu^-(B)$$

Now, $\lambda_\mu^-(B) = \lambda_\mu^-(B^{-1})$, Since λ_μ is a bipolar anti fuzzy HX subgroup of HX group ϑ.
$$\begin{aligned}\lambda_\mu^-(B) &= \lambda_\mu^-(EB^{-1})\\ &= \lambda_\mu^-((A^{-1}A)B^{-1})\\ &= \lambda_\mu^-(A^{-1}(AB^{-1}))\\ &\ge \min\{\lambda_\mu^-(A^{-1}), \lambda_\mu^-(AB^{-1})\}\\ &= \min\{\lambda_\mu^-(A), \lambda_\mu^-(E)\}\\ &= \lambda_\mu^-(A).\end{aligned}$$
$$\lambda_\mu^-(B) \ge \lambda_\mu^-(A)$$

Hence $\lambda_\mu^-(A) = \lambda_\mu^-(B)$

Therefore, $\lambda_\mu^-(AB^{-1}) = \lambda_\mu^-(E) \Rightarrow \lambda_\mu^-(A) = \lambda_\mu^-(B)$. □

Theorem 3.6. Let λ_μ be a bipolar anti fuzzy HX subgroup of a HX group ϑ with identity E, then

i. $\lambda_\mu^+(AB^{-1}) \leq \max \{ \lambda_\mu^+(A), \lambda_\mu^+(B)\}$

ii. $\lambda_\mu^-(AB^{-1}) \geq \min \{ \lambda_\mu^-(A), \lambda_\mu^-(B)\}$ for all $A,B \in \vartheta$

Proof. Let λ_μ be a bipolar anti fuzzy HX subgroup of a HX group ϑ for all $A,B \in \vartheta$

i. $\lambda_\mu^+(AB^{-1}) \leq \max \{ \lambda_\mu^+(A), \lambda_\mu^+(B)\}$

ii. $\lambda_\mu^-(AB^{-1}) \geq \min \{ \lambda_\mu^-(A), \lambda_\mu^-(B)\}$

iii. $\lambda_\mu^+(A^{-1}) = \lambda_\mu^+(A)$, $\lambda_\mu^-(B) = \lambda_\mu^-(A)$

Now

i. $\lambda_\mu^+(AB^{-1}) \leq \max \{ \lambda_\mu^+(A), \lambda_\mu^+(B^{-1})\} = \max \{ \lambda_\mu^+(A), \lambda_\mu^+(B)\}$

$\Leftrightarrow \lambda_\mu^+(AB^{-1}) \leq \max \{ \lambda_\mu^+(A), \lambda_\mu^+(B)\}$

ii. $\lambda_\mu^-(AB^{-1}) \geq \min \{ \lambda_\mu^-(A), \lambda_\mu^-(B^{-1})\} = \min \{ \lambda_\mu^-(A), \lambda_\mu^-(B) \}$

$\Leftrightarrow \lambda_\mu^-(AB^{-1}) \geq \min \{ \lambda_\mu^+(A), \lambda_\mu^+(B)\}$ □

4. Properties of lower level subsets of a bipolar anti fuzzy HX subgroup

In this section, we introduce the concept of lower level subsets of a bipolar anti fuzzy HX subgroup and discuss some of its properties.

Definition 4.1. Let λ_μ be a bipolar anti fuzzy HX subgroup of a HX group ϑ. For any $< \alpha, \beta > \in [0,1]\times[-1,0]$, we define the set $\lambda_{\mu <\alpha, \beta>} = \{ A \in \vartheta / \lambda_\mu^+(A) \leq \alpha$ and $\lambda_\mu^-(A) \geq \beta \}$ is called the $<\alpha, \beta>$ lower level subset of λ_μ or simply the lower level subset of λ_μ.

Theorem 4.2. Let λ_μ is a bipolar anti fuzzy HX subgroup of HX group ϑ then for $< \alpha , \beta > \in [0,1]\times[-1,0]$ such that $\alpha \geq \lambda_\mu^+(E)$, $\beta \leq \lambda_\mu^-(E)$ and $\lambda_{\mu <\alpha, \beta>}$ is a sub HX group of ϑ.

Proof. For all $A ,B \in \lambda_{\mu <\alpha, \beta>}$ we have $\lambda_\mu^+(A) \leq \alpha$, $\lambda_\mu^-(A) \geq \beta$ and $\lambda_\mu^+(B) \leq \alpha$, $\lambda_\mu^-(B) \geq \beta$

Now
$$\begin{aligned} \lambda_\mu^+(AB^{-1}) &\leq \max \{ \lambda_\mu^+(A), \lambda_\mu^+(B)\} \\ &\leq \max \{ \alpha, \alpha \} \\ &= \alpha \end{aligned}$$
$$\Rightarrow \lambda_\mu^+(AB^{-1}) \leq \alpha$$
$$\begin{aligned} \lambda_\mu^-(AB^{-1}) &\geq \min \{ \lambda_\mu^-(A), \lambda_\mu^-(B)\} \\ &\geq \min \{ \beta, \beta \} \\ &= \beta \end{aligned}$$
$$\Rightarrow \lambda_\mu^-(AB^{-1}) \geq \beta$$

Hence $AB^{-1} \in \lambda_{\mu <\alpha, \beta>}$

Hence, $\lambda_{\mu <\alpha, \beta>}$ is a sub HX group of ϑ. □

Theorem 4.3. Let ϑ be a HX group and λ_μ be a fuzzy subset of ϑ such that $\lambda_{\mu<\alpha,\beta>}$ is a sub HX group ϑ for $<\alpha,\beta>\in[0,1]\times[-1,0]$ such that $\alpha \geq \lambda_\mu^+(E)$, $\beta \leq \lambda_\mu^-(E)$, Then λ_μ is a bipolar anti fuzzy HX subgroup of ϑ.

Proof. Let $A, B \in \vartheta$, let $A \in \lambda_{\mu<\alpha_1,\beta_1>} \Rightarrow \lambda_\mu^+(A) = \alpha_1$, $\lambda_\mu^-(A) = \beta_1$ and

$B \in \lambda_{\mu<\alpha_2,\beta_2>} \Rightarrow \lambda_\mu^+(B) = \alpha_2$, $\lambda_\mu^-(B) = \beta_2$

Suppose $\lambda_{\mu<\alpha_1,\beta_1>} < \lambda_{\mu<\alpha_2,\beta_2>}$ then $A, B \in \lambda_{\mu<\alpha_2,\beta_2>}$ As $\lambda_{\mu<\alpha_2,\beta_2>}$ is a subgroup of ϑ, $AB^{-1} \in \lambda_{\mu<\alpha_2,\beta_2>}$

$$\begin{aligned}
\text{Hence} \quad \lambda_\mu^+(AB^{-1}) &\leq \alpha_2 \\
&= \max\{\alpha_1, \alpha_2\} \\
&= \max\{\lambda_\mu^+(A), \lambda_\mu^+(B)\}
\end{aligned}$$

$$\text{Therefore} \quad \lambda_\mu^+(AB^{-1}) \leq \max\{\lambda_\mu^+(A), \lambda_\mu^+(B)\}$$

$$\begin{aligned}
\lambda_\mu^-(AB^{-1}) &\geq \beta_2 \\
&= \min\{\beta_1, \beta_2\} \\
&= \min\{\lambda_\mu^-(A), \lambda_\mu^-(B)\}
\end{aligned}$$

$$\text{Therefore} \quad \lambda_\mu^-(AB^{-1}) \geq \min\{\lambda_\mu^-(A), \lambda_\mu^-(B)\}$$

Hence λ_μ is a bipolar anti fuzzy HX subgroup of ϑ. □

Definition 4.4. Let λ_μ is a bipolar anti fuzzy HX subgroup of a HX group ϑ.The sub HX groups $\lambda_{\mu<\alpha,\beta>}$ for $<\alpha,\beta>\in[0,1]\times[-1,0]$ and $\alpha \geq \lambda_\mu^+(E)$, $\beta \leq \lambda_\mu^-(E)$ are called lower level sub HX groups of λ_μ.

Theorem 4.5. Let ϑ be a HX group and λ_μ be a bipolar anti fuzzy HX subgroup of ϑ. If two lower level sub HX groups $\lambda_{\mu<\alpha,\gamma>}$, $\lambda_{\mu<\beta,\delta>}$ with $\alpha < \beta$ and $\delta < \gamma$ of λ_μ are equal if and only if there is no $A \in \vartheta$ such that $\alpha < \lambda_\mu^+(A) \leq \beta$ and $\delta \leq \lambda_\mu^-(A) < \gamma$.

Proof. Let $\lambda_{\mu<\alpha,\gamma>} = \lambda_{\mu<\beta,\delta>}$ .Suppose that there exists $A \in \vartheta$ such that $\alpha < \lambda_\mu^+(A) \leq \beta$ and $\delta \leq \lambda_\mu^-(A) < \gamma$. Then $\lambda_{\mu<\alpha,\gamma>} \subset \lambda_{\mu<\beta,\delta>}$ Since $A \in \lambda_{\mu<\beta,\delta>}$ but not in $\lambda_{\mu<\alpha,\gamma>}$ which contradicts the hypothesis. Hence there exists no $A \in \vartheta$ such that $\alpha < \lambda_\mu^+(A) \leq \beta$ and $\delta \leq \lambda_\mu^-(A) < \gamma$.

Conversely, let there be no $A \in \vartheta$ such that $\alpha < \lambda_\mu^+(A) \leq \beta$ and $\delta \leq \lambda_\mu^-(A) < \gamma$ Since $\alpha < \beta$ and $\delta < \gamma$, we have, $\lambda_{\mu<\alpha,\gamma>} \subset \lambda_{\mu<\beta,\delta>}$ Let $A \in \lambda_{\mu<\beta,\delta>}$, then $\lambda_\mu^+(A) \leq \beta$ and $\lambda_\mu^-(A) \geq \delta$. Since there exists no $A \in \vartheta$ such that $\alpha < \lambda_\mu^+(A) \leq \beta$ and $\delta \leq \lambda_\mu^-(A) < \gamma$, we have $\lambda_\mu^+(A) \leq \alpha$ and $\lambda_\mu^-(A) \geq \gamma$ which implies $A \in \lambda_{\mu<\alpha,\gamma>}$. i.e. $\lambda_{\mu<\beta,\delta>} \subset \lambda_{\mu<\alpha,\gamma>}$. Hence, $\lambda_{\mu<\alpha,\gamma>} = \lambda_{\mu<\beta,\delta>}$. □

Theorem 4.6. A fuzzy subset λ_μ of ϑ is a bipolar anti fuzzy HX subgroup of HX group ϑ if and only if the lower level subsets $\lambda_{\mu<\alpha,\beta>}$,$<\alpha,\beta> \in$ Image λ_μ are HX subgroups of ϑ.

Theorem 4.7. Any sub HX group H of a HX group ϑ can be realized as a lower level sub HX group of some bipolar anti fuzzy HX subgroup of ϑ

Proof. Let λ be a bipolar fuzzy subset and $A \in \vartheta$, Define

$$\lambda_\mu^+(A) = \begin{cases} 0, & \text{if } A\in H \\ \alpha, & \text{if } A\notin H \end{cases} \quad \text{and} \quad \lambda_\mu^-(A) = \begin{cases} 0, & \text{if } A\in H \\ \beta, & \text{if } A\notin H \end{cases}$$

We shall prove λ be a bipolar anti fuzzy HX subgroup of ϑ

Let $A, B \in \vartheta$

i. Suppose $A, B \in H$ then $AB \in H$ and $AB^{-1} \in H$

$$\lambda_\mu^+(A) = \lambda_\mu^+(B) = 0$$
$$\Rightarrow \lambda_\mu^+(AB^{-1}) = 0 \leq \max\{0,0\} = \max\{\lambda_\mu^+(A), \lambda_\mu^+(B)\}$$

Therefore, $\lambda_\mu^+(AB^{-1}) \leq \max\{\lambda_\mu^+(A), \lambda_\mu^+(B)\}$

And $\lambda_\mu^-(A) = \lambda_\mu^-(B) = 0$

$$\Rightarrow \lambda_\mu^-(AB^{-1}) = 0 \geq \min\{0, 0\} = \min\{\lambda_\mu^-(A), \lambda_\mu^-(B)\}$$

Therefore, $\lambda_\mu^-(AB^{-1}) \geq \min\{\lambda_\mu^-(A), \lambda_\mu^-(B)\}$

ii. Suppose $A \in H$, $B \notin H$ then $AB \notin H$ and $AB^{-1} \notin H$

$$\lambda_\mu^+(A) = 0, \ \lambda_\mu^+(B) = \alpha$$
$$\Rightarrow \lambda_\mu^+(AB^{-1}) = \alpha \leq \max\{0, \alpha\} = \max\{\lambda_\mu^+(A), \lambda_\mu^+(B)\}$$

Therefore, $\lambda_\mu^+(AB^{-1}) \leq \max\{\lambda_\mu^+(A), \lambda_\mu^+(B)\}$

And $\lambda_\mu^-(A) = 0, \lambda_\mu^-(B) = \beta$

$$\Rightarrow \lambda_\mu^-(AB^{-1}) = \beta \geq \min\{0,\beta\} = \min\{\lambda_\mu^-(A), \lambda_\mu^-(B)\}$$

Therefore, $\lambda_\mu^-(AB^{-1}) \geq \min\{\lambda_\mu^-(A), \lambda_\mu^-(B^{-1})\}$

iii. Suppose $A, B \notin H$ then $AB^{-1} \in H$ or $AB^{-1} \notin H$

$$\lambda_\mu^+(A) = \lambda_\mu^+(B) = \alpha \ ; \ \lambda_\mu^-(A) = \lambda_\mu^-(B) = \beta$$

Define

$$\lambda_\mu^+(AB^{-1}) = \begin{cases} 0, & \text{if } AB^{-1}\in H \\ \alpha, & \text{if } AB^{-1}\notin H \end{cases} \quad \text{and} \quad \lambda_\mu^-(AB^{-1}) = \begin{cases} 0, & \text{if } AB^{-1}\in H \\ \beta, & \text{if } AB^{-1}\notin H \end{cases}$$

Let $AB^{-1} \notin H$

$$\lambda_\mu^+(AB^{-1}) = \alpha \leq \max\{\alpha, \alpha\} = \max\{\lambda_\mu^+(A), \lambda_\mu^+(B)\}$$

Therefore, $\lambda_\mu^+(AB^{-1}) \leq \max\{\lambda_\mu^+(A), \lambda_\mu^+(B^{-1})\}$

$$\lambda_\mu^-(AB^{-1}) = 0 \geq \min\{0, 0\} = \min\{\lambda_\mu^-(A), \lambda_\mu^-(B)\}$$

Therefore, $\lambda_\mu^-(AB^{-1}) \geq \min\{\lambda_\mu^-(A), \lambda_\mu^-(B)\}$

Thus in all cases , λ_μ be a bipolar anti fuzzy HX subgroup of ϑ. For this bipolar anti fuzzy HX subgroup $\lambda_{\mu <\alpha, \beta>}$ = H. □

Remark 4.8. As a consequence of the Theorem 4.5, Theorem 4.6, the lower level HX groups of a bipolar anti fuzzy HX sub group λ_μ of a HX group ϑ form a chain. Since $\lambda_\mu^{+}(E) \leq \lambda_\mu^{+}(A)$ and $\lambda_\mu^{-}(E) \geq \lambda_\mu^{-}(A)$ for all A in ϑ. Therefore , $\lambda_{\mu <\alpha_0, \beta_0>}$, $\alpha \in [0,1]$ and $\beta \in [-1,0]$ where $\lambda_\mu^{+}(E) = \alpha_0$, $\lambda_\mu^{-}(E) = \beta_0$ is the smallest HX sub group and we have the chain $\{E\} \subseteq \lambda_{\mu <\alpha_0, \beta_0>} \subseteq \lambda_{\mu <\alpha_1, \beta_1>} \subseteq \lambda_{\mu <\alpha_2, \beta_2>} \subseteq \ldots\ldots\ldots \subseteq \lambda_{\mu <\alpha_n, \beta_n>} = \vartheta$, where $\alpha_0 < \alpha_1 < \alpha_2 < \ldots. < \alpha_n$ and $\beta_0 > \beta_1 > \beta_2 >. \ldots. > \beta_n$.

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