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European Option Pricing Under the Vaseck Model of the Short Rate in Mixed Fractional Brownian Motion Environment

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ABSTRACT

In this study, assume that the stock price obey the stochastic differential equation driven by mixed fractional Brownian motion, and the short rate follows the Vaseck model. Then, the Black-Scholes partial differential equation is obtained under the assumptions by using fractional Ito formula. Finally, the pricing formulae of the European call and put option are obtained by partial differential equation theory. The results of Black-Scholes model is generalized.

Keywords: Option pricing; Vaseck model; Black-Scholes model; mixed fractiona Brownian motion

1. Introduction

The break-through in option valuation theory started with the publication of two seminal papers by Black and Scholes[1]. In the papers the authors introduced a continuous time model of a complete friction-free market where the price of a stock follows a geometric Brownian motion. They presented a self-financing, dynamic trading strategy consisting of a riskless security and a risky stock, which replicates the payoff of an option. Then they argued that the absence of arbitrage dictates that the option price be equal to the cost of setting up the replicating portfolio.

Recently, fractional Brownian motion has been considered to replace Brownian motion in the usual financial models as it has better behaved tails and exhibits long-term dependence while remaining Gaussian. For details about the stochastic analysis theory of fractional Brownian motion, see Ref. [2,3]. The fractional Brownian motion is applied in finance, such as Ref. [4,5,6].

However, all the above option pricing studies assume that the risk-free rate or the short rate is constant during the life of the option. Kung and Lee assume that the short rate follows the Merton model and the stock price is driven by standard Brownian motion. Using their option model, the European call and put option are obtained, see Ref.[7].

Hence, in this study, we incorporate its stochastic nature into fractional B-S model. Specifically, we use the following stochastic process, first proposed by Vaseck, to depict its dynamics and derive explicit pricing formulas for European call and put on a stock.

The paper is organized as follows: In Section 2, we treat the Black-Scholes model that the short rate obey the vaseck model. In Section 3, we derive the formula for the price of a riskless zero-coupon bond paying \$1 at maturity based on Eq. (2). In Section 4, the pricing formulas for European call and put on a stock are obtained. Section 5 contains conclusions.

2. The model

Firstly, we assume that the short rate of the market satisfied the Vasicek model

$$dr_t = \theta(\mu_r - r_t)dt + \sigma_{r1}dW_{H_1}(t) + \sigma_{r2}dW_{H_2}(t), \tag{1}$$

where r_t is the short-term interest rates. θ is the mean-reversion speed. μ_r is the long-term interest rate. σ_{r1} and σ_{r2} are the instantaneous volatility. $W_{H_1}(t)$ and $\{W_{H_2}(t), t \geq 0\}$ are the fractional Brownian motion with Hurst parameter H_1, H_2. Secondly, there are zero-coupon bond and stock in this market. Let B_t be the price of a riskless zero-coupon bond paying 1 \$ at time T.

$$dB(t,r_t) = r_t B_t dt + \sigma_{b1}B(t,r_t)dW_{H_1} + \sigma_{b2}B(t,r_t)dW_{H_2}, \quad B(T,r_T) = 1, \quad t_0 \leq t \leq T, \tag{2}$$

And, the dynamics of the stock price process takes the following form

$$dS_t = \mu S_t dt + \sigma_1 S_t dW_{H_1}(t) + \sigma_2 S_t dW_{H_2}(t), \tag{3}$$

where μ is expectation return rate which is time-dependent. Constant σ_1 and σ_2 are volatility of the stock.

3. Explicit pricing formulas of zero-coupon bond

To solve the value of $B(t,r_t)$, by Eq. (1) and fractional Ito formula[3-5], we have

$$\begin{aligned} dB(t,r_t) &= \frac{\partial B(t,r_t)}{\partial t}dt + \frac{\partial B(t,r_t)}{\partial r_t}dr_t + H_1\sigma_{r1}^2 t^{2H_1-1}\frac{\partial^2 B(t,r_t)}{\partial r_t^2}dt + H_2\sigma_{r2}^2 t^{2H_2-1}\frac{\partial^2 B(t,r_t)}{\partial r_t^2}dt \\ &= \frac{\partial B(t,r_t)}{\partial t}dt + \theta(\mu_r - r_t)\frac{\partial B(t,r_t)}{\partial r_t}dt + \sigma_{r1}\frac{\partial B(t,r_t)}{\partial r_t}dW_{H_1}(t) + \sigma_{r2}\frac{\partial B(t,r_t)}{\partial r_t}dW_{H_2}(t) \\ &\quad + H_1\sigma_{r1}^2 t^{2H_1-1}\frac{\partial^2 B(t,r_t)}{\partial r_t^2}dt + H_2\sigma_{r2}^2 t^{2H_2-1}\frac{\partial^2 B(t,r_t)}{\partial r_t^2}dt. \end{aligned}$$

Compared to Eq. (2), such that

$$\frac{\partial B(t,r_t)}{\partial t}+\theta(\mu_r-r_t)\frac{\partial B(t,r_t)}{\partial r_t}+H_1\sigma_{r1}^2t^{2H_1-1}\frac{\partial^2 B(t,r_t)}{\partial r_t^2}+H_2\sigma_{r2}^2t^{2H_2-1}\frac{\partial^2 B(t,r_t)}{\partial r_t^2}=r_tB(t,r_t)\,.$$

Then the value of zero-coupon bond at time t satisfied

$$\begin{cases}\dfrac{\partial B(t,x)}{\partial t}+\theta(\mu_r-x)\dfrac{\partial B(t,x)}{\partial x}+H_1\sigma_{r1}^2t^{2H_1-1}\dfrac{\partial^2 B(t,x)}{\partial x^2}+H_2\sigma_{r2}^2t^{2H_2-1}\dfrac{\partial^2 B(t,x)}{\partial x^2}=xB(t,x),\\ B(T,x)=1.\end{cases}\tag{4}$$

Make sure that $B(T,x)=1$, let $B(t,x)=\exp\{A_1(t)+xA_2(t)\}$, $A_1(T)=0, A_2(T)=0$, so that

$$\frac{\partial B(t,r_t)}{\partial t}=A_1'(t)B(t,x)+xA_2'(t)B(t,x)\,,\ \frac{\partial B(t,x)}{\partial x}=A_2(t)B(t,x)\,,\ \frac{\partial^2 B(t,x)}{\partial x^2}=A_2(t)^2B(t,x)\,.\tag{5}$$

Compare Eq. (4) and Eq. (5), then

$$\begin{cases}\theta A_2(t)-A_2'(t)+1=0\\ A_1'(t)+\theta\mu_rA_2(t)+(H_1\sigma_{r1}^2t^{2H_1-1}+H_2\sigma_{r2}^2t^{2H_2-1})A_2(t)^2=0\\ A_1(T)=0, A_2(T)=0.\end{cases}$$

Then we conclude that

$$A_1(t)=-\mu_r(T-t)-\mu_r(1-e^{\theta(T-t)})-\int_t^T(H_1\sigma_{r1}^2s^{2H_1-1}+H_2\sigma_{r2}^2s^{2H_2-1})A_2(s)^2\,ds\,,$$

$$A_2(t)=\frac{1-\theta\exp\{-\theta(T-t)\}}{\theta}\,.$$

So that, the explicit solution of Eq. (4) is given by

$$B(t,r_t)=\exp\{A_1(t)+r_tA_2(t)\}\,,B(T,r_T)=1\,.\tag{6}$$

When $\theta=0$ $\sigma_{r1}=0$ $\sigma_{r2}=0$, we have $dr_t=0$,then $r_t=r$. And, the Eq. (6) can be changed as following

$$B(t,r_t)=\exp\{r(T-t)\}\,.$$

4. Explicit pricing formulas for European options

In what follows we introduce some relevant derivatives of two stocks, and show how to obtain the formulae for the value of these derivatives. Let

$$D_1(t)=H_1\sigma_{b1}^2B(t,r_t)^2+H_1\sigma_{b1}^2B(t,r_t)^2\,,\ D_2(t)=H_1\sigma_1^2t^{2H_1-1}+H_2\sigma_2^2t^{2H_2-1}\,,$$

$$D_3(t)=H_1\sigma_{b1}\sigma_2t^{2H_1-1}+H_2\sigma_{b1}\sigma_2t^{2H_2-1},\quad D(t)=D_1(t)+D_2(t)-2D_3(t)\,.$$

In this study, we assume that there are no transaction costs, margin requirements, and taxes; all securities are divisible; security trading is continuous and borrowing and short-selling are permitted without restriction; there are no dividend payouts over the life of the option; and all investors can borrow or lend at the same short rate. Further, we consider the European call option with payoff $(S_T-B(T,r_T)K)^+$. Because of $B(T,r_T)=1$, since $(S_T-B(T,r_T)K)^+$ can be written as

$$(S_T - B(T, r_T)K)^+ = (S_T - K)^+, \tag{7}$$

where T is maturity date, K is exercise price.

Let $C = C(S_t, B(t, r_t), t, K)$ be the call price, which is a function of the stock price S_t the riskless zero-coupon bond price $B(t, r_t)$, and the time t. By Ito's lemma, the change in the call price over an infinitesimal time dt satisfies the following stochastic differential equation:

$$\begin{aligned} dC &= \frac{\partial C}{\partial t}dt + \frac{\partial C}{\partial B(t,r_t)}dB(t,r_t) + D_1(t)\frac{\partial^2 C}{\partial B(t,r_t)^2}dt + \frac{\partial C}{\partial S_t}dS_t \\ &\quad + D_2(t)S_t^2\frac{\partial^2 C}{\partial S_t^2}dt + 2D_3(t)\frac{\partial^2 C}{\partial S_t \partial B(t,r_t)}dt \\ &= [\frac{\partial C}{\partial t} + D_1(t)\frac{\partial^2 C}{\partial B(t,r_t)^2} + D_2(t)S_t^2\frac{\partial^2 C}{\partial S_t^2} + 2D_3(t)\frac{\partial^2 C}{\partial S_t \partial B(t,r_t)}]dt \\ &\quad + \frac{\partial C}{\partial B(t,r_t)}dB(t,r_t) + \frac{\partial C}{\partial S_t}dS_t. \end{aligned} \tag{8}$$

Now we form a hedge portfolio consisting of the stock, the riskless bond, and the call. Let θ_t^0 be the number of shares of the bond, θ_t^1 be the number of the stock, and θ_t^2 be the number of the call. The self-finance hedge is formed such that the value (say, H) of the hedge portfolio is zero. That is

$$H = \theta_t^0 B(t, r_t) + \theta_t^1 S_t + \theta_t^2 C = 0.$$

Hence, we have

$$dH = \theta_t^0 dB(t, r_t) + \theta_t^1 dS_t + \theta_t^2 dc = 0. \tag{9}$$

Substituting Eq. (8) into Eq. (9) and grouping, Eq. (9) becomes

$$\begin{aligned} dH &= \theta_t^2[\frac{\partial C}{\partial t} + D_1(t)\frac{\partial^2 C}{\partial B(t,r_t)^2} + D_2(t)S_t^2\frac{\partial^2 C}{\partial S_t^2} + 2D_3(t)\frac{\partial^2 C}{\partial S_t \partial B(t,r_t)}]dt \\ &\quad + [\theta_t^2\frac{\partial C}{\partial S_t} + \theta_t^1]dS_t + [\frac{\partial C}{\partial B(t,r_t)} + \theta_t^0]dB(t,r_t). \end{aligned} \tag{10}$$

Eq(10) implies that $\theta_t^2\frac{\partial C}{\partial S_t} + \theta_t^1 = 0$, $\frac{\partial C}{\partial B(t,r_t)} + \theta_t^0 = 0$, and

$$\frac{\partial C}{\partial t} + D_1(t)B(t,r_t)^2\frac{\partial^2 C}{\partial B(t,r_t)^2} + D_2(t)S_t^2\frac{\partial^2 C}{\partial S_t^2} + 2D_3(t)S_t B(t,r_t)\frac{\partial^2 C}{\partial S_t \partial B(t,r_t)} = 0.$$

Hence, the price of European call option with payoff $(S_T - B(T, r_T)K)^+$ must satisfied

$$\begin{cases} \dfrac{\partial C}{\partial t} + D_1(t)y^2\dfrac{\partial^2 C}{\partial y^2} + D_2(t)x^2\dfrac{\partial^2 C}{\partial x^2} + 2D_3(t)xy\dfrac{\partial^2 C}{\partial x \partial y} = 0, \\ C(T, x, y) = (x - Ky)^+. \end{cases} \tag{11}$$

Letting

$$\xi = \frac{x}{y} \ , \ F(t,\xi) = \frac{C}{y}, \tag{12}$$

we get

$$C_x = \frac{\partial F}{\partial \xi}, C_y = F - \xi \frac{\partial F}{\partial \xi}, C_{xx} = \frac{1}{y}\frac{\partial^2 F}{\partial \xi^2}, C_{xy} = -\frac{\xi}{y}\frac{\partial^2 F}{\partial \xi^2}, C_{yy} = \frac{\xi^2}{y^2}\frac{\partial^2 F}{\partial \xi^2}. \tag{13}$$

Substituting Eqs. (13) into Eq. (11), we obtain

$$\frac{\partial F}{\partial t} + D(t)\xi^2 \frac{\partial^2 F}{\partial \xi^2} = 0, F(T,\xi) = (\xi - K)^+. \tag{14}$$

Letting

$$z = \ln\frac{\xi}{K} - \int_t^T D(\tau)d\tau \ , s = \int_t^T D(\tau)d\tau \ , F(t,\xi) = KU(s,z) \ , \tag{15}$$

We have

$$\frac{\partial F}{\partial t} = K[-D(t)\frac{\partial U}{\partial s} + D(t)\frac{\partial U}{\partial z}] \ , \ \frac{\partial F}{\partial \xi} = \frac{K}{\xi}\frac{\partial U}{\partial z} \ , \ \frac{\partial^2 F}{\partial \xi^2} = \frac{K}{\xi^2}[\frac{\partial^2 U}{\partial z^2} - \frac{\partial U}{\partial z}]. \tag{16}$$

Substituting Eqs. (16) into Eq. (14), the Eq. (14) can be changed as following

$$\frac{\partial U}{\partial s} = \frac{\partial^2 U}{\partial z^2}, U(0,z) = (e^z - 1)^+. \tag{17}$$

Eq. (17) is a standard one-dimensional heat equation with the explicit solution

$$U(s,z) = \frac{1}{2\sqrt{\pi s}}\int_{-\infty}^{\infty} U(0,\tau)e^{\frac{(\tau - z)^2}{4s}} d\tau \, . \tag{18}$$

Substituting $U(0,z) = (e^z - 1)^+$ into Eq. (14), we have

$$C(s,z) = e^{z+s}\Phi(\frac{z+2s}{\sqrt{2s}}) - \Phi(\frac{z}{\sqrt{2s}}) \, .$$

By the inverse transformation of Eq. (12) and Eq. (15), the price of European call is

$$C(S_t, B(t,r_t), t, K) = S_t\Phi(d_1) - KB(t,r_t)\Phi(d_2) \ , \tag{19}$$

where,

$$d_1 = \frac{\ln S_t - \ln B(t,r_t) - \ln K + \int_t^T D(\tau)d\tau}{\sqrt{2\int_t^T D(\tau)d\tau}} \ , \ d_2 = \frac{\ln S_t - \ln B(t,r_t) - \ln K - \int_t^T D(\tau)d\tau}{\sqrt{2\int_t^T D(\tau)d\tau}} \, .$$

By the same way, the European put is

$$P(S_t, B(t,r_t), t, K) = KB(t,r_t)\Phi(-d_2) - S_t\Phi(-d_1) \, . \tag{20}$$

If the short rate is constant (i.e., θ, σ_{r1} and σ_{r2} in Eq(1) are both 0), then the bond price in Eq. (6) is

$$B(t,r_t) = \exp\{r(T-t)\} \, . \tag{21}$$

Eq. (19) mean than $B(t,r_t)$ is time-function. Then we have

$$\sigma_{r1} = \sigma_{r2} = 0 \ , \ D_1(t) = 0 \ , \ D_3(t) = 0 \ , D(t) = H_1\sigma_1^2 t^{2H_1 - 1} + H_2\sigma_2^2 t^{2H_2 - 1} \, .$$

Substituting them into Eqs. (30) and Eq. (31), we obtain the fractional B-S formulas for European call and put options(see Ref.[3-5])

$$C(S_t, B(t,r_t), t, K) = S_t\Phi(\bar{d}_1) - K\exp\{r(T-t)\}\Phi(\bar{d}_2) \, ,$$

$$P(S_t, B(t,r_t), t, K) = K\exp\{r(T-t)\}\Phi(-\bar{d}_2) - S_t\Phi(-\bar{d}_1)$$

$$d_1 = \frac{\ln S_t - \ln K + r(T-t) + 0.5\sigma_1^2 t^{2H_1} + 0.5\sigma_2^2 t^{2H_2}}{\sqrt{\sigma_1^2 t^{2H_1} + \sigma_2^2 t^{2H_2}}} \quad d_2 = d_1 - \sqrt{\sigma_1^2 t^{2H_1} + \sigma_2^2 t^{2H_2}} \ .$$

Let $H_1 = H_2 = 0.5$, the European call and put in B–S model can be hold(see Ref.[1]).

5. Conclusion

In this paper, we derived a closed-form pricing formula for European call and put. Previous option pricing studies typically assume that the short rate is constant or time-function over the life of the option. In reality, the short rate is evolving randomly through time. Our findings suggest that European call and put on a stock can be calculated when the short rate follows the Vasicek model. It is clear that The Eq. (19) and Eq. (20) are the generalization of the classical Black-Scholes model.

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REFERENCES

1. Black F, Scholes M. The pricing of options and corporate liabilities [J]. Journal of Political Economy, 1973, 81(3): 637-654.
2. Elliott R J, Hoek J. A General Fractional White Noise Theory and Applications to Finance [J], Mathematical Finance, 2003, 13 (2): 301-330.
3. Hu Y, Øksendal B. Fractional White Noise Calculus and Applications to Finance, Infinite Dimensional Analysis [J], Quantum Probability and Related Topics, 2003, 6 (1): 1-32 .
4. Benth F E. On arbitrage-free pricing of weather derivatives based on fractional Brownian motion [J]. Applied Mathematical Finance, 2003, 10 (4): 303-324.
5. Bjork T, Hult H. A note on Wick products and the fractional Black-Scholes model [J]. Finance and Stochastics, 2005, 9 (2): 197-209.
6. Guasoni P. No arbitrage under transaction costs, with fractional Brownian motion and beyond [J], Mathematical Finance, 2006, 16(3): 569-582.
7. Kung J J, Lee L S. Option pricing under the Merton model of the short rate [J], Mathematics And Computers In Simulation, 2009, 80(1):378-386.