Journal of Physical Sciences, Vol. 16, 2012, 75-84
ISSN: 0972-8791, www.vidyasagar.ac.in/journal
Published on 31 December 2012

European Option Pricing in Fractional Jump Diffusion Markets

Dong Yan

Department of Basic, Shaanxi Railway Institute, Wei Nan Shaanxi 714000 China
E-mail: dongyan840214@126.com

Received June 21, 2012; accepted December 18, 2012

ABSTRACT

Previous option pricing research typically assumes that the stock volatility is constant during the life of the option. In this study, we assume the stock volatility in our option valuation model is function of time and stock price. The stock price Process numerically is simulated by using the Monte Carlo method. Then, the numerical option pricing method for European option is hold. Finally, we compare our results with the known results in the linear case, the results show that our method is effective.

Keywords: fractional Brownian motion; Poisson process; incomplete markets; Monte Carlo method

AMS Mathematics Subject Classification (2010): 60H10; 90A06;

1. Introduction

The interest in pricing financial derivatives – including pricing options – arises from the fact that financial derivatives can be used to minimize losses caused by price fluctuations of the underlying assets. This process of protection is called hedging. There is a variety of financial products on the market, such as futures, forwards, swaps and options. In this paper we will concentrate on European Call and Put options.

We recall that a European Call option is a contract where at a prescribed time in the future, known as the expiry date T, the owner of the option, known as the holder, may purchase a prescribed asset, known as the underlying asset S_t, for a prescribed amount, known as the exercise or strike price K. The opposite party, or the writer, has the obligation to sell the asset if the holder chooses to exercise his right. Therefore, the value of the option at expiry, known as the pay-off function, is $C(S_T,T)=(S_T-K)^+$. Reciprocally, a European Put option is the right to sell the asset with the pay-off function $P(S_T,T)=(K-S_T)^+$. While European options can only be exercised in T, American options can be exercised at any time until the expiration, which complicates their pricing process significantly.

Option pricing theory has made a great leap forward since the development of the Black–Scholes option pricing model by Fischer Black and Myron Scholes in [2] in 1973 and previously by Robert Merton in [3]. The solution of the famous (linear) Black–Scholes equation

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV=0\,,$$

provides both the price for a European option and a hedging portfolio that replicates the option assuming that (see [4]):

(a) The price of the asset price or underlying derivative S_t follows a Geometric Brownian motion W(t), meaning that S_t satisfies the following stochastic differential equation (SDE): $dS_t=\mu S_t dt+\sigma S_t dW_t$.

(b) The trend or drift μ (measures the average rate of growth of the asset price), the volatility σ (measures the standard deviation of the returns) and the riskless interest rater are constant for $0\le t\le T$ and no dividends are paid in that time period.

(c) The market is frictionless, thus there are no transaction costs (fees or taxes), the interest rates for borrowing and lending money are equal, all parties have immediate access to any information, and all securities and credits are available at any time and any size. That is, all variables are perfectly divisible and may take any real number.

Moreover, individual trading will not influence the price.

(d) There are no arbitrage opportunities, meaning that there are no opportunities of instantly making a risk-free profit.

Under these assumptions the market is complete, which means that any asset can be replicated with a portfolio of other assets in the market (see [5]). Then, the linear Black–Scholes equation (1) can be transformed into the heat equation and analytically solved to price the option [1].

One can argue that these restrictive assumptions never occur in reality. Due to transaction costs (see [6–8]), large investor preferences (see [9–11]) and incomplete markets [12] they are likely to become unrealistic and the classical model results in strongly or fully nonlinear, possibly degenerate, parabolic diffusion-convection equations, where both the volatility σ and the drift μ can depend on the time t, the stock price S_t or the derivatives of the option price C or P itself.

Recently, some articles have focused on the valuation of European options when the underlying value follows a jump diffusion process or Levy processes which are a fairly large class of continuous time processes with stationary independent increments. For jump diffusion process or Levy processes and their application in finance (see[13-17]).

On the other hand, fractional Brownian motion has been considered to replace Brownian motion in the usual financial models as it has better behaved tails and exhibits long-term dependence while remaining Gaussian. For details about the stochastic analysis theory of fractional Brownian motion, see (18-19). The fractional Brownian motion is applied in finance, such as Ref.[20-22].

In this paper we will be concerned with the nonlinear Black-Scholes model for European options with a non-constant volatility $\sigma=\sigma(t,S_t)$ under the fractional jump-diffusion Environment. The remaining of the paper is organized as follow. The model and some theoretical results are presented in Section 2. In Section 3, we perform numerical simulations for the premium. Concluding remarks are given in Section 4.

2. The model

In this paper we study the pricing problem for an underlying asset price with jumps which is governed by the following stochastic differential equation:

$$\frac{dS_t}{S_t}=r_t dt+\sigma(t,S_t)[a_t\Diamond \mathrm{d}W_t^H+b_t\mathrm{d}(N_t-\lambda t)],\ \{S_t\},t\in[0,T],\ S_0>0 \text{ is given} \qquad (1.1)$$

where, r,a,b are deterministic functions with respect to t . σ is the function of t and S_t such that $1+\sigma(t,S_t)b_t>0$. Here $\{N_t\},t\in[0,T]$ is a Poisson process with deterministic intensity λ and $\{W_t^H\},t\in[0,T]$ is a fractional Brownian motion with Hurst parameter H . $\int_0^t a_\tau\Diamond \mathrm{d}W_\tau^H$ is the wick product for the fractional Brownian motion. Note that the process M defined by $M_t=N_t-\lambda t$ for $t\in[0,T]$ is the compensated process associated to N. We consider a market with two assets: the risky asset S_t given by the Eq. (1.1) to which is related a European call option and a risk-free asset given by

$$dA_t=rA_t dt\ ,\ t\in[0,T],\ A_0=1\,.$$

We work on a probability space (Ω,F,P). M_t and W_t^H are independent and we denote by $\{\mathrm{F}_t\},t\in[0,T]$ the filtration generated by $\{N_t\},t\in[0,T]$ and $\{W_t^H\},t\in[0,T]$. We assume that (1.1) is the price of the asset under the risk-neutral probability P. Recall that a stochastic process is a function of two variables the time $t\in[0,T]$ and the event $\varepsilon\in\Omega$, but in the literature it is common to write S_t , while it means $S_t=S_t(\varepsilon)$. The same interpretation is true for W_t^H , N_t and M_t or any other stochastic process in this paper. To the authors knowledge, it is impossible to find an explicit formula for the solution of the pricing problem. However, the premium can be determined and expressed in the following expectation form (see Ref.[23])

$$C=\exp\{-\int_0^T r_t dt\}E_P[(S_T-k)]^+, \qquad (1.2)$$

where E_P denotes the expected value in a risk-neutral world. Here P is called the equivalent martingale measure. Note that, when $\sigma(t,S_t)=\sigma_t$ Eq. (1.1) is reduced to the well-known cfractional jump modle

$$S_t=S_0\exp\{+\int_0^t(r_s-\sigma_s b_s\lambda_s)\mathrm{d}s-H\int_0^t a_s^2\sigma_s^2 s^{2H-1}\mathrm{d}s+\int_0^t a_s\sigma_s\Diamond \mathrm{d}W_s^H\}\times\prod_{k=1}^{k=N_t}(1+\sigma_{t_k}b_{t_k})\,.$$

Therefore, the expectation in (1.2) can be calculated by integrating over the normal distribution which gives the European-call pricing formula C by Sun, Xue 2009 and Xue, Sun 2010, when σ , a,b are constants.

$$C = S_t e^{-\lambda b\sigma}\sum_{i=0}^{i=\infty}\frac{[(1+\sigma b)\lambda(T-t)]^i}{i!}e^{-\lambda(T-t)}\Phi(d_2^i) - K\exp\{-\int_t^T r_s \mathrm{d}s\}\sum_{i=0}^{i=\infty}\frac{[\lambda(T-t)]^i}{i!}e^{-\lambda(T-t)}\Phi(d_1^i),$$

where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution, and

$$d_1^i = \frac{\ln(\frac{S_t(1+\sigma b)^i}{K}) + \int_t^T r_s \mathrm{d}s - \lambda b\sigma(T-t) - 0.5a^2\sigma^2(T^{2H}-t^{2H})}{a\sigma\sqrt{T^{2H}-t^{2H}}}, \quad d_2^i = d_1^i - a\sigma\sqrt{T^{2H}-t^{2H}}.$$

Similarly, the pricing formula of the European put option P can be written as

$$P = K\exp\{-\int_t^T r_s \mathrm{d}s\}\sum_{i=0}^{i=\infty}\frac{[\lambda(T-t)]^i}{i!}e^{-\lambda(T-t)}\Phi(-d_1^i)$$
$$- S_t e^{-\lambda b\sigma}\sum_{i=0}^{i=\infty}\frac{[(1+\sigma b)\lambda(T-t)]^i}{i!}e^{-\lambda(T-t)}\Phi(-d_2^i)$$

However $\sigma(t,S_t)$ is the function of t and S_t, the expectation function can not be calculated to have an explicit formula because the random variable S_T does not have a known probability density. To surmount this problem, we use Monte Carlo techniques to simulate the premium. The Monte Carlo method is a very effective tool to simulate the prices of financial derivatives that do not have closed explicit formulas. The use of this method in options pricing was initiated by Boyle (1977). Since then it has been used by many researchers in finance. In this paper, we compute the premium and the price of the option at any time $t \in [0,T]$, using the Monte Carlo method.

3. Numerical computing of option prices

In this section we discuss the simulation of the premium (1.2) using the Monte Carlo method. The main steps are summarized below:

Step 1. Simulation of S_T: We select an integer $L > 0$, then we simulate $S_T(i)$ for $i \in (1,2,\cdots,L)$.

Step 2. Monte Carlo solution for the premium: The simulation of the premium via the Monte Carlo method involves the following steps:

(1)For each path $S_T(i)$, compute the payoff $\max\{S_T(i)-K,0\}$.

(2) Calculate the mean of the resulting payoffs $\frac{1}{L}\sum_{i=1}^{L}\max(S_T(i)-K,0)$.

(3) Estimate the price of the option by discounting the mean payoff at the risk-free rate $\frac{1}{L}\sum_{i=1}^{L}\max(S_T(i)-K,0)\exp(\int_0^T r_s ds)$.

In the proceeding subsections, we give the details of the above steps.

3.1. Simulation of S_T

We seek L realizations of S_T:

$$S_T(\omega_1),\cdots,S_T(\omega_i),\cdots S_T(\omega_L),$$

where $\omega_1,\cdots,\omega_i,\cdots\omega_L$ are chosen randomly from Ω. We follow the following algorithm:

(1) Simulate L trajectories for $\{S_t, 0\le t\le T\}$:

$$S_t(1),\cdots,S_t(i),\cdots,S_t(L),$$

where $S_t(i)$ is a simulation of $S_T(\omega_i)$ and $i=1,23,\cdots,L$.

(2)For each i ($i=1,23,\cdots,L$), take the value of $S_t(i)$ at the terminal time: $S_T(i)$. First, we select an integer $M>0$, then we discretize the time interval $[0,T]$ into steps $t_j = j\cdot\Delta t$, $j=0,1,2,\cdots,M$ of identical duration $\Delta t=\frac{T}{M}$:

$$S_{t_0}(i),\cdots,S_{t_j}(i),\cdots,S_{t_M}(i)$$

and thus we get L approximations of S_T:

$$S_{t_M}(1),\cdots,S_{t_M}(2),\cdots,S_{t_M}(L)$$

Let i be fixed in $\{1,\cdots,L\}$. We start by simulating a trajectory $S_t(i)$ of the Brownian motion and a trajectory $N_t(i)$ and then we use Eq. (2.2) to find the approximation $S_{t_M}(i)$ of S_T. . We implement the following steps:

(a). Simulation of the Brownian motion and the Brownian integral.
We simulate $\{W_{t_j}(i), i=1,2,\cdots L, j=0,1,\cdots H\}$ noting the fact that the Brownian motion fulfills:

$$W_0^H(i)=0,\quad W_{t_j}{}^H(i)=W_{t_{j-1}}{}^H(i)+\Delta t^H Z_j(i), i=1,2,\cdots L, j=0,1,\cdots M$$

where $Z_j(i)$ follows a normal distribution $N(0,1)$. We simulate $2L$ uniform random variable $U_j(i)$ and $V_j(i)$, and we use the Box–Muller method

$$Z_j(i)=\sqrt{-\log(U_j(i))}\cdot\cos(2\pi V_j(i)).$$

(b). Simulation of the Poisson Process and the Poisson part.
Regarding the Poisson process, we simulate first the jump times $\{T_k, k\ge 0\}$ of $\{N_t, 0\le t\le T\}$ with intensity λ by $\{T_{N_{t_j}}(i), i=1,2,\cdots L, j=0,1,\cdots M\}$. We are using the following properties of the Poisson process:

$$T_{N_{t_0}}(i)=0,\quad T_{N_{t_j}}(i)=T_{N_{t_{j-1}}}(i)+ExpLaw(\lambda\Delta t), i=1,2,\cdots L, j=0,1,\cdots M$$

where $ExpLaw(\lambda)$ is an exponential random variable which can be written as $ExpLaw(\lambda\Delta t)=\frac{-1}{\lambda\Delta t}\log(U)$, and U is a uniform random variable. A trajectory of the Poisson process $N_{t_j}(i), i=1,2,\cdots L,\ \ j=0,1,\cdots M$ is then determined by using:

$$N_{t_0}=1, N_{t_j}(i)=\sum_{k=0}^{j} I_{\{T_k(i)\le t_j\}}, i=1,2,\cdots L, j=0,1,\cdots M\ .$$

Since, from the last equation, and recalling the properties of the fractional Brownian motion, we get

$$S_{t_j}(i)-S_{t_{j-1}}(i)=\Delta t\cdot r_{t_{j-1}}S_{t_{j-1}}(i)+\sigma(t_{j-1},S_{t_{j-1}}(i))a_{t_{j-1}}\Delta t^H Z_j(i)$$
$$+\sigma(t_{j-1},S_{t_{j-1}}(i))b_{t_{j-1}}(N_{t_j}(i)-N_{t_{j-1}}(i)).$$

3.2. Monte Carlo solution for the premium

We have from the previous subsections L realizations for S_T, so we can apply the Monte Carlo method to compute the premium numerically using

$$\exp\{-\int_0^T r_s ds\}\cdot\frac{1}{L}\sum_{i=1}^{i=L}\max(S_T(i)-K,0)$$

To reduce the computational time we reduce the variance by using the antithetic variable method. This technique consists of computing two values of the premium C. The first value C_1 is calculated as described above and the second value C_2 is calculated similarly as C_1 with changing the sign of all the random samples from the standard normal distribution. Then C is obtained by taking the average of C_1 and C_2.

The standard error of the estimate premium is then $\frac{s_C}{L}$, where s_C is the standard deviation of the estimate premium and L is the number of trials. A 95% confidence interval for the premium is

$$\mu_C-\frac{1.96s_C}{\sqrt{L}}<C<\mu_C+\frac{1.96s_C}{\sqrt{L}}$$

where μ_C is the mean of the estimated premium.

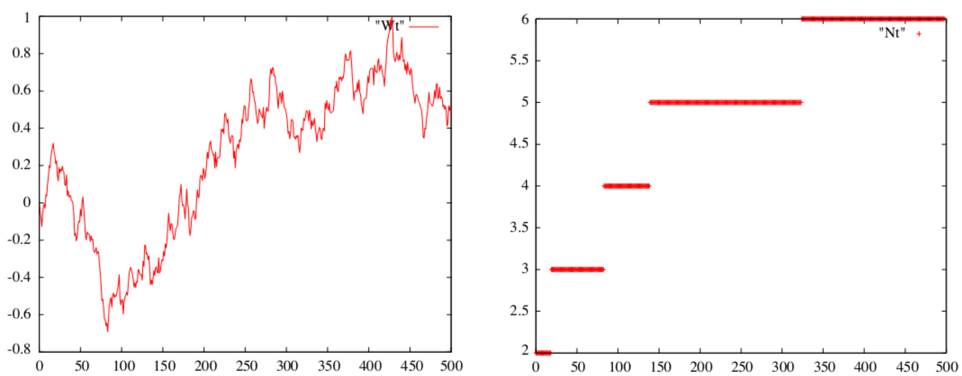

Figure 1. Fractional Brownian motion and Poisson process

Now, we present the numerical results of the premium by the Monte Carlo simulation when

$$T = 1,\ \lambda_t = 0.01t + 3;\ r_t = 0.01(2 + 8\sin(\pi t) + \cos(\frac{\pi t}{2})),$$

$a_t = 0.1, b_t = 0.3, \sigma_t = 0.1 + \sqrt{t}$, $S_0 = 7$ and $K = 7.5$. Notice that, the parameters T and λ are used to simulate trajectories for the Brownian motion and for the Poisson process with number of realizations $H = 500$ (see Fig. 1). Then, we simulate trajectories for the stock price at

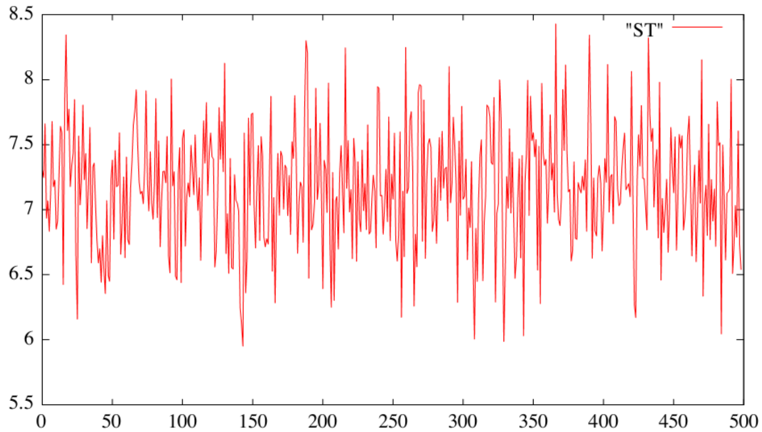

Figure 2. Realizations of the asset price for $H = 0.53, T = 1, \lambda_t = 0.01t + 3$, $r_t = 0.1 + \sin(\frac{\pi t}{2})$, $a_t = 0.25$, $b_t = 0.3$, $\sigma_t = \sqrt{t} + 0.01$ and $S_0 = 7$

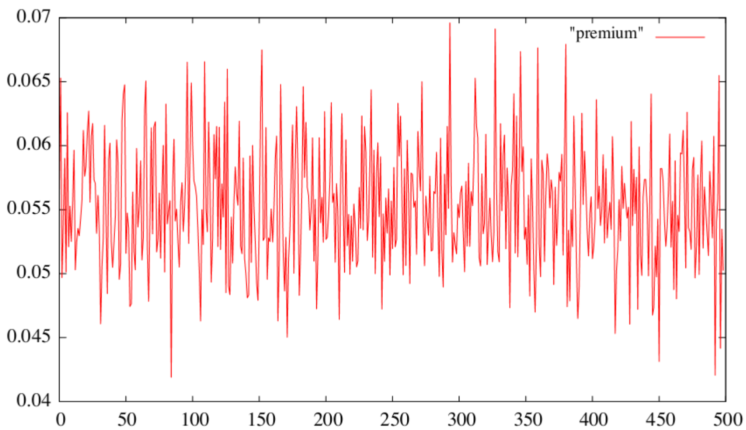

Figure 3. Realizations of the asset premium for d $K = 7.5$

terminal time $T = 1$ with $H = 500$ (see Fig. 2) and for the premium with number of realizations $L = 500$ (see Fig. 3). It is found that, the standard error of the estimate premium is 0.237×10^{-3}. A 95% confidence interval for the premium is therefore given by

$$5.469\times10^{-2} < C < 5.562\times10^{-2}.$$

We also provide the premium as a function of the stock price at $t=0$ for two different values of the strike K = 7.5 and K = 9 with number of realizations $L=500$, see Fig. 4.

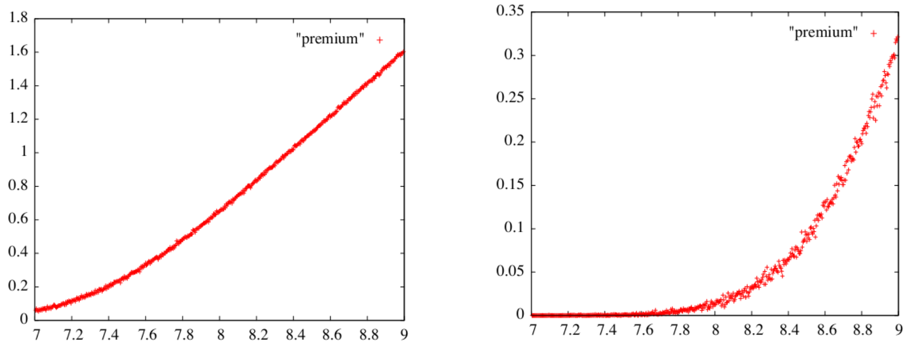

Figure 4. the premium as a function of the stock price at time $t=0$ for $K=7.5$ (left) and $K=9$ (right)

We now obtain the results by using Monte Carlo method to deal with the linear model. For all remain calculations we used the following parameters: $S_0=100$ $K=100$, $T=1$, $a=1$, $b=0$, $H=0.5$.

In this linear case, the call option pricing formula can be hold by the following closed form. We compare our method with the real value given by the closed form.

	$L=10^4$	$L=10^5$	$L=10^6$	$L=10^7$	Real value
$r=0.1, \sigma=0.1$	10.2100	10.2816	10.3165	10.3052	10.3082
$r=0.1, \sigma=0.15$	11.5588	11.6360	11.6806	11.6657	11.6691
$r=0.1, \sigma=0.2$	13.1411	13.2332	13.2837	13.2659	13.2697
$r=0.1, \sigma=0.25$	14.8259	14.9357	14.9921	14.9715	14.9758
$r=0.1, \sigma=0.3$	16.5691	16.6909	16.7528	16.7293	16.7341
$r=0.02, \sigma=0.2$	8.8163	8.8872	8.9259	8.9132	8.9160
$r=0.06, \sigma=0.2$	10.8734	10.9582	11.0014	10.9863	10.9895
$r=0.1, \sigma=0.2$	13.1411	13.2332	13.2837	13.2659	13.2697
$r=0.14, \sigma=0.2$	15.5750	15.6777	15.7367	15.7165	15.7211
$r=0.18, \sigma=0.2$	18.1309	18.2561	18.3200	18.2978	18.3031

Table1. Numerical values by Mont Carlo method

We see that in the linear case the Mont Carlo method yield a very accurate result (see Table 1).

4. Conclusion

European option pricing in fractional jump diffusion markets

In this paper, a fractional jump diffusion model is considered for option pricing. The pricing problem for such a model does not have a closed formula since the market is incomplete. However, since it imitates financial crashes, it is a more realistic approach. The price of a European option is simulated numerically by using the Monte Carlo method.

REFERENCES

1. P. Wilmott, S. Howison, J. Dewynne, The Mathematics of Financial Derivatives, Cambridge University Press, New York, 1995.
2. F. Black, M. Scholes, The pricing of options and corporate liabilities, Polit. Econ. 81 (3) (1973) 637–654.
3. R.C. Merton, Theory of rational option pricing, Bell Economics, 4 (1) (1973) 141–183.
4. R. Seydel, Tools for Computational Finance, 2nd edition, Springer, Berlin, 2004.
5. D. Tavella, C. Randall, Pricing Financial Instruments -The Finite Difference Method, New York, 2000.
6. H.E. Leland, Option pricing and replication with transactions costs, Finance, 40 (1985) 1283–1301.
7. P. Boyle, T. Vorst, Option replication in discrete time with transaction costs, Finance, 47 (1) (1992) 271–293.
8. G. Barles, H.M. Soner, Option pricing with transaction costs and a nonlinear Black–Scholes equation Stochastic Finance, 2 (1998) 369–397.
9. R. Frey, P. Patie, Risk Management for Derivatives in Illiquid Markets: A Simulation Study, Springer, Berlin, 2002.
10. R. Frey, A. Stremme, Market volatility and feedback effects from dynamic hedging, Mathematics Finance, 4 (1997) 351–374.
11. P. Sch¨ onbucher, P. Wilmott, The feedback–effect of hedging in illiquid markets, SIAM Applied Mathematics,61 (2000) 232–272.
12. H.M. Soner, N. Touzi, Super replication under gamma constraints, SIAM Control Optimal,39 (1) (2001) 73–96.
13. Cont R, Tankov P. Financial Modelling with Jump Processes, Chapman and Hall, London, 2004.
14. Schoutens W. Levy Processes in Finance: Pricing Financial Derivatives, Wiley, New York, 2003.
15. Ball C, Torous W. On jumps in common stock prices and their impact on call option pricing, J. Finance, 40 (1995) 155-173.
16. Corcuera J M, Nualart D, Schoutens W. Completion of a levy market by power-jump assets, Finance Stochast., 9 (2005) 109–127.
17. Luciano E, Schoutens W. A multivariate jump-driven financial asset model, Quantitative Finance, 6(5) (2006) 385-402.

18. Elliott R J, Hoek J. A General Fractional White Noise Theory and Applications to Finance, Mathematical Finance, 13 (2) (2003) 301-330.
19. Hu Y, Øksendal B. Fractional White Noise Calculus and Applications to Finance, Infinite Dimensional Analysis, Quantum Probability and Related Topics, 6 (1) (2003) 1-32 .
20. Benth F E. On arbitrage-free pricing of weather derivatives based on fractional Brownian motion, Applied Mathematical Finance, 10 (4) (2003) 303-324.
21. Bjork T, Hult H. A note on Wick products and the fractional Black-Scholes model, Finance and Stochastics, 9(2) (2005) 197-209.
22. Guasoni P. No arbitrage under transaction costs, with fractional Brownian motion and beyond, Mathematical Finance, 16(3) (2006) 569-582.
23. Harrison J.M, Kreps D.M. martingales and stochastic integrals in mutiperiod securities markets, Economics theory, 20(3) (1979) 381-408.