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# $\alpha$-Ideals in a Distributive Nearlattice

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**ABSTRACT**

In this paper, the authors give the concept of $\alpha$-ideals in a distributive nearlattice. They provide a number of characterizations of $\alpha$-ideals in a nearlattice. They prove that a nearlattice $S$ with $0$ is disjunctive if and only if its every ideal is an $\alpha$-ideal. They also show that $S$ is sectionally quasi-complemented if and only if each prime $\alpha$-ideal is a minimal prime ideal. Finally $S$ is generalized Stone if and only if each prime ideal contains a unique prime $\alpha$-ideal.

***Keywords:*** $\alpha$-ideal, Annulets, Disjunctive nearlattice, Quasi-complemented nearlattice, Generalized Stone nearlattice.

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## 1. Introduction

W. H. Cornish in [3] has studied the $\alpha$-ideals in a lattice $L$ with $0$. On the other hand, Bigard [2] has studied $\alpha$-ideals in the context of lattice ordered groups. In this paper we study the $\alpha$-ideals of nearlattices.

In a lattice $L$ with $0$, set of all ideals of the form $(x]^*$ can be made into a lattice $A_0(L)$. Where $(x]^* = \{y \in L / y \wedge x = 0\}$. By [3] $(x]^*$ is called an annulet of $L$ and $A_0(L)$ denotes the lattice of annulets of $L$. For an ideal $J$ in $L$, [3] has defined $\alpha(J) = \{(x]^* : x \in J\}$ and for a filter $F$ in $A_0(L)$,

$\alpha^{\leftarrow}(F)=\{x\in L:(x]^*\in F\}$. It is easy to see that $\alpha(J)$ is a filter in $A_0(L)$ and $\alpha^{\leftarrow}(F)$ is an ideal in $L$. An ideal $J$ in $L$ is called an $\alpha$*-ideal* if $\alpha^{\leftarrow}\alpha(J)=J$.

By a *near lattice*, we mean a meet semilattice with the property that any two elements possessing a common upper bound have a supremum. A nearlattice S is called a *distributive nearlattice* if for all $x,y,z\in S$, $x\wedge(y\vee z)=(x\wedge y)\vee(x\wedge z)$, provided $y\vee z$ exists. A nonempty subset $I$ of a nearlattice $S$ is called an *ideal* if

i) For $x,y\in I$, $x\vee y\in I$, provided $x\vee y$ exists, and

ii) For $x\in I, t\leq x$ ($t\in S$) implies $t\in I$.

An ideal $P$ of a nearlattice $S$ is called a prime ideal if for $x,y\in S$, $x\wedge y\in P$ implies either $x\in P$ or $y\in P$.

A non empty subset $F$ of $S$ is called a filter if

i) for all $x,y\in F$, $x\wedge y\in F$, and

ii) $t\in S, t\geq x$ *and* $x\in F$ imply $t\in F$.

For a distributive nearlattice $S$ with 0, $I(S)$ denotes the set of all ideals, which is a distributive lattice and also pseudocomplemented.

Recently [5] have studied the annulets in a nearlattice. In this paper, we study the $\alpha$-ideals in a nearlattice and generalize several results of [3].

## 2. $\alpha$-ideals

**Proposition 2.1.** *Let $S$ be a distributive near lattice with $0$, then the following hold:*

(i) *For an ideal $I$ in $S$, $\alpha(I)=\{(x]^*/x\in I\}$ is a filter in $A_0(S)$.*

(ii) *For a filter $F$ in $A_0(S)$, $\alpha^{\leftarrow}(F)=\{x\in S/(x]\in F\}$ is an ideal in $S$*

(iii) *If $I_1, I_2$ are ideals in $S$ then $I_1\subseteq I_2$ implies $\alpha(I_1)\subseteq\alpha(I_2)$; and if $F_1, F_2$ are filters in $A_0(S)$ then $F_1\subseteq F_2$ implies that $\alpha^{\leftarrow}(F_1)\subseteq\alpha^{\leftarrow}(F_2)$.*

(iv) *The map $I\rightarrow\alpha^{\leftarrow}\alpha(I)=\{\alpha^{\leftarrow}(\alpha(I))\}$ is a closure operation on the lattice of ideals, that is,*

**Proof.** (i). By [5, Prop. 2.1], $A_0(S)$ is a join semilattice with the lower bound property. Let $(x]^*,(y]^*\in\alpha(I)$, and $(t]^*\in A_0(S)$, where $x,y\in I, t\in S$. Then $((t]^*\underline{\vee}(x]^*)\wedge((t]^*\underline{\vee}(y]^*)=(t\wedge x]^*\wedge(t\wedge y]^*=((t\wedge x)\vee(t\wedge y)]^*\in\alpha(I)$, as

$(t\wedge x)\vee(t\wedge y)\in I$. Also, if $(x]^*\in\alpha(I)$ and $(t]^*\in A_0(S)$ with $(x]^*\subseteq(t]^*$, then $(t]^*=((t]^*\underline{\vee}(x]^*)=(t\wedge x]^*\in\alpha(I)$ so $\alpha(I)$ is a filter in $A_0(S)$.

(ii). Let $x, y \in \alpha^{\leftarrow}(F)$ and $t \in S$, then $(x]^*, (y]^* \in F$, and $(t]^* \in A_0(S)$. Since $F$ is a filter of $A_0(S)$, so $((t]^* \underline{\vee} (x]^*) \wedge ((t]^* \underline{\vee} (y]^* \in F$ implies that $((t \wedge x) \vee (t \wedge y)]^* \in F$ implies that $(t \wedge x) \vee (t \wedge y) \in \alpha^{\leftarrow}(F)$.

Also, if $x \in \alpha^{\leftarrow}(F)$ and $t \in S$ with $t \le x$, then $(t]^* \supseteq (x]^*$ and $(x]^* \in F$ implies that $(t]^* \in F$. So $t \in \alpha^{\leftarrow}(F)$. Hence $\alpha^{\leftarrow}(F)$ is an ideal in $S$.

(iii). Let $(x]^* \in \alpha(I_1)$, then $x \in I_1 \subseteq I_2$ implies that $(x]^* \in \alpha(I_2)$ implies that $\alpha(I_1) \subseteq \alpha(I_2)$. Let $x \in \alpha^{\leftarrow}(F_1)$, then $(x]^* \in F_1 \subseteq F_2$ implies that $x \in \alpha^{\leftarrow}(F_2)$ implies that $\alpha(F_1) \subseteq \alpha(F_2)$.

(iv) is trivial. □

In a join semilattice $S_1$ with the lower bound property ( i. e. $S_1$ is a dual nearlattice) a non-empty subset $F$ of $S_1$ is called a *filter* if

(i) For any $x, y \in F$, $x \wedge y \in F$ if $x, y \in F$, and $x \wedge y$ exists and

(ii) $x \in F$ and $y \ge x$ $(y \in S_1)$ implies that $y \in F$.

Observe that this definition is dual to the definition of an ideal in a nearlattice. Now we give an equivalent definition of a filter in a dual nearlattice which is very easy to prove. This will be needed for further development of this section.

**Theorem 2.2.** *In a dual nearlattice $S_1$ a non-empty subset $F$ of $S_1$ is a filter if and only if*

(i) *For $f \in F$ and $x \ge f$ $(x \in S_1)$ implies $x \in F$ and*

(ii) *For any $f_1, f_2 \in F$ and $x \in S_1, (x \vee f_1) \wedge (x \vee f_2) \in F$.* □

An ideal $I$ of a nearlattice $S$ is called an $\alpha$ **-**ideal if $\alpha^{\leftarrow}\alpha(I) = I$. That is, $\alpha$ -ideals are simply the closed elements with respect to the closure operation of the Proposition 2.1.

**Proposition 2.3.** *The $\alpha$ -ideals of a nearlattice $S$ with $0$ form a complete distributive lattice isomorphic to the lattice of filters, ordered by set inclusion in $A_0(S)$.*

**Proof.** Let $\{I_i\}$ be any class of $\alpha$ -ideals of $S$. Then $\alpha^{\leftarrow}\alpha(I_i) = I_i$ for all $i$. By Proposition 2.1 (iv), $\cap I_i \subseteq \alpha^{\leftarrow}\alpha(\cap I_i)$. Again $\alpha^{\leftarrow}\alpha(\cap I_i) \subseteq \alpha^{\leftarrow}\alpha(I_i) = I_i$ for all $i$ implies that $\alpha^{\leftarrow}\alpha(\cap I_i) \subseteq \cap I_i$, and so $\alpha^{\leftarrow}\alpha(\cap I_i) = \cap I_i$. Thus $\cap I_i$ is an $\alpha$ -ideal. Trivially, lattice of $\alpha$ -ideals is distributive. Hence $\alpha$ -ideals form a complete distributive lattice. For an $\alpha$ -ideal $I$ , $\alpha^{\leftarrow}\alpha(I) = I$. Also, it is easy to see that for any filter $F$ of $A_0(S)$, $\alpha^{\leftarrow}\alpha(F) = F$. Moreover, by Proposition 2.1(iii), both $\alpha$ and $\alpha^{\leftarrow}$ are isotone. Hence the lattice of $\alpha$-ideals of $S$ is isomorphic to the lattice of filters of $A_0(S)$. □

**Corollary 2.4.** *Let $S$ be a distributive lattice with $0$. Then the set of prime $\alpha$-ideals of $S$ are isomorphic to the set of prime filters of $A_0(S)$.* □

Now we give a characterization of $\alpha$-ideals of a nearlattice which generalizes [3, Proposition 3.3].

**Proposition 2.5.** *For an ideal $I$ in a distributive nearlattice $S$ with $0$ the following conditions are equivalent:*
(i) *$I$ is an $\alpha$-ideal.*
(ii) *For $x, y \in S$, $(x]^* = (y]^*$ and $x \in I$ implies $y \in I$.*
(iii) *$I = \cup_{x \in I}(x]^*$ (where $\cup$ = set theoretic union).*

**Proof.** (i) implies (ii). Suppose $I$ is an $\alpha$-ideal, then $\alpha^{\leftarrow}\alpha(I) = I$. Let $x, y \in S$, $(x]^* = (y]^*$ and $x \in I$. So, $(x]^* \in \alpha(I)$ implies that $(y]^* \in \alpha(I)$ implies that $y \in \alpha^{\leftarrow}\alpha(I) = I$.
(ii) implies (i). Let $I$ be an ideal of $S$. $I \subseteq \alpha^{\leftarrow}\alpha(I)$ is always true. Suppose $x \in \alpha^{\leftarrow}\alpha(I)$ then $(x]^* \in \alpha(I)$ implies that $(x]^* = (y]^*$ for some $y \in I$. So by (ii) $x \in I$ implies that $\alpha^{\leftarrow}\alpha(I) \subseteq I$ implies that $\alpha^{\leftarrow}\alpha(I) = I$.
(ii) implies (iii). Clearly $I \subseteq \cup_{x \in I}(x]^{**}$. If $x \in I$ and $y \in (x]^{**}$ then $(x]^* \subseteq (y]^*$ implies that $(y]^* = (x]^* \underline{\vee} (y]^* = (x \wedge y]^*$. Then $x \wedge y \in I$ implies that $y \in I$. Thus $\cup_{x \in I}(x]^{**} \subseteq I$. So $I = \cup_{x \in I}(x]^{**}$.
(iii) implies (ii). If $x, y \in S$, $(x]^* = (y]^*$ and $x \in I$, then $(x]^{**} = (y]^{**}$ implies that $y \in (x]^{**} \subset I$ implies that $y \in I$. This completes the proof. □

A distributive nearlattice $S$ with $0$ is called *disjunctive* if $0 \le x < b$ implies there is an element $x \in S$ such that $x \wedge a = 0$ where $0 < x \le b$.

By [1] we know that a nearlattice $S$ with $0$ is disjunctive if and only if $(x]^* = (y]^*$ implies $x = y$ for some $x, y \in S$.

**Proposition 2.6.** *In a distributive nearlattice $S$ with $0$ the following conditions are equivalent:*
(i) *Each ideal is an $\alpha$-ideal.*
(ii) *Each prime ideal is an $\alpha$-ideal.*
(iii) *$S$ is disjunctive.*

**Proof.** (i) implies (ii). Suppose $P$ is a prime ideal of $S$, then by (i) $P$ is an $\alpha$-ideal, that is, $\alpha^{\leftarrow}\alpha(P) = P$. Let $I$ be any ideal of $S$ then we have $I = \cap(P / P \supseteq I)$ implies $\alpha^{\leftarrow}\alpha(I) = \alpha^{\leftarrow}\alpha(\cap(P / P \supseteq I))$

$=\cap(\alpha^{\leftarrow}\alpha(P)/P\supseteq I)=\cap(P/P\supseteq I)=I$ implies that $\alpha^{\leftarrow}\alpha(I)=I$. So $I$ is an $\alpha$ -ideal.
(ii) implies (i) is trivial.
(i) implies (iii). For any $x,y\in S$, let $(x]^*=(y]^*$. Since $(x]$ is an $\alpha$ -ideal, so by definition of $\alpha$ -ideal , $y\in(x]$.Therefore, $y\le x$. Similarly $x\le y$, and so $x=y$. Hence $S$ is disjunctive. (iii) implies (i). Suppose $I$ is any ideal of $S$. By 2.1, $I\subseteq\alpha^{\leftarrow}\alpha(I)$. For the reverse inclusion, let $x\in\alpha^{\leftarrow}\alpha(I)$. Then by definition $(x]^*\in\alpha(I)$, and so $(x]^*=(y]^*$ for some $y\in I$. This implies $x=y$, as $S$ disjunctive. So $x\in I$, and hence $\alpha^{\leftarrow}\alpha(I)=I$ Therefore, $I$ is an $\alpha$ -ideal of $S$. ☐

Proposition 2.3 implies that there is an order isomorphism between the prime $\alpha$ -ideals of $S$ and the prime filters of $A_0(S)$. It is not hard to show that each $\alpha$ -ideal is an intersection of prime $\alpha$ -ideals.

Following theorem is a generalization of [3, Theorem 3.6]. We need the following lemma.

**Lemma 2.7.** *A distributive near lattice $S$ with $0$ is relatively complemented if and only if every prime filter is an ultra filter (Proper and maximal).*

**Proof.** By [4, Theorem 2.11] $S$ is relatively complemented if and only if its prime ideals are unordered. Thus the result follows. ☐

A prime ideal $P$ of a nearlattice $S$ is called a *minimal prime ideal* if it does not properly contain any other prime ideal.

By [6] we know that a distributive nearlattice $S$ with $0$ is called a *normal nearlattice* if every prime ideal of $S$ contains a unique minimal prime ideal.

A distributive pseudocomplemented lattice $L$ is called a *Stone lattice* if for each $x\in L$, $x^*\vee x^{**}=1$. The concept of Stone lattice is not possible in a general nearlattice with $0$. We can talk about generalized Stone nearlattices. A distributive nearlattice $S$ with $0$, is called a generalized Stone nearlattice if interval $[0,x]$ for each $x\in S$ is a Stone lattice. Moreover, $S$ is called generalized Stone if $(x]^*\vee(x]^{**}=S$ for each $x\in S$. Of course, every generalized Stone nearlattice is normal.

A distributive nearlattice $S$ with $0$ is called quasi-complemented if for each $x\in S$, there exists $x'\in S$ such that $x\wedge x'=0$ and $((x]^*\vee(x']^*)^*=(0]$. $S$ is called sectionally quasi-complemented if each interval $[0,x]$, $x\in S$ is quasi-complemented.

**Theorem 2.8.** *Let $S$ be a distributive near lattice with $0$. Then the following conditions are equivalent:*
(i) *$S$ is sectionally quasi-complemented.*

(ii) *Each prime* $\alpha$ *-ideal is a minimal prime ideal.*
(iii) *Each* $\alpha$ *-ideal is an intersection of minimal prime ideals.*
*Moreover, the above conditions are equivalent to* $S$ *being quasi-complemented if and only if there is an element* $d \in S$ *such that* $(d]^* = (0]^*$.

**Proof.** (i) implies (ii). Suppose $S$ is sectionally quasi-complemented. Then by [5, Theorem 2.7], $A_0(S)$ is relatively complemented. Hence its every prime filter is an ultra filter. Then by Corollary 2.4, each prime $\alpha$ -ideal is a minimal prime ideal.
(ii) implies (iii). It is not hard to show that each ideal of $S$ is an intersection of prime $\alpha$ -ideals. This shows (ii) implies (iii).
(ii) implies (i). Suppose (ii) holds. Then by Corollary 2.4, each prime filter of $A_0(S)$ is maximal. Then by Lemma 2.7, $A_0(S)$ is relatively complemented, and so by [5, Proposition 2.7], $S$ is sectionally quasi-complemented. □

We conclude the paper with the following result which is a generalization of [3, Theorem 3.7].

**Theorem 2.9.** A near lattice $S$ with $0$ is a generalized Stone near lattice if and only if each prime ideal contains a unique prime $\alpha$ -ideal.

**Proof.** Since minimal prime ideals are $\alpha$ -ideals, so by the given condition every prime ideal contains a unique minimal prime ideal. Hence $S$ is normal. Also, by the given condition each prime $\alpha$ -ideal contains a unique prime $\alpha$ -ideal. That is each prime $\alpha$ -ideal contains no other prime $\alpha$ -ideals than itself. Since each minimal prime ideal is also prime $\alpha$ -ideal, so by the condition, each prime $\alpha$ -ideal is itself a minimal prime ideal. Hence by Theorem 2.8, $S$ is a sectionally quasi-complemented. Therefore, by [6, Theorem 2.3], $S$ is generalized Stone.

Conversely, if $S$ is generalized Stone then by [6, Theorem 1.6], $S$ is normal. So each prime ideal contains a unique minimal prime ideal. Thus the result follows as each minimal prime ideal is a prime $\alpha$ -ideal. □

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