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On Fourth Order More Critically Damped Nonlinear Differential Systems

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ABSTRACT

A technique is developed for obtaining the transient response of fourth order more critically damped nonlinear systems. The results obtained by the presented technique agree with the numerical results obtained by the fourth order Runge-Kutta method nicely. An example is solved to illustrate the method.

AMS Subject Classification 34E05

Keywords: Perturbation, Transient response, More critically damped Systems.

1. Introduction

The control of micro vibration has become a growing research field due to the demand of high-performance systems and the advent of micro and nanotechnology in various scientific and industrial fields, such as semiconductor manufacturing, biomedical engineering, aerospace-equipments, and high-precision measurements. In micro and nanotechnology a small vibration is an important factor, as, due to a small vibration the produced equipment may be defective. So, in micro and nano-technological industries, vibration is not desirable. But vibration is unavoidable. It may arise in different way, such as, earth quake, direct disturbance etc. So, vibration control in micro and nano-technological industries is very essential. In micro and nano-technological industries we keep watch that vibrations come to its equilibrium position in minimum time. The more critically damped systems come to equilibrium position in minimum time. So, more critically damped systems play an important role in micro and nano-technological industries.

To investigate the transient behavior of vibrating systems the Krylov-Bogoliubov-Mitropolskii (KBM) [4, 5] method is an extensively used method. Originally, the method was developed for obtaining the periodic solutions of second order nonlinear differential systems with small nonlinearities. Later, the method extended by Popov [9] to investigate the solutions of nonlinear systems in presence of strong linear damping effects. Owing to physical importance Popov's results were rediscovered by Mendelson [6]. Murty *et al*. [7] developed a technique based on the method of Bogoliubov's to obtain the transient response of second and fourth order over-damped nonlinear systems. Later, Murty [8] presented a unified KBM method for second order nonlinear systems which covers the undamped, damped and over-damped cases. Sattar [12] has found an asymptotic solution of a second order critically damped nonlinear system. Shamsul [14] has developed a new asymptotic solution for both over-damped and critically damped nonlinear systems.

First, Shamsul and Sattar [13] developed a perturbation technique based on the work of KBM for obtaining the solution of third order critically damped nonlinear systems. Later, Shamsul [15] has investigated solutions of third order critically nonlinear systems whose unequal eigenvalues are in integral multiple. In article [15] Shamsul has also investigated solutions of third order more critically damped nonlinear systems. Shamsul [17] has also presented a perturbation technique for solving a third order over-damped system based on the KBM method when two roots of the linear equation are almost equal (rather than equal) and one root is small. Rokibul *et al.* [10] found a new technique for obtaining the solutions of third order critically damped nonlinear systems.

In article [7], Murty *et al*. also extended the KBM method for solving fourth order over-damped nonlinear systems. But their method is too much complex and laborious. Akbar *et al.* [1] presented an asymptotic method for fourth order over-damped nonlinear systems which is simple, systematic and easier than the method presented in [7], but the results obtained by [1] is same as the results obtained by [7]. Later, Akbar *et al.* [2] extended the method presented in [1] for fourth order damped oscillatory nonlinear systems. First, Rokibul *et al.* [11] extended the KBM method for obtaining the response of fourth order critically damped nonlinear systems. But none one of the above author's investigated solutions of fourth order more critically damped nonlinear systems.

In the present article we have developed a technique for obtaining the solutions of fourth order more critically damped nonlinear systems.

2.The method

Consider a fourth order weakly nonlinear ordinary differential system

$$x^{(4)} + p_1\dddot{x} + p_2\ddot{x} + p_3\dot{x} + p_4x = -\varepsilon f(x,\dot{x},\ddot{x},\dddot{x}) \qquad (1)$$

where $x^{(4)}$ denote the fourth derivative and over dots denote first, second and third derivative of x with respect to *t*; p_1, p_2, p_3, p_4 are constants, ε is the small parameter and $f(x,\dot{x},\ddot{x},\dddot{x})$ is the given nonlinear function. As the equation is fourth order so there are four real negative eigenvalues, and three of the eigenvalues are equal (for more critically damped). Suppose the eigenvalues are $-\lambda,-\lambda,-\lambda,-\mu$. When $\varepsilon = 0$, the equation (1) becomes linear and the solution of the corresponding linear equation is

$$x(t,0)=(a_0+b_0\,t+c_0\,t^2)e^{-\lambda t}+d_0\,e^{-\mu t} \tag{2}$$

where a_0, b_0, c_0, d_0 are constants of integration.

When $\varepsilon \neq 0$, following [16] an asymptotic solution of the equation (1) is sought in the form

$$x(t,\varepsilon)=(a+b\,t+c\,t^2)e^{-\lambda t}+d\,e^{-\mu t}+\varepsilon\,u_1(a,b,c,\,d,\,t)+\cdots \tag{3}$$

where a, b, c, d the functions of t and satisfy the first order differential equations

$$\begin{aligned}\dot{a}(t)&=\varepsilon\;A_1(a,b,c,d,t)+\cdots\\ \dot{b}(t)&=\varepsilon\;B_1(a,b,c,d,t)+\cdots\\ \dot{c}(t)&=\varepsilon\;C_1(a,b,c,d,t)+\cdots\\ \dot{d}(t)&=\varepsilon\;D_1(a,b,c,d,t)+\cdots\end{aligned} \tag{4}$$

We only consider first few terms in the series expansion of (3) and (4), we evaluate the functions u_i and A_i, B_i, C_i, D_i, $i=1,2,\cdots,n$ such that a, b, c and d appearing in (3) and (4) satisfy the given differential equation (1) with an accuracy of order ε^{n+1}. In order to determine these unknown functions it is customary in the KBM method that the correction terms, u_i, $i=1,2,\cdots,n$ must exclude terms (known as secular terms) which make them large. Theoretically, the solution can be obtained up to the accuracy of any order of approximation. However, owing to the rapidly growing algebraic complexity for the derivation of the formulae, the solution is in general confined to a lower order, usually the first [8].

Now differentiating the equation (3) four times with respect to *t*, substituting the value of x and the derivatives $\dot{x}, \ddot{x}, \dddot{x}, x^{(4)}$ in the original equation (1), utilizing the relations presented in (4) and finally equating the coefficients of ε, we obtain

$$\begin{aligned}&e^{-\lambda t}\left(\frac{\partial}{\partial t}+\mu-\lambda\right)\left(\frac{\partial^2 A_1}{\partial t^2}+3\frac{\partial B_1}{\partial t}+6\,C_1+t\left(\frac{\partial^2 C_1}{\partial t^2}+6\frac{\partial C_1}{\partial t}\right)+t^2\;\frac{\partial^2 C_1}{\partial t^2}\right)\\ &+e^{-\mu t}\left(\frac{\partial}{\partial t}+\lambda-\mu\right)^3 D_1+\left(\frac{\partial}{\partial t}+\lambda\right)^3\left(\frac{\partial}{\partial t}+\mu\right)u_1=-f^{(0)}(a,b,c,d,t)\end{aligned} \tag{5}$$

where $f^{(0)}(a,b,c,d,t)=f(x_0,\dot{x}_0,\ddot{x}_0,\dddot{x}_0)$ and $x_0=(a+b\,t+c\;t^2)e^{-\lambda t}+d\,e^{-\mu t}$.

Now, we expand the functional $f^{(0)}$ in the Taylor's series of the form (see also [12-15] for details)

$$f^{(0)} = (bt + ct^2)^0 \sum_{i,j=0}^{\infty} F_0(a,d)\, e^{-(i\lambda + j\mu)t} + (bt + ct^2)^1 \sum_{i,j=0}^{\infty} F_1(a,d)\, e^{-(i\lambda + j\mu)t}$$
$$+ (bt + ct^2)^2 \sum_{i,j=0}^{\infty} F_2(a,d)\, e^{-(i\lambda + j\mu)t} + (bt + ct^2)^3 \sum_{i,j=0}^{\infty} F_3(a,d)\, e^{-(i\lambda + j\mu)t} + \cdots \tag{6}$$

Thus, using (6), the equation (5) becomes

$$e^{-\lambda t}\left(\frac{\partial}{\partial t} + \mu - \lambda\right)\left\{\frac{\partial^2 A_1}{\partial t^2} + 3\frac{\partial B_1}{\partial t} + 6C_1 + t\left(\frac{\partial^2 B_1}{\partial t^2} + 6\frac{\partial C_1}{\partial t}\right) + t^2\ \frac{\partial^2 C_1}{\partial t^2}\right\}$$
$$+ e^{-\mu t}\left(\frac{\partial}{\partial t} + \lambda - \mu\right)^3 D_1 + \left(\frac{\partial}{\partial t} + \lambda\right)^3\left(\frac{\partial}{\partial t} + \mu\right) u_1 = -\left\{\sum_{i,j=0}^{\infty} F_0(a,d)\, e^{-(i\lambda + j\mu)t}\right.$$
$$+ (bt + ct^2)^1 \sum_{i,j=0}^{\infty} F_1(a,d)\, e^{-(i\lambda + j\mu)t} + (bt + ct^2)^2 \sum_{i,j=0}^{\infty} F_2(a,d)\, e^{-(i\lambda + j\mu)t}$$
$$\left. + (bt + ct^2)^3 \sum_{i,j=0}^{\infty} F_3(a,d)\, e^{-(i\lambda + j\mu)t} + \cdots \right\} \tag{7}$$

KBM [4, 5], Murty *et al.* [7], Sattar [12], Shamsul and Sattar [13], Shamsul [15, 17] imposed the condition that u_1 can not contain the fundamental terms (the solution (2) is called generating solution of (1) and its terms are called fundamental terms) of $f^{(0)}$. Therefore, equation (7) can be separated for unknown functions u_1 and $A_1, B_1, C_1 \quad D_1$ in the following way:

$$e^{-\lambda t}\left(\frac{\partial}{\partial t} + \mu - \lambda\right)\left\{\frac{\partial^2 A_1}{\partial t^2} + 3\frac{\partial B_1}{\partial t} + 6C_1 + t\left(\frac{\partial^2 B_1}{\partial t^2} + 6\frac{\partial C_1}{\partial t}\right) + t^2\ \frac{\partial^2 C_1}{\partial t^2}\right\}$$
$$+ e^{-\mu t}\left(\frac{\partial}{\partial t} + \lambda - \mu\right)^3 D_1 = -\left\{\sum_{i,j=0}^{\infty} F_0(a,d)\, e^{-(i\lambda + j\mu)t} + (bt + ct^2)^1 \sum_{i,j=0}^{\infty} F_1(a,d)\, e^{-(i\lambda + j\mu)t}\right\} \tag{8}$$

And

$$\left(\frac{\partial}{\partial t} + \lambda\right)^3\left(\frac{\partial}{\partial t} + \mu\right) u_1 = -\left\{(bt + ct^2)^2 \sum_{i,j=0}^{\infty} F_2(a,d)\, e^{-(i\lambda + j\mu)t}\right.$$
$$\left. + (bt + ct^2)^3 \sum_{i,j=0}^{\infty} F_3(a,d)\, e^{-(i\lambda + j\mu)t} + \cdots \right\} \tag{9}$$

Now equating the coefficients of t^0, t^1 and t^2; from equation (8), we obtain

$$e^{-\lambda t}\left(\frac{\partial}{\partial t}+\mu-\lambda\right)\frac{\partial^2 C_1}{\partial t^2}=-c\sum_{i,j=0}^{\infty}F_1(a,d)\,e^{-(i\lambda+j\mu)t} \tag{10}$$

$$e^{-\lambda t}\left(\frac{\partial}{\partial t}+\mu-\lambda\right)\left(\frac{\partial^2 B_1}{\partial t^2}+6\frac{\partial C_1}{\partial t}\right)=-b\sum_{i,j=0}^{\infty}F_1(a,d)\,e^{-(i\lambda+j\mu)t} \tag{11}$$

And

$$e^{-\lambda t}\left(\frac{\partial}{\partial t}+\mu-\lambda\right)\left(\frac{\partial^2 A_1}{\partial t^2}+3\frac{\partial B_1}{\partial t}+6C_1\right)$$

$$+e^{-\mu t}\left(\frac{\partial}{\partial t}+\lambda-\mu\right)^3 D_1=-\sum_{i,j=0}^{\infty}F_0(a,d)\,e^{-(i\lambda+j\mu)t} \tag{12}$$

Solving the equation (10), we obtain

$$C_1=\sum_{i,j=0}^{\infty}\frac{c\,F_1(a,d)\,e^{-((i-1)\lambda+j\mu)t}}{(i\lambda+(j-1)\mu)((i-1)\lambda+j\mu)^2} \tag{13}$$

Substituting the value of C_1 from (13) into equation (11) and solving, we obtain

$$B_1=-6\sum_{i,j=0}^{\infty}\frac{c\,F_1(a,d)\,e^{-((i-1)\lambda+j\mu)t}}{((i-1)\lambda+j\mu)^3\,(i\lambda+(j-1)\mu)}-\sum_{i,j=0}^{\infty}\frac{b\,F_1(a,d)\,e^{-((i-1)\lambda+j\mu)t}}{((i-1)\lambda+j\mu)^2\,(i\lambda+(j-1)\mu)} \tag{14}$$

Now substituting the value of C_1 from (13) and B_1 from (14) into equation (12), we obtain

$$e^{-\lambda t}\left(\frac{\partial}{\partial t}+\mu-\lambda\right)\frac{\partial^2 A_1}{\partial t^2}+e^{-\mu t}\left(\frac{\partial}{\partial t}+\lambda-\mu\right)^3 D_1$$

$$=-12\sum_{i,j=0}^{\infty}\frac{c\,F_1(a,d)\,e^{-(i\lambda+j\mu)t}}{((i-1)\lambda+j\mu)^2}-3\sum_{i,j=0}^{\infty}\frac{b\,F_1(a,d)\,e^{-(i\lambda+j\mu)t}}{((i-1)\lambda+j\mu)}-\sum_{i,j=0}^{\infty}F_0(a,d)\,e^{-(i\lambda+j\mu)t} \tag{15}$$

Now, we have only one equation (15) for obtaining the unknown functions A_1 and D_1. Therefore, to separate the equation (15) for obtaining the unknown functions A_1 and D_1, we need to impose some restrictions and thus the value of A_1

and D_1 can be found subject to the condition that the coefficients in the solution of A_1 and D_1 do not become large (see also [3, 15] for details). This completes the determination of A_1, B_1, C_1 and D_1.

Since $\dot{a}, \dot{b}, \dot{c}, \dot{d}$ are proportional to small parameter ε, so they are slowly varying functions of time *t* and as a first approximation, we may consider them as constants in the right hand side. This assumption was first made by Murty *et al*. [7]. Thus the solutions of the equation (4) become

$$a = a_0 + \varepsilon \int_0^t A_1(a_0, b_0, c_0, d_0, t)\, dt$$

$$b = b_0 + \varepsilon \int_0^t B_1(a_0, b_0, c_0, d_0, t)\, dt$$

$$c = c_0 + \varepsilon \int_0^t C_1(a_0, b_0, c_0, d_0, t)\, dt$$

$$d = d_0 + \varepsilon \int_0^t D_1(a_0, b_0, c_0, d_0, t)\, dt \tag{16}$$

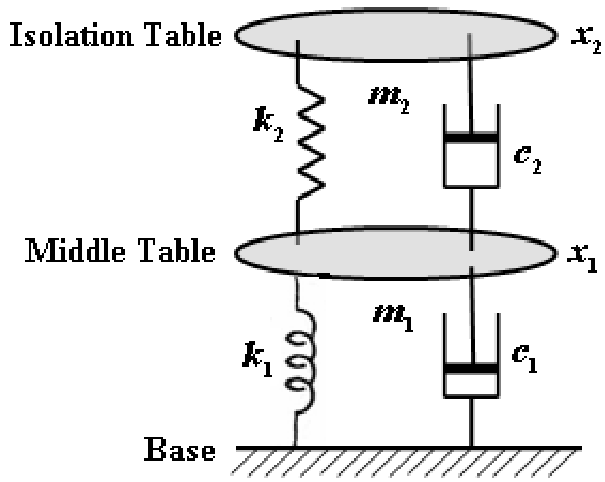
Equation (9) is an inhomogeneous linear ordinary differential equation; therefore it can be solved by the well-known operator method.

Substituting the value of a, b, c, d and u_1 in the equation (3), we shall get the complete solution of (1).

Therefore, the determination of the first order improved solution is completed.

2.1 Example

The figure of the isolation (vibration free) table which is extensively used in the semiconductor manufacturing, biomedical engineering, aerospace-equipments, and high-precision measurements is given below.



Here m_2 and m_1 are the mass of the isolation and middle table respectively, k_2 and k_1 are spring constants, c_2 and c_1 are damping coefficients and x_2, x_1 are the displacement of the isolation and middle table respectively due to disturbances.

The governing equation of the isolation table is

$$m_1 \ddot{x}_1 + c_1 \dot{x}_1 + k_1 x_1 + c_2(\dot{x}_1 - \dot{x}_2) + k_2(x_1 - x_2) = -F_c$$

$$m_2 \ddot{x}_2 + c_2(\dot{x}_2 - \dot{x}_1) + k_2(x_2 - x_1) = F_c + F_d \tag{17}$$

Simplifying the equation (17), we obtain

$$x^{(4)} + p_1 \dddot{x} + p_2 \ddot{x} + p_3 \dot{x} + p_4 x = -\varepsilon x^3 \tag{18}$$

where $p_1 = \{c_2 m_1 + (c_1 + c_2) m_2\}/(m_1 m_2)$, $p_2 = \{k_2 m_1 + (k_1 + k_2) m_2 + c_1 c_2\}/(m_1 m_2)$, $p_3 = \{k_1 c_2 + k_2 c_1\}/(m_1 m_2)$, $p_4 = (k_1 k_2)/(m_1 m_2)$ and $\varepsilon =$

When $p_1 = p_1 p_2$, the three eigenvalues of the corresponding linear equation of (18) become equal. *i. e.* the system (18) becomes more critically damped.

Here $f = x^3$. Therefore,

$$\begin{aligned} f^{(0)} &= a^3 e^{-3\lambda t} + 3a^2 d\, e^{-(2\lambda+\mu)t} + 3a\, d^2\, e^{-(\lambda+2\mu)t} + d^3 e^{-3\mu t} \\ &+ (bt + c\, t^2)^1 \left(3a^2 e^{-3\lambda t} + 6a\, d\, e^{-(2\lambda+\mu)t} + 3d^2\, e^{-(\lambda+2\mu)t}\right) \\ &+ (bt + c\, t^2)^2 \left(3a e^{-3\lambda t} + 3d\, e^{-(2\lambda+\mu)t}\right) + (bt + ct^2)^3 e^{-3\lambda t} \end{aligned}$$

For equation (18), the equations (10)-(12) and equation (9) respectively become

$$e^{-\lambda t}\left(\frac{\partial}{\partial t}+\mu-\lambda\right)\frac{\partial^2 C_1}{\partial t^2}=-\left\{3a^2\,c\,e^{-3\lambda t}+6\,a\,c\,d\;e^{-(2\lambda+\mu)t}+3\,c\,d^2\;e^{-(\lambda+2\mu)t}\right\} \tag{19}$$

$$e^{-\lambda t}\left(\frac{\partial}{\partial t}+\mu-\lambda\right)\left(\frac{\partial^2 B_1}{\partial t^2}+6\frac{\partial C_1}{\partial t}\right)=-\left\{3a^2be^{-3\lambda t}+6a\,b\,d\;e^{-(2\lambda+\mu)t}+3b\,d^2\;e^{-(\lambda+2\mu)t}\right\} \tag{20}$$

$$e^{-\lambda t}\left(\frac{\partial}{\partial t}+\mu-\lambda\right)\left(\frac{\partial^2 A_1}{\partial t^2}+3\frac{\partial B_1}{\partial t}+6C_1\right)+e^{-\mu t}\left(\frac{\partial}{\partial t}+\lambda-\mu\right)^3 D_1$$
$$=-\left\{a^3e^{-3\lambda t}+3\,a^2d\;e^{-(2\lambda+\mu)t}+3a\,d^2\;e^{-(\lambda+2\mu)t}+d^3e^{-3\lambda_4 t}\right\} \tag{21}$$

And

$$\left(\frac{\partial}{\partial t}+\lambda\right)^3\left(\frac{\partial}{\partial t}+\mu\right)u_1=-\left\{b^3t^3e^{-3\lambda t}+6\,a\,b\,c\,t^3\,e^{-3\lambda t}+3b^2\,c\,t^4\,e^{-3\lambda t}\right.$$
$$+3\,a\,c^2\,t^4\,e^{-3\lambda t}+3\,b\,c^2\;t^5\,e^{-3\lambda t}+c^3\;t^6\,e^{-3\lambda t}+6\,b\,c\,d\,t^3\,e^{-(2\lambda+\mu)t}$$
$$\left.+3c^2\,d\,t^4\,e^{-(2\lambda+\mu)t}+3a\,b^2t^2e^{-3\lambda t}+3d\,b^2t^2\,e^{-(2\lambda+\mu)t}\right\} \tag{22}$$

The solution of the equation (19) is

$$C_1=l_1\;a^2\,c\,e^{-2\lambda t}+l_2\,a\,c\,d\;e^{-(\lambda+\mu)t}+l_3\,c\,d^2\,e^{-2\mu t} \tag{23}$$

where $l_1=(3PL^2)/4$, $l_2=(3Q^2L)/2$, $l_3=(3QM^2)/4$ and

$$P=\frac{1}{3\lambda-\mu},\qquad Q=\frac{1}{\lambda+\mu},\qquad L=\frac{1}{\lambda},\qquad M=\frac{1}{\mu}.$$

Putting the value of C_1 from the equation (22) in the equation (19), we obtain

$$B_1=m_1\;a^2\,c\,e^{-2\lambda t}+m_2\;\;a\,c\,d\,e^{-(\lambda+\mu)t}+m_3\,c\,d^2\,e^{-2\mu t}$$
$$+m_4\,a^2be^{-2\lambda t}+m_5a\,b\,d\;e^{-(\lambda+\mu)t}+m_6b\,d^2\;e^{-2\mu t} \tag{23}$$

where $m_1=9PL^3/4$, $m_2=18Q^3L$, $m_3=9QM^3/4$,
$m_4=3PL^2/4$,
$m_5=3Q^2L$, $m_6=3QM^2/4$.

Substituting the values of B_1 and C_1 into equation (21), we shall get an equation for unknown functions A_1 and D_1. To separate the equation (21) for determining the unknown functions A_1 and D_1, in this article we considered the relation $\lambda \approx 3\mu$ exists among the eigenvalues (see also [13, 15] for details). *i. e.* the unequal eigenvalue λ is the multiple of μ. This type of relation ($\lambda \approx 3\mu$) appears intuitively in the symmetric problems. Since our problem is symmetric, therefore consideration of such type of relation is logical. Therefore, under this relation, we obtain

$$\begin{aligned} e^{-\lambda t}\left(\frac{\partial}{\partial t}+\mu-\lambda\right)\frac{\partial^2 A_1}{\partial t^2} &= 6m_1\lambda(\mu-3\lambda)a^2 c\, e^{-3\lambda t} \\ &-12\lambda(\lambda+\mu)m_2\, a\,c\,d\, e^{-(2\lambda+\mu)t} - 6\mu(\lambda+\mu)m_3\, c\,d^2\, e^{-(\lambda+2\mu)t} \\ &+6\lambda(\mu-3\lambda)m_4\, a^2 b e^{-3\lambda t} - 6\lambda(\lambda+\mu)m_5 a\,b\,d\, e^{-(2\lambda+\mu)t} \\ &-6\mu(\lambda+\mu)m_6 b\,d^2\, e^{-(\lambda+2\mu)t} - 6(\mu-3\lambda)l_1 a^2\, c\, e^{-3\lambda t} \\ &+24\lambda l_2\, a\,c\,d\, e^{-(2\lambda+\mu)t} + 6(\lambda+\mu)l_3\, c\,d^2\, e^{-2\mu t} - a^3 e^{-3\lambda t} \\ &-3a^2 d\, e^{-(2\lambda+\mu)t} - 3a\,d^2\, e^{-(\lambda+2\mu)t} \end{aligned} \tag{25}$$

$$e^{-\mu t}\left(\frac{\partial}{\partial t}+\lambda-\mu\right)^3 D_1 = -d^3 e^{-3\mu t} \tag{26}$$

The particular solutions of (25) and (26) respectively become

$$\begin{aligned} A_1 &= n_1\, a^2\, c e^{-2\lambda t} + n_2\, a\,c\,d\, e^{-(\lambda+\mu)t} + n_3\, c\,d^2\, e^{-2\mu t} + n_4\, a^2 b e^{-2\lambda t} \\ &+ n_5 a\,b\,d\, e^{-(\lambda+\mu)t} + n_6 b\,d^2\, e^{-2\mu t} + n_7\, a^2\, c\, e^{-2\lambda t} + n_8\, a\,c\,d\, e^{-(\lambda+\mu)t} \\ &+ n_9\, c\,d^2\, e^{-2\mu t} + n_{10} a^3 e^{-2\lambda t} + n_{11} a^2 d\, e^{-(\lambda+\mu)t} + n_{12}\, a\,d^2\, e^{-2\mu t} \end{aligned} \tag{27}$$

$$D_1 = p_1\, d^3 e^{-2\mu t} \tag{28}$$

Where $n_1 = 27PL^4/8$, $n_2 = 18Q^4 L$, $n_3 = 27QM^4/8$,
$n_4 = 9PL^3/8$ $n_5 = 9Q^3 L$, $n_6 = 9QM^3/8$, $n_7 = -9PL^4/8$,
$n_8 = -9Q^4 L$,
$n_9 = -9QM^4/8$, $n_{10} = -PL^2/4$, $n_{11} = 3Q^2 L/2$,

$$n_{12} = 3QM^2/4, \qquad p_1 = \frac{P^3\,Q^3}{(2P-Q)^3}.$$

The solution of the equation (22) for u_1 is

$$\begin{aligned}u_1 =& \left(r_1\,t^3 + r_2\,t^2 + r_3\,t + r_4\right)\left(b^3 + 6\,a\,b\,c\right)e^{-3\,\lambda\,t} + \left(r_5\,t^4 + r_6\,t^3 + r_7\,t^2 + r_8\,t + r_9\right)\\ &\times (b^2\,c + a\,c^2)\,e^{-3\lambda\,t} + \left(r_{10}\,t^5 + r_{11}\,t^4 + r_{12}\,t^3 + r_{13}\,t^2 + r_{14}\,t + r_{15}\right)b\,c^2\,e^{-3\lambda\,t}\\ &+ \left(r_{16}\,t^6 + r_{17}\,t^5 + r_{18}\,t^4 + r_{19}\,t^3 + r_{20}\,t^2 + r_{21}\,t + r_{22}\right)c^3\,e^{-3\lambda\,t}\\ &+ \left(r_{23}\,t^3 + r_{24}\,t^2 + r_{25}\,t + r_{26}\right)b\,c\,d\,e^{-(\mu+2\,\lambda)\,t}\\ &+ \left(r_{27}\,t^4 + r_{28}\,t^3 + r_{29}\,t^2 + r_{30}\,t + r_{31}\right)c^2\,d\,e^{-(\mu+2\,\lambda)\,t}\\ &+ \left(r_{32}\,t^2 + r_{33}\,t + r_{34}\right)ab^2\,e^{-3\lambda\,t} + \left(r_{35}\,t^2 + r_{36}\,t + r_{37}\right)b^2de^{-(\mu+2\,\lambda)\,t}\end{aligned} \tag{29}$$

where $r_1 = -P\,L^3/8$, $\quad r_2 = r_1\left(3P + 9L/2\right)$,

$r_3 = r_1\left(6P^2 + 9PL + 9L^2\right)$,

$r_4 = r_1\left(6P^3 + 9P^2L + 9PL^2 + 15L^3/2\right)$, $\qquad r_5 = -3PL^2/8$,

$r_6 = r_5\left(4P + 6L\right)$, $\qquad r_7 = r_5\left(12P^2 + 18PL + 18L^2\right)$,

$r_8 = r_5\left(24P^3 + 36P^2L + 36PL^2 + 30L^3\right)$,

$r_9 = r_5\left(24P^4 + 36P^3L + 36P^2L^2 + 30PL^3 + 45L^4)/2\right)$,

$r_{10} = -3PL^3/8$,

$r_{11} = r_{10}\left(5P + 15L/2\right)$, $\qquad r_{12} = r_{10}\left(20\,P^2 + 30PL + 30L^2\right)$,

$r_{13} = r_{10}\left(60P^3 + 90P^2L + 90PL^2 + 75L^3\right)$

$r_{14} = r_{10}\left(120P^4 + 180P^3L + 180P^2L^2 + 150PL^3 + 225L^4/2\right)$,

$r_{15} = r_{10}\left(120P^5 + 180\,P^4L + 180\,P^3L^2 + 150P^2L^3 + 150PL^4 + 315L^5/4\right)$

,

$r_{16} = -PL^3/8$, $\qquad r_{17} = r_{16}\left(6P + 9L/2\right)$,

$r_{18} = r_{16}\left(30P^2 + 45PL + 45L^2\right)$,

$r_{19} = r_{16}\left(120P^3 + 180\,P^2L + 180PL^2 + 150L^3\right)$,

$r_{20} = r_{16}\left(360P^4 + 540P^3L + 540P^2L^2 + 450PL^3 + 675L^4/2\right)$,

$r_{21} = r_{16}\left(720P^5 + 1080P^4L + 1080P^3L^2 + 900P^2L^3 + 675PL^4 + 945L^5/2\right)$

$$r_{22} = r_{16}\left(720P^6 + 1080\,P^5L + 1080P^4L^2 + 900\,P^3L^3 + 675\,P^2\,L^4 + 945P\,L^5/2 + 315\,L^5\right),$$

$$r_{23} = -3Q^3 L/2, \ r_{24} = r_{23}\left(3L/2 + 9Q\right),$$
$$r_{25} = r_{23}\left(3L^2/2 + 9QL + 9Q^2\right),$$
$$r_{26} = r_{23}\left(3L^3/4 + 9QL/2 + 9Q^2 L + 60Q^3\right), \qquad r_{27} = -3Q^3 L/2,$$
$$r_{28} = r_{27}\left(2L + 12Q\right), \qquad r_{29} = r_{27}\left(3L^2 + 18QL + 72Q^2\right),$$
$$r_{30} = r_{28}\left(3L^3 + 18QL^2 + 72Q^2 L + 240Q^3\right),$$
$$r_{31} = r_{28}\left(3L^4/2 + 9QL^3 + 36Q^2 L^2 + 120Q^3 L + 360Q^4\right),$$
$$r_{32} = -3PL^3/8,$$
$$r_{33} = r_{32}\left(3L + 2P\right), \qquad r_{34} = r_{32}\left(L^2 + 3PL + 2P^2\right),$$
$$r_{35} = -3Q^3 L/2,$$
$$r_{36} = r_{35}\left(6Q + L\right), \qquad r_{37} = r_{35}\left(12Q^2 + 3QL + L^2/2\right).$$

Substituting the values of A_1, B_1, C_1, D_1 from the equations (27), (24), (23) and (28) into equation (16), we obtain

$$\begin{aligned}
a = a_0 + \varepsilon \Bigg\{ & \frac{n_1 a^2 c\left(1 - e^{-2\lambda t}\right)}{2\lambda} + \frac{n_2 a c d\left(1 - e^{-(\lambda+\mu)t}\right)}{(\lambda+\mu)} + \frac{n_3 c d^2\left(1 - e^{-2\mu t}\right)}{2\mu} \\
& + \frac{n_4 a^2 b\left(1 - e^{-2\lambda t}\right)}{2\lambda} + \frac{n_5 a b d\, e^{-(\lambda+\mu)t}}{(\lambda+\mu)} + \frac{n_6 b d^2\left(1 - e^{-2\mu t}\right)}{2\mu} \\
& + \frac{n_7 a^2 c\left(1 - e^{-2\lambda t}\right)}{2\lambda} + \frac{n_8 a c d\left(1 - e^{-(\lambda+\mu)t}\right)}{(\lambda+\mu)} + \frac{n_9 c d^2\left(1 - e^{-2\mu t}\right)}{2\mu} \\
& + \frac{n_{10} a^3\left(1 - e^{-2\lambda t}\right)}{2\lambda} + \frac{n_{11} a^2 d\left(1 - e^{-(\lambda+\mu)t}\right)}{(\lambda+\mu)} + \frac{n_{12} a d^2\left(1 - e^{-2\mu t}\right)}{2\mu} \Bigg\}
\end{aligned}$$

$$\begin{aligned}
b = b_0 + \varepsilon \Bigg\{ & \frac{m_1 a^2 c\left(1 - e^{-2\lambda t}\right)}{2\lambda} + \frac{m_2 a c d\left(1 - e^{-(\lambda+\mu)t}\right)}{(\lambda+\mu)} + \frac{m_3 c d^2\left(1 - e^{-2\mu t}\right)}{2\mu} \\
& + \frac{m_4 a^2 b\left(1 - e^{-2\lambda t}\right)}{2\lambda} + \frac{m_5 a b d\left(1 - e^{-(\lambda+\mu)t}\right)}{(\lambda+\mu)} + \frac{m_6 b d^2\left(1 - e^{-2\mu t}\right)}{2\mu} \Bigg\}
\end{aligned} \tag{30}$$

$$c = c_0 + \varepsilon \left\{ \frac{l_1 a^2 c\left(1 - e^{-2\lambda t}\right)}{2\lambda} + \frac{l_2 a c d\left(1 - e^{-(\lambda+\mu)t}\right)}{(\lambda+\mu)} + \frac{l_3 c d^2\, e^{-2\mu t}}{2\mu} \right\}$$

$$d = d_0 + \varepsilon \frac{p_4 d_0^3 e^{-2\mu t}}{2\mu}$$

Therefore, we obtain the first approximate solution of the equation (17) as

$$x(t,\varepsilon) = (a + b\,t + c\,t^2) e^{-\lambda t} + d\, e^{-\mu t} + \varepsilon u_1 (a, b, c, d, t) \tag{31}$$

where $a\ b, c, d$ are given by the equation (30) and u_1 given by (29).

3. Result and Discussion

It is usual to compare the perturbation solution to the numerical solution to test the accuracy of the approximate solution. Let us consider $k_1 = 15.5,\ k_2 = 75.75,\ k_3 = 118.625, k_4 = 42.25$. Thus we have $\lambda = 3.1,\ \mu = 1.0$. We have computed $x(t,\varepsilon)$ by (31) in which a, b, c, d are computed by equation (30) and u_1 is computed by equation (29) when $\varepsilon = 0.1$ together with two sets of initial conditions $a_0 = 0.5,\ b_0 = 0.0,\ c_0 = 0.3,\ d_0 = 0.1$ [or $x(0) = 0.599982,\ \dot{x}(0) = -1.749542,\ \ddot{x} = 5.902058$ $\dddot{x}(0) = -21.860722$] and $a_0 = 0.4,\ b_0 = 0.0,\ c_0 = 0.4,\ d_0 = 0.1$ [or $x(0) = 0.499969,\ \dot{x}(0) = -1.439448,\ \ddot{x} = 5.140599,\ \dddot{x}(0) = -20.739441$] for various values of *t* and the results are presented in the Table **I** and **II** respectively. The corresponding numerical solution (designated by *x**) have been computed by a fourth order Runge-Kutta method. As we have truncated the series (3) from ε^2 in the solution (31), so errors should occur 1% when $\varepsilon = 0.1$. But from table **I** and **II**, we see that errors are smaller than 1%.

Table **I**

t	x	x^*	Errors%
0.0	0.599982	0.599982	0.00000
0.5	0.163559	0.163550	0.00550
1.0	0.056936	0.056904	0.05623
1.5	0.022951	0.022912	0.17021
2.0	0.010342	0.010310	0.31037
2.5	0.005155	0.005133	0.43711
3.0	0.002788	0.002774	0.50468
3.5	0.001592	0.001583	0.56854
4.0	0.000938	0.000932	0.64377
4.5	0.000561	0.000558	0.53763
5.0	0.000338	0.000336	0.59523

Initial values are $a_0 = 0.5,\ b_0 = 0.0,\ c_0 = 0.3,\ d_0 = 0.1$ and $\varepsilon = 0.1$

x Computed by (31) x^* is computed by *Runge-Kutta* method.

Table **II**

t	x	x^*	Errors%
0.0	0.499969	0.499969	0.00000
0.5	0.147644	0.147639	0.00338
1.0	0.056938	0.056921	0.02986
1.5	0.024147	0.024127	0.08289
2.0	0.010951	0.010935	0.14631
2.5	0.005381	0.005370	0.20484
3.0	0.002861	0.002854	0.24526
3.5	0.001614	0.001609	0.31075
4.0	0.000944	0.000941	0.31880
4.5	0.000563	0.000561	0.35650
5.0	0.000339	0.000338	0.29585

Initial values are $a_0 = 0.4,\ b_0 = 0.0,\ c_0 = 0.4,\ d_0 = 0.1$ and $\varepsilon = 0.1$
x Computed by (31)
x^* is computed by *Runge-Kutta* method

Again for $\lambda = 4.6,\ \mu = 1.5$, we have computed $x(t,\varepsilon)$ by (31) in which a, b, c, d are computed by equation (30) and u_1 is computed by equation (29) when $\varepsilon = 0.1$ together with another two sets of initial conditions $a_0 = 0.5$, $b_0 = 0.0$, $c_0 = 0.3$, $d_0 = 0.1$ [or $x(0) = 0.599999$, $\dot{x}(0) = -2.549924$, $\ddot{x} = 12.004210$, $\dddot{x}(0) = -60.204288$] and $a_0 = 0.4$, $b_0 = 0.0$, $c_0 = 0.4$, $d_0 = 0.1$ [or $x(0) = 0.499999$, $\dot{x}(0) = -2.089921$, $\ddot{x} = 10.088197$, $\dddot{x}(0) = -53.230522$] for various values of t and the results are presented in the Table **III** and **IV** respectively.

Table **III**

t	x	x^*	Errors%
0.0	0.599999	0.599999	0.00000
0.5	0.092656	0.092655	0.00107
1.0	0.023288	0.023280	0.03436
1.5	0.008250	0.008244	0.07278
2.0	0.003495	0.003492	0.08591
2.5	0.001592	0.001591	0.06285
3.0	0.000744	0.000743	0.13485
3.5	0.000350	0.000350	0.00000

4.0	0.000165	0.000165	0.00000
4.5	0.000078	0.000078	0.00000
5.0	0.000037	0.000037	0.00000

Initial values are $a_0 = 0.5,\quad b_0 = 0.0,\quad c_0 = 0.3,\quad d_0 = 0.1$ and $\varepsilon = 0.1$

x Computed by (31) x^* is computed by *Runge-Kutta* method.

Table **IV**

t	x	x^*	Errors%
0.0	0.499999	0.499999	0.00000
0.5	0.085137	0.085138	0.00117
1.0	0.023288	0.023284	0.01717
1.5	0.008376	0.008373	0.03582
2.0	0.003525	0.003524	0.02837
2.5	0.001598	0.001597	0.06261
3.0	0.000745	0.000744	0.13440
3.5	0.000350	0.000350	0.00000
4.0	0.000165	0.000165	0.00000
4.5	0.000078	0.000078	0.00000
5.0	0.000037	0.000037	0.00000

Initial values are $a_0 = 0.4,\quad b_0 = 0.0,\quad c_0 = 0.4,\quad d_0 = 0.1$ and $\varepsilon = 0.1$

x Computed by (31) x^* is computed by *Runge-Kutta* method.

If we do not change the ratio (the ratio is $\lambda : \mu \approx 3:1$ and by considering $\lambda = 3.1, \mu = 1$) but increase the difference (by considering $\lambda = 4.6, \mu = 1.5$) we see that the results become more near to the numerical results than the previous results.

4. Conclusion

In presence of strong linear damping forces, approximate solutions of a fourth order more critically-damped nonlinear system have been found base on the KBM method. The solutions obtained by this method show good coincidence with corresponding numerical values.

REFERENCES

1. Ali Akbar, M., A.C. Paul and M.A. Sattar, An Asymptotic Method of Krylov-Bogoliubov-Mitropolskii for Fourth Order Over-Damped Nonlinear Systems, Ganit: J. Bangladesh Math. Soc., Vol. **22**, PP. 83-96, 2002.
2. Ali Akbar, M., M. Shamsul Alam and M. A. Sattar, Asymptotic Method for Fourth Order Damped Nonlinear Systems, Ganit, J. Bangladesh Math. Soc., Vol. **23**, pp. 72-80, 2003.

3. Ali Akbar, M., M. Shamsul Alam and M. A. Sattar, A Simple Technique for Obtaining Certain Over-damped Solutions of an *n*-th Order Nonlinear Differential Equation, Soochow Journal of Mathematics Vol. **31(2)**, pp.291-299, 2005.
4. Bogoliubov, N. N. and Yu. Mitropolskii, Asymptotic Methods in the Theory of Nonlinear Oscillations, Gordan and Breach, New York, 1961.
5. Krylov, N. N. and N. N. Bogoliubov, Introduction to Nonlinear Mechanics, Princeton University Press, New Jersey, 1947.
6. Mendelson, K. S., Perturbation Theory for Damped Nonlinear Oscillations, J. Math. Physics, Vol. **2**, pp. 3413-3415, 1970.
7. Murty, I. S. N., B. L. Deekshatulu and G. Krishna, On an Asymptotic Method of Krylov-Bogoliubov for Over-damped Nonlinear Systems, J. Frank. Inst., Vol. **288**, pp. 49-65, 1969.
8. Murty, I. S. N., A Unified Krylov-Bogoliubov Method for Solving Second Order Nonlinear Systems, Int. J. Nonlinear Mech. Vol. **6**, pp. 45-53, 1971.
9. Popov, I. P., A Generalization of the Bogoliubov Asymptotic Method in the Theory of Nonlinear Oscillations (in Russian), Dokl. Akad. USSR Vol. **3**, pp. 308-310, 1956.
10. Rokibul Islam, M., M. Ali Akbar and M. Samsuzzoha, A New Technique for Third Order Critically Damped Non-linear Systems, J. Appl. Math. Sci. Res. Vol. 4(5), 2008.
11. Rokibul Islam, M., M. Sharif Uddin, M. Ali Akbar, M. Azmol Huda and S. H. S. Hossain, New Technique for Fourth Order Critically Damped Non-linear Systems, Calcutta Math. Soc., 2008.
12. Sattar, M. A., An asymptotic Method for Second Order Critically Damped Nonlinear Equations, J. Frank. Inst., Vol. **321**, pp. 109-113, 1986.
13. Shamsul Alam, M. and M. A. Sattar, An Asymptotic Method for Third Order Critically Damped Nonlinear Equations, J. Mathematical and Physical Sciences, Vol. **30**, pp. 291-298, 1996.
14. Shamsul Alam, M., Asymptotic Methods for Second Order Over-damped and Critically Damped Nonlinear Systems, Soochow Journal of Math. Vol. **27**, pp. 187-200, 2001.
15. Shamsul Alam, M., Bogoliubov's Method for Third Order Critically Damped Nonlinear Systems, Soochow J. Math. Vol. **28**, pp. 65-80, 2002.
16. Shamsul Alam, M., A Unified Krylov-Bogoliubov-Mitropolskii Method for Solving *n*-th Order Nonlinear Systems, J. Frank. Inst. Vol. **339**, pp. 239-248, 2002.
17. Shamsul Alam M., Asymptotic Method for Non-oscillatory Nonlinear Systems, Far East J. Appl. Math., Vol. **7**, pp. 119-128, 2002.