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Fundamental Group of Generalized Sum of Spaces

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ABSTRACT

Firstly, generalized sum and generalized connected sum of topological spaces have been defined and then some of the important properties of these generalized sums are studied in this paper. In a special case when the generalized sum $X_1 + \cdots + X_n$ is arcwise connected, the fundamental group $\pi(X_1 + \cdots + X_n)$ has been determined in that situation.

Keywords. Generalized sum, Generalized connected sum, compatible spaces, connectedness, fundamental group, free product.

1. Introduction

Sum is an important and a particular tool to construct new spaces. Properties of spaces and surfaces obtained by making a number of sums and connected sums are studied by many mathematicians. Recently, a very few of types of particular sums and connected sums of spaces have been studied in [1] and [2]. A sum of two topological spaces has been defined and studied also by S. Majumdar and M. Assaduzzaman in [3]. In this paper, the definition of sum for topological spaces is to be generalized and then some properties of this generalized sum are studied. Finally, the fundamental group of the constructed generalized sum is discussed also in a particular case.

2. Necessary definitions

For any two topological spaces (X, T_1) and (Y, T_2), their sum $X+Y$ is the space ($X \cup Y = X+Y$, T), where T is the topology generated by $T_1 \cup T_2$. Generalizing this definition, one can define that the sum $X_1 + \cdots + X_n$, called the *generalized sum*, is the topological space $(X_1 \cup \cdots \cup X_n, T)$ where, (X_1, T_1), (X_2, T_2), … , (X_n, T_n) are topological spaces and the topology T is generated by $T_1 \cup \cdots\cdots \cup T_n$. In other words, $X = X_1 + \cdots\cdots + X_n = X_1 \cup \cdots\cdots \cup X_n$

$=\{U_1 \cup \cdots\cdots \cup U_n \big| U_i \in T_i\}$. If all the X_i's are open subspaces of *X*, *X* is called *usual extension* of the X_i's. A usual extension exists if (i) $X_1 \cap \cdots\cdots \cap X_n$ is open in each X_i's, and (ii) the class of all intersections of X_i's with the open sets in X_j's is identical with the class of intersections of X_j's with the open sets in X_i's for any pair of distinct indices *i* and *j*. Topological spaces (X_1, T_1), (X_2, T_2), ... , (X_n, T_n) are called *compatible for sum* if they completely satisfy (i) and (ii). Similarly, the connected sum in [2] can be generalized and this *generalized connected sum* can be denoted by $X_1 \# \cdots \# X_n$.

A topological space *X* is said to be *locally compact at a point x* if there is some open set *U* containing x whose closure $\overline{U}$ is compact. The space is *locally compact* if it is locally compact at each of its points. A space *X* is said to be *locally connected at a point x* if for every open set *U* containing *x* there is a connected open set *V* containing *x* and contained in *U*. The space is *locally connected* if it is locally connected at each of its points. For two points *x*, *y* in a topological space *X*, a path joining *x* and *y* is a continuous map $f:[0,1] \to X$ such that $f(0)=x$, $f(1)=y$. The space X is *path connected* if any two points of *X* can be joined by a path.

Let *X* be a topological spaces and $x_0 \in X$. A loop at x_0 is a continuous map $f: I \to X$ such that $f(0)=f(1)=x_0$. Two loops *f* and g at x_0 are said to be *homotopic*, written $f \cong g$, if there exists a continuous map $F: I \times I \to X$ such that $F((0,t))=f(t)$ and $F((1,t))=g(t)$ for each $t \in I$. Then $\cong$ is an equivalence relation on the set of loops at x_0. We denote the equivalence class of f by [f]. A multiplication is defined on these equivalence classes by [*g*][*f*]=[g$*$f], where $(g*f)(t)=2t$ when $0 \le t \le \frac{1}{2}$ and $(g*f)(t)=1-2t$ when $\frac{1}{2} \le t \le 1$. This multiplication is well defined and is associative. If 1 denotes the constant map $1: I \to X$ with $1(t)=x_0$, for each $t \in I$, then [1][*f*]=[*f*][1]=[*f*], and if $g: I \to X$ is the loop given by g(*t*)= f(1-*t*), then [*f*][g]=[g][*f*]=[1]. Thus the equivalence classes of the loops at x_0 form a group. This group is called the *fundamental group* of *X* with base point at x_0 and it is usually denoted by $\pi(X, x_0)$.

If X is path connected, then $\pi(X,x_0)\cong\pi(X,x_1)$ for every pair of points x_0 and x_1 in X. In this situation, we write $\pi(X)$ to denote the fundamental group of X without mentioning the base point. For example: If $X=S^1$ the unit circle, then $\pi(X)=$ Z, the set of integers.

Let $\{G_i : i\in I\}$ be a collection of groups, and assume there is given for each index i, a homomorphism φ_i of G_i into a fixed group G. We say that G is the free product of the groups G_i with respect to the homomorphisms φ_i if and only if the following condition holds:

For any group H and any homomorphisms $\varphi_i : G_i \to H, i\in I$, there is a group homomorphism $f : G\to H$ such that $\psi_i = f\varphi_i$ for each $i\in I$.

3. Some Properties of $X_1+\cdots+X_n$ and $X_1\#\cdots\# X_n$

Theorem 1. *Let X_1, ..., X_n be compact spaces, then the generalized sum $X_1+\cdots+X_n$ is compact.*

Proof. Let $\{G_\alpha\}$ be any open cover of $X_1+\cdots+X_n$. Then $\{G_\alpha\}$ is also an open cover of each X_i since each X_i can be considered as the subspace of $X_1+\cdots+X_n$ Since each X_i is compact, there exists a finite sub-cover $\{G_{\alpha_i}{}^i\}$ of each X_i. Then clearly $\bigcup_{i=1}^{n}\{G_{\alpha_j}{}^i\}$ is also an open finite sub-cover of $X_1+\cdots+X_n$. Hence the theorem.

Lemma 2. *If X is locally compact and Y, a closed subspace of X then Y is locally compact*.

Proof. Let $y\in Y$. Since $y\in X$ and X is locally compact, there exists an open set V in X such that $y\in V$ and $\overline{V}$ is compact in X. Then $V\cap Y$ is open in Y and $y\in V\cap Y$. Let $\{W_\alpha\}$ be an open cover of $\overline{(V\cap Y)}_Y$ in Y. Then, for all α, $W_\alpha = U_\alpha\cap Y$, for some U_α which is open in X. Thus $\{U_\alpha\}$ is an open cover of $\overline{(V\cap Y)}_Y$ in X. Since $\overline{(V\cap Y)}_X$ is a closed subset of $\overline{V}$ in X, $\overline{(V\cap Y)}_X$ is compact in X. Hence there exist $U_{\alpha_1}\cdots\cdots U_{\alpha_n}$ such that $\overline{(V\cap Y)}_Y\subseteq U_{\alpha_1}\cup\cdots\cdots\cup U_{\alpha_n}$. Hence $\overline{(V\cap Y)}_Y\subseteq W_{\alpha_1}\cup\cdots\cdots\cup W_{\alpha_n}$. Therefore Y is locally compact.

Applying this theorem directly to the generalized sum $X_1+\cdots+X_n$, we certainly get the following result:

Theorem 3. *If* $X_1 + \cdots + X_n$ *is locally compact, then each of the* X_i *'s is locally compact*.

Since each X_i is a subspace of $X_1 + \cdots + X_n$, so from a standard result it follows that:

Theorem 4. $X_1 + \cdots + X_n$ *is connected if and only if each* X_i $(i = 1, \cdots, n)$ *is connected provided that* $X_1 \cap \cdots \cap X_n \neq \Phi$.

Theorem 5. *If* X_i*'s are locally connected then the sum* $X_1 + \cdots + X_n$ *is locally connected.*

Proof. Let $z \in \sum X_i$. If W is an open set in $X_1 \oplus \cdots \oplus X_n$ with $z \in W$, then $W = U_1 \cup \cdots \cup U_n$ with each U_i open in the corresponding X_i respectively. If $z \in U$, there exists connected open set $U_i^{/}$ in X_i (i=1, 2, …, n) with $z \in U_i^{/}$. Since $U_i^{/}$'s are open in $X_1 \oplus \cdots \oplus X_n$, so the statement of the theorem holds.

Lemma 6. *If X is locally connected and R is an equivalence relation on X, then the quotient space* $\frac{X}{R}$ *is locally connected.*

Proof. Let $\pi : X \to \frac{X}{R}$ denote the mapping given by $\pi(x) = cls\, x$ .Then π is continuous, open and onto. Let $x \in X$, and let $\overline{U}$ be an open set in $\frac{X}{R}$ such that $cls\, x \in \overline{U}$. Then $\overline{U} = \pi(U)$, for some open set U in X such that $x \in U$. Since X is locally connected, there exists a connected open set U' in X such that $x \in U'$. Then, $\pi(U')$ is a connected open set in $\frac{X}{R}$ and $cls\, x \in \pi(U')$. Hence $\frac{X}{R}$ is locally connected.

Let X_i's be locally connected spaces. Since from the definition of connected sum in [2], it can be written $X_1 \# \cdots \# X_n = \frac{X_1 \cup \cdots \cup X_n}{R}$, where R is an equivalence relation defined as in the definition in [2]. Thus by the theorem 5 and lemma 6, it is clear that $X_1 \# \cdots \# X_n$ is locally connected. Hence we have the following results:

Theorem 7. (i) *If X_i's $(i = 1, \cdots, n)$ are locally connected, then $X_1 \# \cdots \# X_n$ is also locally connected.*

(ii) *If X_i's $(i = 1, \cdots, n)$ are path connected, then $X_1 \# \cdots \# X_n$ is path connected.*

4. Fundamental Group of $X_1 + \cdots + X_n$

Sometimes a group, named the fundamental group, is associated with a topological space. It is topological invariant and serves to classify spaces to some degree. We shall now study with the problem that how one can find the fundamental group of the generalized sum $X_1 + \cdots + X_n$ of n compatible topological spaces. To do this we recall the well known and famous theorem of H. Seifert and E. Van Kampen. This theorem asserts that, if U and V are both open sets of arcwise connected space X so that $X = U \cup V$, and $U \cap V$ is nonempty connected, then the fundamental group $\pi(X)$ of X is completely determined by the following ways:

For group homeomorphisms $\varphi_1 : \pi(U \cap V) \to \pi(U)$, $\varphi_2 : \pi(U \cap V) \to \pi(V)$; there must exist group homomorphisms $\psi_1 : \pi(U) \to \pi(X)$, $\psi_2 : \pi(V) \to \pi(X)$ and $\psi : \pi(U \cap V) \to \pi(X)$ such that $\psi_1\varphi_1 = \psi_2\varphi_2 = \psi$, where $\varphi_1, \varphi_2, \psi_1, \psi_2$ and ψ are induced by inclusion maps. In other words, $\pi(X)$ is the possible freest group so that the above conditions are satisfied. Now we suppose that the fundamental groups $\pi(X_i)$ of the arcwise connected spaces X_i(i = 1, 2, … , n) for which $X_1 \cap \cdots \cap X_n \neq \Phi$ are known then $\pi(X_1 + \cdots + X_n)$ can be found in the following sense:

Since $X_1, \cdots, X_n$ are arcwise connected open subspaces of $X_1 + \cdots + X_n$ such that $X_1 \cap \cdots \cap X_n$ is also non-empty connected and $X_1 + \cdots + X_n = X_1 \cup \cdots \cup X_n$, so by Seifert-Van Kampen theorem $\pi(X_1 + \cdots + X_n)$ is the possible freest group so that the following are true:

For group homomorphisms $\varphi_i : \pi(X_1 \cap \cdots \cap X_n) \to \pi(X_i),\ i = 1, \cdots, n$; there must exist group homomorphisms $\psi_i : \pi(X_i) \to \pi(X_1 + \cdots + X_n)$, and $\psi : \pi(X_1 \cap \cdots\cdots \cap X_n) \to \pi(X_1 + \cdots + X_n)$ such that $\psi_i\varphi_i = \psi$, where φ_i, ψ_i and ψ are induced by inclusion maps.

Therefore the Seifert-Van Kampen theorem for the sum $X_1 + \cdots + X_n$ can be stated more precisely in the following form:

Theorem 8. *Let $X_1, \cdots, X_n$ be path connected compatible spaces such that $X_1 \cap \cdots \cap X_n \neq \Phi$ and are path connected also. If H be a group and ρ, ρ_i $(i = 1, \cdots, n)$ are any homomorphisms such that the diagram*

$$
\begin{tikzcd}
 & \pi(X_1) \arrow[ddr, "\rho_1"] & \\
 & \pi(X_2) \arrow[dr, "\rho_2"] & \\
\pi(\cap X_i) \arrow[uur, "\varphi_1"] \arrow[ur, "\varphi_2"] \arrow[rr, "\boxed{\rho}"] \arrow[dr, "\varphi_n"'] & & H \\
 & \pi(X_n) \arrow[ur, "\rho_n"'] &
\end{tikzcd}
$$

is commutative where the homomorphisms φ_i are induced by the inclusion maps. Then there exists a unique homomorphism $\sigma : \pi(X_1 + \cdots + X_n) \to H$ such that the following three diagrams are commutative:

$$
\begin{tikzcd}
 & \pi(\sum X_i) \arrow[d, "\sigma"] \\
\pi(X_i) \arrow[ur, "\psi_i"] \arrow[r, "\rho_i"'] & H
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \pi(\sum X_i) \arrow[d, "\sigma"] \\
\pi(\cap X_i) \arrow[ur, "\psi"] \arrow[r, "\rho"'] & H
\end{tikzcd}
$$

and

where ψ, ψ_i are also homomorphisms induced by inclusion maps.

Thus the fundamental group $\pi(X_1 + \cdots + X_n)$ of the generalized sum $X_1 + \cdots + X_n$ is determined through the last theorem. From the expression of the theorem shown by the diagrams, it can be stated that the group $\pi(X_1 + \cdots + X_n)$ is generated by the union of the images $\psi_\lambda[\pi(X_\lambda)]$, where $\lambda = 1, 2, \ldots, n$. The above result yields us immediately the following corollary.

Corollary. If $\pi(X_i \cap \cdots \cap X_n) = 1$ i.e., if $X_i \cap \cdots \cap X_n$ is simply connected, then $\pi(X_i \cap \cdots \cap X_n)$ is the free product of the $\pi(X_i)$'s under the homomorphisms ψ_i's i.e., $\pi(X_i + \cdots + X_n) = \pi(X_1) * \cdots * \pi(X_n)$, where * denotes the free product.

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