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New Iterative Algorithms for Minimization of Nonlinear Functions

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ABSTRACT

In this paper, we suggest new iterative algorithms for minimization of nonlinear functions by using the decomposition technique. Then the comparative study of the new algorithms with the Newton's Algorithm is established by means of examples.

Keywords: Iterative methods; Decomposition technique; Newton's algorithm; Two-step algorithms; Three-step algorithms; Convergence

1. Introduction

Optimization problems with or without constraints arise in various fields such as science, engineering, economics, management sciences, etc., where numerical information is processed. In recent times, many problems in business situations and engineering designs have been modeled as an optimization problem for taking optimal decisions. In fact, numerical optimization techniques[15] have made deep in to almost all branches of engineering and mathematics.

A wide class of problem which arise in several branches of pure and applied science can be studied in the general framework of the nonlinear equations $g(x) = 0$ Due to their importance, several numerical methods [1-7, 9-11, 16] have been suggested and analyzed under certain conditions. These numerical methods have been constructed using different techniques such as Taylor series, homotopy perturbation method and its variant forms, quadrature formula, variational iteration method and decomposition method. To implement the decomposition method, one has to calculate the so-called Adomian polynomial, which is itself a difficult problem. To overcome these difficulties, several other techniques have been suggested and analyzed for solving the nonlinear equations. One of the decompositions is due to Daftardar-Gejji and Jafari [12]. To apply this technique, we first use the new series representation of the nonlinear function, which is obtained by using the quadrature formula and the fundamental theorem of calculus. Using essentially the idea of He[13], we rewrite the nonlinear equation as a coupled system of nonlinear equations. Applying the decomposition of Daftardar-Gejji and Jafari [12], we are able to construct some new iterative methods for solving the nonlinear equations. Recently Mohammad Aslam, et. al.[8] introduced some new iterative methods for nonlinear equations. In this paper, we propose new iterative algorithms namely New Algorithm-I and New Algorithm-II for minimization of nonlinear functions. Then the comparative study is made between new algorithms and Newton's algorithm by means of examples.

2. New Iterative Methods

In this section, we introduce two new iterative numerical algorithms for minimization of nonlinear real valued and twice differentiable real functions by using the concept of different decomposition technique.

Consider the nonlinear optimization problem

$$\text{Minimize } \{f(x),\ x \in R,\ f : R \to R\}$$

where f is a non-linear twice differentiable function.

2.1. Two step and Three step Iterative Algorithms

Consider the function $G(x) = x - (g(x)/g'(x))$ where $g(x) = f'(x)$. Here $f(x)$ is the function to be minimized. $G'(x)$ is defined around the critical point x^* of $f(x)$ if $g'(x^*) = f''(x^*) \neq 0$ and is given by

$$G'(x) = g(x)g''(x)/g'(x).$$

If we assume that $g''(x^*) \neq 0$, we have $G'(x^*) = 0$ iff $g(x^*) = 0$.

Here we consider the nonlinear equation g(x) = 0 --- (2.1) Using the quadratute formula and the fundamental theorem of calculus, the equation (2.1) can be written as

$$g(x) = g(\gamma) + (x - \gamma)\left[\frac{g'(\gamma) + 2\, g'(\frac{\gamma + x}{2}) + g'(x)}{4}\right] = 0$$

where γ is an initial guess sufficiently close to α, which is a simple root of the nonlinear equation (2.1). Using the idea of He [13], one can rewrite the nonlinear equation (2.1) as a coupled system

$$g(\gamma) + (x - \gamma)\left[\frac{g'(\gamma) + 2\, g'(\frac{\gamma + x}{2}) + g'(x)}{4}\right] + h(x) = 0 \quad \text{---(2.2)}$$

$$h(x) = g(x) - g(\gamma) - (x-\gamma)\left[\frac{g'(\gamma) + 2\,g'(\frac{\gamma + x}{2}) + g'(x)}{4}\right] \qquad \text{---(2.3)}$$

From (2.2) we, have

$$x = \gamma - 4\left[\frac{g(\gamma) + h(x)}{g'(\gamma) + 2\,g'(\frac{\gamma+x}{2}) + g'(x)}\right] = c + N(x) \qquad \text{--(2.4)}$$

where $c = \gamma$ ---(2.5)

$$N(x) = -4\left[\frac{g(\gamma) + h(x)}{g'(\gamma) + 2\,g'(\frac{\gamma+x}{2}) + g'(x)}\right] \qquad \text{---(2.6)}$$

It is clear that the operator *N(x)* is nonlinear $x = \sum_{i=0}^{\infty} x_i$ ---(2.7)

The nonlinear operator N can be decomposed as

$$N x = N x_0 + \sum_{i=1}^{\alpha}\{ N(\sum_{j=0}^{i} x_j) - N(\sum_{j=0}^{i-1} x_j) \} \qquad \text{---(2.8)}$$

Combining (2.4), (2.7) and (2.8) we have

$$\sum_{i=0}^{\infty} x_i = c + N x_0 + \sum_{i=1}^{\alpha}\{ N(\sum_{j=0}^{i} x_j) - N(\sum_{j=0}^{i-1} x_j) \} \qquad \text{---(2.9)}$$

We have the following iterative scheme.

$$x_0 = c$$

$$x_1 = N(x_0)$$

$$x_2 = N(x_0 + x_1) - N(x_0) \qquad \text{---(2.10)}$$

.....

$$x_{m+1} = N(\sum_{j=0}^{m} x_j) - N(\sum_{j=0}^{m-1} x_j), \quad m = 1, 2, ...$$

Then

$$x_1 + x_2 + \dots + x_{m+1} = N(x_0 + x_1 + \dots + x_m), \; m = 1, 2, \dots \qquad \text{--(2.11)}$$

$$x = c + \sum_{i=1}^{\infty} x_i \qquad \text{--(2.12)}$$

It can be shown that the series, $\sum_{i=0}^{\infty} x_i$ converges absolutely and uniformly to a unique solution of equation (2.5) see [12].

From (2.6) and (2.11), we get $x_0 = c = \gamma$. ---(2.13)

From (2.3), (2.7) and using the idea of Yun [17],

we obtain, $h(x_0) = 0$ ---(2.14)

$$x_1 = N(x_0) = -4\left[\frac{g(\gamma) + h(x_0)}{g'(\gamma) + 2\, g'(\frac{\gamma + x_0}{2}) + g'(x_0)}\right]$$

$$x_1 = N(x_0) = -4\left[\frac{g(\gamma)}{g'(\gamma) + 2\, g'(\frac{\gamma + x_0}{2}) + g'(x_0)}\right] \qquad \text{---(2.15)}$$

Note that x is approximated by

$$X_m = x_0 + x_1 + x_2 + \dots + x_m, \text{ where } \lim_{m \to \infty} X_m = x \qquad \text{---(2.16)}$$

For m = 0, $x \approx X_0 = x_0 = c = \gamma$ ---(2.17)

For m = 1,

$$x \approx X_0 = x_0 + x_1 = \gamma - 4\left[\frac{g(\gamma)}{g'(\gamma) + 2\, g'(\frac{\gamma + x_0}{2}) + g'(x_0)}\right]$$

$$= x_0 - \frac{g(x_0)}{g'(x_0)} \qquad \text{---(2.18)}$$

This formulation allows us to suggest the following one-step iterative method for minimization of nonlinear functions.

Algorithm 2.1: For a given x_0, compute the approximate solution x_{n+1} by the following iterative scheme.

$$x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)}, \quad g'(x_n) \neq 0, \quad n = 0, 1, 2, \dots$$

It is well known Newton method for minimization of nonlinear functions which has second order convergence.
From (2.18), we get,

$$x_0 + x_1 - \gamma = \frac{-g(\gamma)}{g'(\gamma)} \qquad \text{---(2.19)}$$

From (2.3), (2.7) and using the idea of Yun[17], we have

$$h(x_0 + x_1) = g(x_0 + x_1) - g(\gamma) - (x_0 + x_1 - \gamma)\left[\frac{g'(\gamma) + 2\,g'(\frac{\gamma + x_0 + x_1}{2}) + g'(x_0 + x_1)}{4}\right]$$

$$h(x_0 + x_1) = g(x_0 + x_1) - g(\gamma) + \frac{g(\gamma)}{4g'(\gamma)}\left[g'(\gamma) + 2\,g'(\frac{\gamma + x_0 + x_1}{2}) + g'(x_0 + x_1)\right] \qquad \text{---(2.20)}$$

$$x_1 + x_2 = N(x_0 + x_1) = -4\left[\frac{g(\gamma) + h(x_0 + x_1)}{\left[g'(\gamma) + 2\,g'(\frac{\gamma + x_0 + x_1}{2}) + g'(x_0 + x_1)\right]}\right]$$

$$= \frac{-g(\gamma)}{g'(\gamma)} - \left[\frac{4\,g(x_0 + x_1)}{\left[g'(\gamma) + 2\,g'(\frac{\gamma + x_0 + x_1}{2}) + g'(x_0 + x_1)\right]}\right] \qquad \text{---(2.21)}$$

For m = 2,

$$x \approx X_2 = x_0 + x_1 + x_2 = c + N(x_0 + x_1)$$

$$= \gamma - \frac{g(\gamma)}{g'(\gamma)} - \left[\frac{4\,g(x_0 + x_1)}{\left[g'(\gamma) + 2\,g'(\frac{\gamma + x_0 + x_1}{2}) + g'(x_0 + x_1)\right]}\right] \qquad \text{---(2.22)}$$

Using this relation, we can suggest the following two-step iterative method for minimization of nonlinear functions.

Algorithm 2.2

For a given x_0, compute the approximate solution x_{n+1}, by the following iterative scheme:

$$y_n = x_n - \frac{g(x_n)}{g'(x_n)}, \qquad --(2.23)$$

$$x_{n+1} = y_n - \left[\frac{4\,g(y_n)}{\left[g'(x_n) + 2\,g'(\frac{x_n + y_n}{2}) + g'(y_n) \right]} \right], \quad n=0,1,2.... \qquad ---(2.24)$$

From (2.22), we obtain

$$x_0 + x_1 + x_2 - \gamma = \frac{-g(\gamma)}{g'(\gamma)} - \frac{4\,g(x_0 + x_1)}{g'(\gamma) + 2\,g'(\frac{\gamma + x_0 + x_1}{2}) + g'(x_0 + x_1)} \qquad ---(2.25)$$

$$h(x_0 + x_1 + x_2) = g(x_0 + x_1 + x_2) - g(\gamma) - (x_0 + x_1 + x_2 - \gamma) \times \left[\frac{g'(\gamma) + 2\,g'(\frac{\gamma + x_0 + x_1 + x_2}{2}) + g'(x_0 + x_1 + x_2)}{4} \right]$$

$$= g(x_0 + x_1 + x_2) - g(\gamma) - \frac{1}{4}\left(-\frac{g(\gamma)}{g'(\gamma)} - \frac{4\,g(x_0 + x_1)}{g'(\gamma) + 2\,g'(\frac{\gamma + x_0 + x_1}{2}) + g'(x_0 + x_1)} \right) \times \left[g'(\gamma) + 2\,g'(\frac{\gamma + x_0 + x_1 + x_2}{2}) + g'(x_0 + x_1 + x_2) \right] \qquad ---(2.26)$$

$$x_1 + x_2 + x_3 = N(x_0 + x_1 + x_2) = -4\left[\frac{g(\gamma) + h(x_0 + x_1 + x_2)}{g'(\gamma) + 2\,g'(\frac{\gamma + x_0 + x_1 + x_2}{2}) + g'(x_0 + x_1 + x_2)} \right]$$

$$= \frac{-g(\gamma)}{g'(\gamma)} - \frac{4\,g(x_0 + x_1)}{g'(\gamma) + 2\,g'(\frac{\gamma + x_0 + x_1}{2}) + g'(x_0 + x_1)} - \frac{4\,g(x_0 + x_1 + x_2)}{g'(\gamma) + 2\,g'(\frac{\gamma + x_0 + x_1 + x_2}{2}) + g'(x_0 + x_1 + x_2)} \qquad ---(2.27)$$

For m = 3,

$$x \approx X_3 = x_0 + x_1 + x_2 + x_3 = c + N(x_0 + x_1 + x_2)$$

$$= \gamma - \frac{g(\gamma)}{g'(\gamma)} - \frac{4\, g(x_0 + x_1)}{g'(\gamma) + 2\, g'(\frac{\gamma + x_0 + x_1}{2}) + g'(x_0 + x_1)} - \frac{4\, g(x_0 + x_1 + x_2)}{g'(\gamma) + 2\, g'(\frac{\gamma + x_0 + x_1 + x_2}{2}) + g'(x_0 + x_1 + x_2)} \quad \text{---(2.28)}$$

Using this formulation, we can suggest the following three step iterative method for minimization of nonlinear functions.

Algorithm 2.3: For a given x_0, compute the approximate solution x_{n+1} by the following iterative scheme:

$$y_n = x_n - \frac{g(x_n)}{g'(x_n)} \quad \text{--- (2.29)}$$

$$z_n = y_n - \frac{4\, g(y_n)}{g'(x_n) + 2\, g'(\frac{x_n + y_n}{2}) + g'(y_n)} \quad \text{---(2.30)}$$

$$x_{n+1} = z_n - \frac{4\, g(z_n)}{g'(x_n) + 2\, g'(\frac{x_n + z_n}{2}) + g'(z_n)} \quad n = 0,1,2,\ldots.. \quad \text{--(2.31)}$$

New Algorithm - I

Since $g(x) = f'(x)$, the equations (2.23) and (2.24) becomes

$$y_n = x_n - \frac{f'(x_n)}{f''(x_n)}$$

$$x_{n+1} = y_n - \frac{4\, f'(y_n)}{\left[f''(x_n) + 2\, f''(\frac{x_n + y_n}{2}) + f''(y_n) \right]} \quad n = 0,1,2....$$

New Algorithm – II

Since $g(x) = f'(x)$, the equations (2.29), (2.30) and (2.31) becomes

$$y_n = x_n - \frac{f'(x_n)}{f''(x_n)}$$

$$z_n = y_n - \frac{4\, f'(y_n)}{f''(x_n) + 2\, f''(\frac{x_n + y_n}{2}) + f''(y_n)}$$

$$x_{n+1} = z_n - \frac{4\, f'(z_n)}{f''(x_n) + 2\, f''(\frac{x_n + z_n}{2}) + f''(z_n)} \qquad n = 0,1,2,.....$$

3. Convergence Analysis

In this section, we consider the convergence criteria of the iterative methods developed in section 2. In a similar way, one can consider the convergence of other Algorithms.

Theorem 3.1.: Let $\alpha \in I$ be a simple zero of sufficiently differentiable function $g : I \subseteq R \to R$ for an open interval I. If x_0 is sufficiently close to α , then the iterative methods defined by Algorithm 2.3 has fourth order convergence.

Proof : Refer the article [8]

The convergence analysis of Algorithm 2.2 follows [8] as the convergence in theorem 3.1.

4. Numerical Illustrations

Example 4.1: Consider the function $f(x) = x^3 - 2x - 5$. The minimized value of the function is 0.816497. The following table depicts the number of iterations needed to converge to the minimized value for all the new algorithms with three initial values $x_0 = 1$, $x_0 = 2$, and $x_0 = 3$.

Iterations	Newton's Algorithm	New Algorithm- I	New Algorithm-II
0	1.000000	1.000000	1.000000
1	0.833333	0.818182	0.816673
2	0.816667	0.816497	0.816497
3	0.816497		
4.			

Iterations	Newton's Algorithm	New Algorithm-I	New Algorithm-II
0	2.000000	2.000000	2.000000
1	1.166667	0.947368	0.871857
2	0.869048	0.817164	0.816498
3	0.818085	0.816497	0.816497
4.	0.816498		
5.	0.816497		

Iterations	Newton's Algorithm	New Algorithm-I	New Algorithm-II
0	3.000000	3.000000	3.000000
1	1.611111	1.192771	1.021020
2	1.012452	0.827292	0.816757
3	0.835460	0.816497	0.816497
4.	0.816712		
5.	0.816497		

Example 4.2: Consider the function $f(x) = x^4 - x - 10$. The minimized value of the function is 0.629961. The following table depicts the number of iterations needed to converge to the minimized value for all the new algorithms with three initial values $x_0 = 1$, $x_0 = 2$, and $x_0 = 3$.

Iterations	Newton's algorithm	New Algorithm-I	New Algorithm-II
0	1.000000	1.000000	1.000000
1	0.750000	0.675926	0.649494
2	0.648148	0.630173	0.629961
3	0.630466	0.629961	
4	0.629961		
Iterations	**Newton's algorithm**	**New Algorithm-I**	**New Algorithm-II**
0	2.000000	2.000000	2..000000
1	1.354167	1.094315	0.961188
2	0.948222	0.703474	0.644283
3	0.724830	0.630759	0.629961
4	0.641836	0.629961	
5	0.630179		
6	0.629961		
Iterations	**Newton's algorithm**	**New Algorithm-I**	**New Algorithm-II**
0	3.000000	3.000000	3.000000
1	2.009259	1.599535	1.382586
2	1.360148	0.905037	0.738834
3	0.951810	0.653781	0.630340
4	0.726525	0.629992	0.629961
5	0.642227	0.629961	
6	0.630193		
7	0.629961		

Example 4.3: Consider the function $f(x) = xe^{x} - 1$. The minimized value of the function is -1. The following table depicts the number of iterations needed to converge to the minimized value for all the new algorithms with three initial values $x_0 = 1$, $x_0 = 2$, and $x_0 = 3$.

Iterations	Newton's algorithm	New Algorithm- I	New Algorithm-II
0	1.000000	1.000000	1.000000
1	0.333333	-0.008114	-0.208820
2	-0.238095	-0.719836	-0.902685
3	-0.670528	-0.985252	-0.999923
4	-0.918350	-0.999997	-1.000000
5	-0.993836	-1.000000	
6	-0.999962		
7	-1		

Iterations	Newton's algorithm	New Algorithm-I	New Algorithm-II
0	2.000000	2.000000	2.000000
1	1.250000	0.845809	0.597600
2	0.557692	-0.130446	-0.490100
3	-0.051330	-0.785608	-0.972552
4	-0.538159	-0.992756	-0.999999
5	-0.854409	-1.000000	-1.000000
6	-0.981421		
7	-0.999661		
8	-0.999999		
9	-1.000000		

Iterations	Newton's algorithm	New Algorithm-I	New Algorithm-II
0	3.000000	3.000000	3.000000
1	2.200000	1.759418	1.484172
2	1.438095	0.633103	0.167228
3	0.728954	-0.293034	-0.742732
4	0.095395	-0.862006	-0.997161
5	-0.427368	-0.997846	-1.000000
6	-0.791491	-1.000000	
7	-0.964025		
8	-0.998751		
9	-0.999999		
10	-1.000000		

5. Conclusion

In this paper, we have proposed two algorithms namely New Algorithm-I and New Algorithm-II which are two-step and three-step iterative algorithms for minimization nonlinear functions by using a different decomposition technique. It is clear from the numerical results that the rate convergence of New algorithms I and II are faster than Newton's method. In future, we may extend the new algorithms for constrained optimization problems.

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