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On Semiderivations in Prime Gamma Rings

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ABSTRACT

Let M be a prime gamma ring. Let d : M → M be a semiderivation associated with a function g : M → M. We prove that d must be an ordinary derivation or of the form d(x) = pδ(x − g(x)) for all x∈M, δ∈Γ, where p is an element of the extended centroid of M. We have also seen that g must necessarily be an endomorphism.

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1. Introduction

Let *M* and Γ be additive abelian groups. *M* is called a Γ-ring if for all x,y,z∈*M*, *α,β*∈Γ the following conditions are satisfied:

(i) $x\beta y \in M$,

(ii) $(x+y)\alpha z = x\alpha z + y\alpha z$, $x(\alpha+\beta)y = x\alpha y + x\beta y$, $x\alpha(y+z) = x\alpha y + x\alpha z$,

(iii) $(x\alpha y)\beta z = x\alpha(y\beta z)$.

A Γ-ring M is called prime if xΓMΓy = 0 implies that x = 0 or y = 0.

The structure of semi-derivations of prime rings has been studied by C. L. Chuang [5]. He proved a structure theorem with the help of extended centroid of the classical associative rings. The same results have been obtained by M. Bresar [4].

A. Firat [8] obtained some results of prime rings with semi-derivations. J. Bergen and P. Grzesczuk [2] studied the commutativity properties of semiprime rings by means of skew (semi)-derivations. H. E. Bell and W. S. Martindale III [1] worked on prime rings with semiderivations and investigated the commutativity properties. J. C. Chang [5] generalized some results of prime rings with derivations to the prime rings with semi-derivations.

In this paper, we prove that $d(x) = p\delta(x - g(x))$ for all $x\in M$, $\delta\in\Gamma$, where p is the element of the extended centroid of a Γ-ring M, d is a semiderivation on M associated with a function g on M.

2. Semiderivations in Prime Γ-rings

An additive mapping D from M to M is called a derivation if $D(x\alpha y) = D(x)\alpha y + x\alpha D(y)$ holds for all $x, y\in M$, $\alpha\in\Gamma$.

Let M be a Γ-ring. An additive mapping $d: M \to M$ is called a semiderivation associated with a function $g: M \to M$ if, for all $x,y\in M$, $\alpha\in\Gamma$,

$d(x\alpha y) = d(x)\alpha g(y) + x\alpha d(y) = d(x)\alpha y + g(x)\alpha d(y)$; (ii) $d(g(x)) = g(d(x))$.

If g = I, i.e., an identity mapping of M, then all semi-derivations associated with g are merely ordinary derivations. If g is any endomorphism of M, then other examples of semi-derivations are of the form $d(x) = x - g(x)$.

Example 2.1
Let M_1 be a $\Gamma_1$ring and M_2 be a Γ_2-ring. Consider $M = M_1 \times M_2$ and $\Gamma = \Gamma_1 \times \Gamma_2$.

Define addition and multiplication on M and Γ by

$$(m_1, m_2) + (m_3, m_4) = (m_1 + m_3, m_2 + m_4)$$

$$(\alpha_1, \alpha_2) + (\alpha_3, \alpha_4) = (\alpha_1 + \alpha_3, \alpha_2 + \alpha_4)$$

$$(m_1, m_2)(\alpha_1, \alpha_2)(m_3, m_4) = (m_1\alpha_1 m_3, m_2\alpha_2 m_4),$$

for every $(m_1, m_2), (m_3, m_4)\in M$ and $(\alpha_1, \alpha_2), (\alpha_3, \alpha_4)\in\Gamma$.

Under these addition and multiplication M is a Γ-ring. Let $\delta: M_1 \to M_1$ be an additive map and $\tau: M_2 \to M_2$ be a left and right $M_2\Gamma$-module which is not a derivation.

Define $d : M \to M$ such that $d((m_1, m_2)) = (0, \tau(m_2))$ and $g: M \to M$ such that $g((m_1, m_2)) = (\delta(m_1), 0)$, $m_1\in M_1$, $m_2\in M_2$. Then it is clear that d is a semi-derivation of M (with associated map g) which is not a derivation.

We refer to [8,9] for the definitions of centroid , extended centroid of Γ-rings.

Lemma 2.2 Let M be a prime Γ-ring and Let $I \neq 0$ be an ideal of M. If $d \neq 0$ is a semiderivation on M, then $d \neq 0$ on I.

Proof. Suppose $d(I) = 0$. Then for $r\in I$, $x\in M$, we have $0 = d(r\alpha x) = d(r)\alpha g(x) + r\alpha d(x) = r\alpha d(x)$. *Replace r by rβy, we get* $r\beta y\alpha d(x) = 0$, *for all* $r\in I$, $x,y\in M$, $\alpha,\beta\in\Gamma$. *By the primeness of M,* $r=0$ *or* $d(x) = 0$. But $I \neq 0$, we get $d(x) = 0$ for all $x\in M$.

Lemma 2.3. Let M be a prime Γ-ring and Let I $\neq 0$ be an ideal of M. If d $\neq 0$ is a semiderivation on M and a∈M such that aβd(r) = 0, for all r∈I and β∈Γ, *then a* = 0.

Proof. By Lemma 2.2 we may pick r∈*I* such that d*(r)* ≠ 0. For s∈I we see that 0 = *aβd(sαr)* = *aβ(d(s)αg(r)* + sαd(r)) = aβsαd(r) , for α,β∈Γ.
By the primeness of M, a = 0.

Lemma 2.4 . Let M be a prime Γ-ring and Let I $\neq 0$ be an ideal of M. If d $\neq 0$ is a semiderivation on M, then d(d(I))≠ 0.

Proof. Suppose d(d(I*))* = 0. Then for r,s∈I, we exploit the definition *of d* in different ways to obtain

(1) 0 = d(d(rαs*))* = d(d(r)αs + g(r)αd(s)) = d(d(r))αs + g(d(r))αd(s) + d(g(r)αd(s))

(2) 0 = d(d(rαs*))* = d(d(r)αs + g(r)αd(s)) = d(d(r))αg(s) + d(r)αd(s) + d(g(r)αd(s)).

Subtraction of (2) from (1) yields
(3) (g(d(r)) – d(r))αd(s) = 0, r,s∈I, α∈Γ.
An application of Lemma 2.3 to (3) then says that d(d*(r))* = *d(r)* for all r∈*I.*
Again for r,s∈I, α∈Γ, we may also write
0 = d(d(rαs*))* = d(d(r)αs + g(r)αd(s)) = d(d(r))αg(s) + d(r)αd(s) + d(g(r))αg(d(s))
whence we have
(4) d(r)αd(s) + d(g(r))αg(d(s)) = 0, for all r,s∈I, α∈Γ.
Since d*(g(r)*) = *g(d(r)*) = d(r) for all r∈I and characteristic M ≠ 2, we conclude from (4) that d*(r)αd(s)* = 0 for all r,s∈I, α∈Γ. Another application of Lemma 2.3 asserts that d*(r)* = 0 for all r∈I, which then contradicts Lemma 2.2.

Lemma 2.5 Let M be a prime Γ-ring and Let d : M → M be a semiderivation with associated function g: M → M. If there exists a nonzero ideal I of M for which I ∩g(M) = 0, then there exists p∈C such that d(x) = pδ(x — g(x)) for all x ∈M, δ∈Γ, where C is the extended centroid of M.

Proof. Let *W be* the ideal generated by
$\sum_i r_i\alpha_i(x_i - g(x_i))\beta_i s_i, \textit{for all}\, r_i, s_i, x_i \in I, \alpha_i, \beta_i \in \Gamma$ and note that (otherwise *g* would be the identity mapping in contradiction to I∩ *g(M)* =0.
We define a mapping ϕ: W → M according to the rule:
$\sum_i r_i\alpha_i(x_i - g(x_i))\beta_i s_i \rightarrow \sum_i r_i\alpha_i d(x_i)\beta_i s_i \; \textit{where}\, r_i, s_i \in I, x_i \in M, \alpha_i, \beta_i \in \Gamma$.
Now we have to show that ϕ is well defined. Suppose that

$\sum_i r_i\alpha_i(x_i - g(x_i))\beta_i s_i = 0, \textit{for all}\; r_i, s_i, x_i \in I, \alpha_i, \beta_i \in \Gamma$. We have to prove that

$\sum_i \mathrm{r}_i\alpha_i d(x_i)\beta_i s_i = 0, \textit{where}\; r_i, s_i \in I,\; x_i \in M,\; \alpha_i, \beta_i \in \Gamma$.

Applying the semiderivation d to $\sum_i r_i\alpha_i(x_i - g(x_i))\beta_i s_i = 0.$ we see that

$0 = d(\sum_i (r_i\alpha_i x_i\beta_i s_i - r_i\alpha_i g(x_i)\beta_i s_i))$

$= \sum_i [r_i\alpha_i d(x_i\beta_i s_i) + d(r_i)\alpha_i g(x_i\beta_i s_i) - d((r_i)\alpha_i g(x_i))\beta_i g(s_i) - r_i\alpha_i g(x_i)\beta_i d(s_i)]$

$= \sum_i [r_i\alpha_i d(x_i)\beta_i s_i + r_i\alpha_i g(x_i)\beta_i d(s_i) + d(r_i)\alpha_i g(x_i)\beta_i g(s_i) - d(r_i)\alpha_i g(x_i)\beta_i g(s_i)$

$- g(r_i)\alpha_i d(g(x_i))\beta_i g(s_i) - r_i\alpha_i g(x_i)\beta_i d(s_i)]$

$= \sum_i r_i\alpha_i d(x_i)\beta_i s_i - \sum_i g(r_i)\alpha_i g(d(x_i))\beta_i g(s_i)$

$= \sum_i r_i\alpha_i d(x_i)\beta_i s_i - g(\sum_i r_i\alpha_i d(x_i)\beta_i s_i$

Therefore $\sum_i r_i\alpha_i d(x_i)\beta_i s_i = g(\sum_i r_i\alpha_i d(x_i)\beta_i s_i) \in I \cap g(M)$ whence

$\sum_i r_i\alpha_i d_1(x_i)\beta_i s_i = 0$ and ϕ is well defined. By the nature of the extended centroid *C* it follows that there exists $p \in C$ such that $p\delta w = \phi(w)$ for all $w \in W, \delta \in \Gamma$. Now, regarding M as a subring of the central closure *C(M),* we have for all $r,s \in I$, $x \in M$, $\alpha,\beta,\delta \in \Gamma$, $r\alpha p\delta(x - g(x))\beta s = p\delta(r\alpha(x - g(x))\beta s) = \phi(r\alpha(x - g(x))\beta v) = r\alpha d(x)\beta s$.
This implies that $(p\delta(x - g(x)) - d(x))\alpha r\beta s = 0$.
From the primeness of M we thus see that $d(x) = p\delta(x - g(x))$ for all $x \in M, \delta \in \Gamma$.

Theorem 2.6 Let d be a semiderivation of a prime Γ-ring M associated with the (endomorphism) mapping g: M → M. Then either one of the following two cases holds:
(1) There exists an element p in the extended centroid of M such that $d(x) = p\delta(x - g(x))$ for all $x \in M$, $\delta \in \Gamma$,
(2) The endomorphism g is an identity mapping and d is an ordinary derivation.

Proof. Set $d_1(x) = x - g(x)$ for $x \in M$. Then d_1 is also a semiderivation of M associated with the ring endomorphism g. Let
$U = \{\sum_i r_i\alpha_i d(x_i)\beta_i s_i : r_i, s_i, x_i \in M,\; \alpha_i, \beta_i \in \Gamma \;\textit{and}\; \sum_i r_i\alpha_i d_1(x_i)\beta_i s_i = 0\}$.

Then U is obviously a two-sided ideal of M. Let $r_i, s_i, x_i \in M$, $\alpha_i, \beta_i \in \Gamma$ be such that $\sum_i r_i\alpha_i d_1(x_i)\beta_i s_i = \sum_i r_i\alpha_i(x_i - g(x_i))\beta_i s_i = 0.$ Applying the semiderivation d to

$\sum_i r_i\alpha_i(x_i - g(x_i))\beta_i s_i = 0.$ and using the defining identities (i), (ii) for the semiderivation d to expand the resulting expression, we compute, as in Lemma 2.5.

0 = d($\sum_i r_i\alpha_i d_1(x_i)\beta_i s_i$) = d($\sum_i (r_i\alpha_i x_i\beta_i s_i - r_i\alpha_i g(x_i)\beta_i s_i)$)

=

$\sum_i [r_i\alpha_i d(x_i\beta_i s_i) + d(r_i)\alpha_i g(x_i\beta_i s_i) - d(r_i\alpha_i g(x_i))\beta_i g(s_i) - r_i\alpha_i g(x_i)\beta_i d(s_i)].$

$$\sum_i \begin{matrix}[r_i\alpha_i d(x_i)\beta_i s_i + r_i\alpha_i g(x_i)\beta_i d(s_i) + d(r_i)\alpha_i g(x_i)\beta_i g(s_i) - d(r_i)\alpha_i g(x_i)\beta_i g(s_i) \\ - g(r_i)\alpha_i d(g(x_i))\beta_i g(s_i) - r_i\alpha_i g(x_i)\beta_i d(s_i)]\end{matrix}.$$

= $\sum_i r_i\alpha_i d(x_i)\beta_i s_i - \sum_i g(r_i)\alpha_i g(d(x_i))\beta_i g(s_i)$.= $\sum_i r_i\alpha_i d(x_i)\beta_i s_i - g(\sum_i r_i\alpha_i d(x_i)\beta_i s_i)$.

Therefore $\sum_i r_i\alpha_i d(x_i)\beta_i s_i = g(\sum_i r_i\alpha_i d(x_i)\beta_i s_i)$ whenever $\sum_i r_i\alpha_i d_1(x_i)\beta_i s_i = 0$.

That is, $d_1(u) = u - g(u) = 0$ for all $u \in U$. If the two-sided ideal U is nonzero, then by Lemma 2.3, $d_1 = 0$ on M and hence $g(x) = x$ for all $x \in M$. Thus g is the identity endomorphism of M and d is merely an ordinary derivation of M, as desired. Now, assume that U = 0.

That is, for any $r_i,s_i,x_i \in M$, $\alpha_i,\beta_i \in \Gamma$, $\sum_i r_i\alpha_i d_1(x_i)\beta_i s_i = 0$ implies $\sum_i r_i\alpha_i d(x_i)\beta_i s_i = 0$.

Let W be the two-sided ideal $\{\sum_i r_i\alpha_i d_1(x_i)\beta_i s_i : r_i, s_i, x_i \in M,\ \alpha_i, \beta_i \in \Gamma\}$.

Then the mapping ϕ defined on W according to the rule ϕ: $\sum_i r_i\alpha_i d_1(x_i)\beta_i s_i \to \sum_i r_i\alpha_i d_1(x_i)\beta_i s_i$, where $r_i,s_i,x_i \in M$, $\alpha_i,\beta_i \in \Gamma$, is well defined. But ϕ is obviously an M_Γ-bimodule map of W into M. By the definition of the extended centroid of M, there exists an element p in the extended centroid of M such that $\phi(w) = p\delta w$ for all $w \in W$, $\delta \in \Gamma$. In particular, for all $r,s,x \in M$, $\alpha,\beta \in \Gamma$, $r\alpha d(x)\beta s = \phi(r\alpha d_1(x)\beta s) = p\delta(r\alpha d_1(x)\beta s) = r\alpha(p\delta d_1(x))\beta s$. It follows from the primeness of M that $d(x) = p\delta d_1(x) = p\delta(x - g(x))$ for all $x \in M$, $\alpha,\beta \in \Gamma$, as required.

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