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Commutativity of Two Torsion Free σ-Prime Gamma Rings with Nonzero Derivations

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ABSTRACT

Let M be a 2-torsion free σ-prime Γ-ring and d a nonzero derivation on M. Then M is commutative with the help of the condition $[d(x),x]_\alpha \in Z(M)$, for all $x \in M$ and $\alpha \in \Gamma$. Let I be a nonzero σ-ideal of M and d a nonzero derivation on M commuting with σ. Then M is commutative in both conditions $[d(x),d(y)]_\alpha=0$ and $d(x\alpha y) = d(y\alpha x)$, for all $x,y \in I$ and $\alpha \in \Gamma$.

Keywords: n-torsion free, σ-ideals, derivations, σ-prime Γ-rings.

1 Introduction

Let M and Γ be additive abelian groups. M is said to be a Γ-ring if there exists a mapping $M X \Gamma X M \rightarrow M$ (sending (x, α, y) into $x\alpha y$) such that

(a) $(x + y)\alpha z = x\alpha z + y\alpha z$,
$x(\alpha + \beta)y = x\alpha y + x\beta y$,
$x\alpha(y + z) = x\alpha y + x\alpha z$,
(b) $(x\alpha y)\beta z = x\alpha(y\beta z)$,
for all $x,y,z \in M$ and $\alpha,\beta \in \Gamma$.

A subset A of a Γ-ring M is a left(right) ideal of M if A is an additive subgroup of M and $(M\Gamma A)$, the set of all $m\alpha a$ such that $m \in M, \alpha \in \Gamma, a \in A$, $(A\Gamma M)$ is contained in A. The centre of M is denoted by Z(M),the set of all $m \in M$ such that $a\alpha m=m\alpha a$ for all $a \in M$ and $\alpha \in \Gamma$. M is commutative if $a\alpha b=b\alpha a$, for all $a,b \in M$ and $\alpha \in \Gamma$. M is prime if $a\Gamma M\Gamma b=0$ with $a,b \in M$, then a=0 or b=0. M is σ-prime if $a\Gamma M\Gamma b= a\Gamma M\Gamma\sigma(b) =0$ with $a,b \in M$, then a=0 or b=0. An ideal I of M is σ-ideal if σ(I)=I. We denote the commutator $x\alpha y-y\alpha x$ by $[x,y]_\alpha$. M is n-torsion free if nm=0 for all $m \in M$ implies m=0, where n is an integer.An additive mapping $d:M\rightarrow M$ is a derivation if $d(a\alpha b)=a\alpha d(b) + d(a)\alpha b$, a left derivation if $d(a\alpha b)=a\alpha d(b) + b\alpha d(a)$, a Jordan derivation if d(aαa)=aαd(a) + d(a)αa, a Jordan left derivation if d(aαa)=2aαd(a) , for all a,b$\in$M and α$\in$Γ.

Y.Ceven[4] concerned on Jordan left derivation on completely prime Γ-rings. He investigated the existence of a nonzero Jordan left derivation on a completely prime Γ-ring that makes the Γ-ring commutative with an assumption. With the same assumption, he proved that every Jordan left derivation on a completely prime Γ –ring is a left derivation on it. In this paper, he gave an example of Jordan left derivation for Γ-rings.

Mustafa Asci and Sahin Ceran [8] worked on a nonzero left derivation d on a prime Γ-ring M for which M is commutative with the conditions d(U)$\subseteq$U and d^2(U)$\subseteq$Z, where U is an ideal of M and Z is the centre of M. They also proved the commutativity of M by the nonzero left derivation d_1 and right derivation d_2 on M with the conditions d_2(U)$\subseteq$U and d_1d_2(U)$\subseteq$Z.

In [11], Sapanci and Nakajima defined a derivation and a Jordan derivation on Γ-rings and investigated a Jordan derivation on a certain type of completely prime Γ-ring which is a derivation. They also gave examples of a derivation and a Jordan derivation on Γ-rings.

Bresar and Vukman [3] showed that the existence of a nonzero Jordan left derivation on R into X implies R is commutative, where R is a ring and X is 2-torsion free and 3-torsion free left R-module.

Qing Deng [5] worked on Jordan left derivations d of prime ring R of characteristic not 2 into a nonzero faithful and prime left R-module X. He proved the commutativity of R with Jordan left derivation d.

Md. Ashraf and Rehman [1] worked on Lie ideals and Jordan left derivations of prime rings. They proved that if d is an additive mapping on a 2-torsion free prime ring R satisfying d(u^2)=2ud(u),for all u$\in$U, where U is a Lie ideal of R such that $u^2\in$U, for all u$\in$U, then d(uv) = ud(v) + vd(u), for all u$\in$U.

L.Oukhtite and S.Salhi [10] studied on derivations in σ-prime rings. They showed that R is a 2-torsion free σ-prime ring and d:R→R is a nonzero derivation with [d(x),x] $\in$Z(R), for x$\in$R, then R is commutative. They also proved that if d commutes with σ, then R is commutative for the conditions that [d(x),d(y)] = 0 and d(xy) = d(yx), for all x,y$\in$I,where I is a σ-ideal of R.

In our paper, we follow the results of L.Oukhtite and S.Salhi [10] in gamma rings. We prove that if d is a nonzero derivation on a 2-torsion free σ-prime Γ-ring M and $[d(x),x]_\alpha$ $\in$Z(M), for all x$\in$M and α$\in$Γ, then M is commutative. We also investigate a nonzero derivation d which commutes with σ on M for which M is commutative in both conditions $[d(x),d(y)]_\alpha$ = 0 and d(xαy) = d(yαx), for all x,y$\in$I and α$\in$Γ,where I is a σ-ideal of M.

Throughout this paper we shall use the mark (*) for aαbβc=aβbαc, for all a,b,c$\in$M and α,β$\in$Γ.

In order to prove our main result, we shall state and prove some lemmas as primary results .

2 Primary Results

Lemma 2.1 Let $U\not\subset Z(M)$ be a Lie ideal of a 2-torsion free σ-prime Γ-ring M. Then there exists an ideal I of M such that $[I,M]_\alpha \subseteq U$ but $[I,M]_\alpha \not\subset Z(M)$, for all $\alpha\in\Gamma$.

Proof . Since M is 2-torsion free and $U\not\subset Z(M)$,by results in [6], we can show that $[U,U]_\alpha \neq 0$ and $[I,M]_\alpha\subseteq U$,where $I = I\alpha[U,U]_\alpha\alpha M\neq 0$ is an ideal of M generated by $[U,U]_\alpha$. Now, $U\not\subset Z(M)$ implies $[I,M]_\alpha\not\subset Z(M)$; for if $[I,M]_\alpha \subseteq Z(M)$ then $[I,[I,M]_\alpha]_\alpha = 0$, which gives $I \subseteq Z(M)$ and ,since $I\neq 0$ is an ideal of M, so M=Z(M). □

Lemma 2.2 Let $U\not\subset Z(M)$ be a Lie ideal of a 2-torsion free σ-prime Γ-ring M and $a,b\in M$ such that $a\alpha U\beta b=a\alpha U\beta\sigma(b)$, for all $\alpha,\beta\in\Gamma$. Then a=0 or b=0.

Proof. Since M is a σ-prime Γ-ring, there exists an ideal I of M such that $[I,M]_\alpha \subseteq U$ but $[I,M]_\alpha\not\subset Z(M)$, for all $\alpha\in\Gamma$, Lemma 2.1. Now, for $u\in U, y\in I$ and $m\in M$, we have $[y\alpha a\alpha u,m]_\alpha \in [I,M]_\alpha \subseteq U$, and so

$0=a\alpha[y\alpha a\alpha u,m]_\beta\beta b$
$= a\alpha[y\alpha a\alpha u,m]_\beta\beta\sigma(b)$
$=a\alpha[y\alpha a,m]_\alpha\beta u\beta\sigma(b) + a\alpha y\beta a\alpha[u,m]_\alpha\beta\sigma(b)$,by (*)
$= a\alpha[y\alpha a,m]_\alpha\beta u\beta\sigma(b)$,since $a\alpha[u,m]_\alpha\in a\alpha U\beta b= a\alpha U\beta\sigma(b)$
$=a\alpha y\alpha a\alpha m\beta u\beta\sigma(b) - a\alpha m\alpha y\alpha a\beta u\beta\sigma(b)$
$= a\alpha y\alpha a\alpha m\beta u\beta\sigma(b) – a\alpha m\alpha y\beta a\alpha u\beta\sigma(b)$,by (*)
$= a\alpha y\alpha a\alpha m\beta u\beta\sigma(b)$, by assumption.

Thus $a\alpha I\alpha a\alpha M\beta U\beta\sigma(b) = 0$.

If $I\neq 0$ then $U\beta\sigma(b) = 0$, by the σ-primeness of M. Now, if $u\in U$ and $m\in M$ then $(u\alpha m - m\alpha u)\in U$ and hence $(u\alpha m - m\alpha u)\beta b=0$ gives $u\alpha m\beta b=0$,that is $u\alpha M\beta b=0$. As $U\neq 0$, we have b=0. Proceding in the same way we may reach to the decision that if $b\neq 0$ then a=0. □

Lemma 2.3 Let M be a σ-prime Γ-ring satisfying (*) and I a nonzero σ-ideal of M. Let d be a nonzero derivation on M commuting with σ. If $[x,M]_\alpha\alpha I\beta d(x) = 0$,for all $x\in I$ and $\alpha,\beta\in\Gamma$, then M is commutative.

Proof. Let $x\in I$. Since $t=x-\sigma(x) \in I$, we have $[t,m]_\alpha\alpha I\beta d(t) = 0$, for all $m\in M$ and $\alpha,\beta\in\Gamma$. For $t\in Sa_\sigma(M)$, we get $[t,m]_\alpha\alpha I\beta d(t) = \sigma([t,m]_\alpha)\alpha I\beta d(t) = 0$, for all $m\in M$ and $\alpha,\beta\in\Gamma$ and by Lemma 2.2, $d(t) =0$ or $[t,m]_\alpha = 0$, for all $m\in M$ and $\alpha\in\Gamma$.

Suppose that $d(t) = 0$. Then $d(x) = d(\sigma(x))$. Therefore, $[x,m]_\alpha\alpha I\beta d(x) = [x,m]_\alpha\alpha I\beta\sigma(d(x)) = 0$ and by Lemma 2.2, we have $d(x) = 0$ or $[x,m]_\alpha = 0$, for all $m\in M$ and $\alpha\in\Gamma$.i.e., either $d(x) = 0$ or $x\in Z(M)$. If $[t,m]_\alpha = 0$, for all $m\in M$ and $\alpha\in\Gamma$, then $t\in Z(M)$ and thus $[x,m]_\alpha =[\sigma(x),m]_\alpha$, for all $m\in M$ and $\alpha\in\Gamma$. Hence $[x,m]_\alpha\alpha I\beta d(x) = \sigma([x,m]_\alpha)\alpha I\beta d(x) = 0$. Again by Lemma 2.2, $d(x) = 0$ or $x\in Z(M)$.

Finally, for each $x\in I$ either $d(x) = 0$ or $x\in Z(M)$. Consider G_1,the set of all $x\in I$ such that $d(x) = 0$ and G_2, the set of all $x\in I$ such that $x\in Z(M)$. It is clear that G_1

and G_2 are additive subgroups of I and hence by Brauer's trictk, I = G_1 or I = G_2. If I = G_1, then d(x) = 0, for all $x \in I$. For any $n \in M$, we replace x by nαx in d(x) = 0 to get xαd(n) = 0 for all $x \in I$ and $\alpha \in \Gamma$ and so that Iαd(n) = 0, for all $n \in M$ and $\alpha \in \Gamma$. In particular, using (*), we get 1αIβd(n) = σ(1)αIβd(n) = 0, for all $n \in M$ and $\alpha,\beta \in \Gamma$. By Lemma 2.2, d = 0, a contradiction. Hence, I = G_2 so that $I \subseteq Z(M)$. Let $m,n \in M$ and $x \in I$. Then by (*), we have mαnβx = mαxβn = nαmβx. Now, from mαnβx = mαxβn = nαmβx, we have $[m,n]_\alpha \alpha I = 0$ and then $[m,n]_\alpha \alpha I\beta 1 = [m,n]_\alpha \alpha I \beta \sigma(1) = 0$. This gives $[m,n]_\alpha = 0$, for all $m,n \in M$ and $\alpha \in \Gamma$, by Lemma 2.2 and so M is commutative. □

Lemma 2.4 Let M be a 2-torsion free σ-prime Γ-ring which satisfies (*) and I a nonzero σ-ideal of M. If d is a derivation on such that d^2(I) = 0 and commutes with σ on M, then d = 0.

Proof. First suppose that d is nonzero. Let $m_0 \in M$ such that $d(m_0) \neq 0$. For any $x \in I$, we have $d^2(x) = 0$. Replacing x by xαy in $d^2(x) = 0$, we get

$$d^2(x)\alpha y + 2d(x)\alpha d(y) + x\alpha d^2(y), \quad (1)$$

for all $x,y \in I$ and $\alpha \in \Gamma$.

Using the facts that $d^2(x) = 0$ and M is 2-torsion free in (1), we get d(x)αd(y) = 0, for all $x,y \in I$ and $\alpha \in \Gamma$ so that d(x)αd(I) = 0. In particular, $d(x)\alpha d(y\beta m_0) = d(x)\alpha y \beta d(m_0) = 0$, for all $y \in I$ and $\alpha,\beta \in \Gamma$ and therefore $d(x)\alpha I \beta d(m_0) = 0$. Since d commutes with σ, the fact that I is a σ-ideal gives $\sigma(d(x))\alpha I \beta d(m_0) = 0$. Consequently $d(x)\alpha I\beta d(m_0) = \sigma(d(x))\alpha I \beta d(m_0) = 0$, for all $x \in I$ and $\alpha,\beta \in \Gamma$. By Lemma 2.2, we get

$$d(x) = 0, \quad (2)$$

for all $x \in I$.

Replacing x by $x\alpha m_0$ in (2), we get $x\alpha d(m_0)$, for all $x \in I$ and $\alpha \in \Gamma$ so that $I\alpha d(m_0) = 0$. In particular, $1\alpha I\beta d(m_0) = \sigma(1)\alpha I \beta d(m_0) = 0$ so that $d(m_0) = 0$, a contradiction. Consequently, d = 0. □

The main results state and prove as follows.

Theorem 2.1 Let M be a 2-torsion free σ-prime Γ-ring satisfying (*) and let d: M → M be a nonzero derivation. If $[d(x),x]_\alpha \in Z(M)$, for all $x \in M$ and $\alpha \in \Gamma$, then M is commutative.

Proof. Replacing x by x + y in $[d(x),x]_\alpha \in Z(M)$, we get

$$[d(x),y]_\alpha + [d(y),x]_\alpha \in Z(M), \quad (3)$$

Replacing y by xαx in (3) and using the fact that M is 2-torsion free, we have $x\alpha[d(x),x]_\alpha \in Z(M)$. Hence $[m,x]_\alpha \alpha [d(x),x]_\alpha = 0$, for all $x \in M$ and $\alpha \in \Gamma$. Replacing m by d(x) in $[m,x]_\alpha \alpha[d(x),x]_\alpha = 0$, we get $[d(x),x]_\alpha \alpha [d(x),x]_\alpha = 0$. Now, for $[d(x),x]_\alpha \in Z(M)$, we get $[d(x),x]_\alpha \alpha M \beta [d(x),x]_\alpha \alpha \sigma([d(x),x]_\alpha) = 0$, for all $x \in M$ and $\alpha,\beta \in \Gamma$. Since $[d(x),x]_\alpha \alpha \sigma([d(x),x]_\alpha) \in Sa_\sigma(M)$ and M is σ-prime, then $[d(x),x]_\alpha = 0$ or $[d(x),x]_\alpha \alpha \sigma([d(x),x]_\alpha) = 0$. Suppose that $[d(x),x]_\alpha \alpha \sigma([d(x),x]_\alpha) = 0$. Then by

$[d(x),x]_\alpha \in Z(M)$, we have $[d(x),x]_\alpha \alpha M\beta[d(x),x]_\alpha = [d(x),x]_\alpha \alpha M\beta\sigma([d(x),x]\alpha) = 0$ and by the semiprimeness of M , we get

$$[d(x),x]_\alpha = 0, \tag{4}$$

for all $x \in M$ and $\alpha \in \Gamma$ and so

$$[d(x),y]_\alpha + [d(y),x]_\alpha = 0, \tag{5}$$

for all $x,y \in M$ and $\alpha \in \Gamma$.
Replacing y by xαy in (5), we have

$$[d(x), x\alpha y]_\alpha + [d(x\alpha y),x]_\alpha = d(x)\alpha[x,y]_\alpha = 0, \tag{6}$$

for all $x,y \in M$ and $\alpha \in \Gamma$.
Replacing y by yβz in (6) and using (*), we have $d(x)\alpha y\beta[x,z]_\alpha = 0$, for all $x,y,z \in M$ and $\alpha,\beta \in \Gamma$ and hence $d(x)\alpha M\beta[x,z]_\alpha = 0$, for all $x,z \in M$ and $\alpha,\beta \in \Gamma$. In particular,

$$d(\sigma(x))\alpha M\beta[\sigma(x),\sigma(z)]_\alpha = \sigma(d(x))\alpha M\beta\sigma([x,z]_\alpha) = 0, \tag{7}$$

since d commutes with σ.
Applying σ in (7) and (*), we obtain $[x,z]_\alpha \alpha M\beta d(x) = 0$, for all $x,z \in M$ and $\alpha,\beta \in \Gamma$. Hence by Lemma 2.3, we can conclude that M is commutative. □

Theorem 2.2 Let M be a 2-torsion free σ-prime Γ-ring which satisfies (*) and I a nonzero σ-ideal of M. If d: M → M is a nonzero derivation such that $[d(x),d(y)]_\alpha = 0$, for all $x,y \in I$ and $\alpha \in \Gamma$ and commutes with σ, then M is commutative.
Proof. We have

$$[d(x),d(y)]_\alpha = 0, \tag{8}$$

for all $x,y \in I$ and $\alpha \in \Gamma$.
Replacing y by xαy in (8), we get

$$d(x)\alpha[d(x),y]_\alpha + [d(x),x]_\alpha \alpha d(y) = 0, \tag{9}$$

for all $x,y \in I$ and $\alpha \in \Gamma$.
Now, for $m \in M$, we replace y by yβm in (9) and use (*) to get

$$d(x)\alpha y\beta[d(x),m]_\alpha + [d(x),x]_\alpha = 0, \tag{10}$$

for all $x,y \in I$ and $\alpha,\beta \in \Gamma$.
Replacing m by d(z) in (10) and by (*), we have

$$[d(x),x]_\alpha \alpha y\beta d^2(z) = 0, \tag{11}$$

for all $x,y,z \in I$ and $\alpha,\beta \in \Gamma$.
Since d commutes with σ and I is a σ-ideal, (11) becomes $[d(x),x]_\alpha \alpha I\beta d^2(z) = \sigma([d(x),x]_\alpha)\alpha I\beta d^2(z) = 0$ and so by Lemma 2.2, we have either $d^2(z) = 0$, for all $z \in I$ or $[d(x),x]_\alpha = 0$, for all $x \in I$ and $\alpha \in \Gamma$. If $d^2(z) = 0$, for all $z \in I$, then by Lemma 2.4, we have d = 0, which is a contradiction. So suppose that

$$[d(x),x]_\alpha = 0, \tag{12}$$

for all $x \in I$ and $\alpha \in \Gamma$.
Replacing x by x + y in (12), we obtain

$$[d(x),y]_\alpha + [d(y),x]_\alpha = 0, \tag{13}$$

for all $x,y \in I$ and $\alpha \in \Gamma$.
Replacing y by yαx in (13), we have $[y,x]_\alpha \alpha d(x) = 0$ and so

$$[x,y]_\alpha \alpha d(x) = 0, \tag{14}$$

for all $x \in I$ and $\alpha \in \Gamma$.

For any $m \in M$, we replace y by $m\beta y$ and use (*), we obtain $[x,m]_\alpha \alpha y \beta d(x) = 0$, for all $x,y \in I$ and $\alpha,\beta \in \Gamma$ and so $[x,m]_\alpha \alpha I \beta d(x) = 0$, for all $x \in I, m \in M$ and $\alpha,\beta \in \Gamma$. Hence by Lemma 2.3, M is commutative. □

Theorem 2.3 Let M be a 2-torsion free σ-prime Γ-ring which satisfies (*) and I be a nonzero σ-ideal of M. Suppose that $d: M \to M$ is a nonzero derivation such that $d(x\alpha y) = d(y\alpha x)$, for all $x,y \in I$ and $\alpha \in \Gamma$ and d commutes with σ. Then M is commutative.

Proof. Let $x,y,z \in I$. Since $d[x,y]_\alpha = 0$, for all $x \in I$ and $\alpha \in \Gamma$, the condition $d([x,y]_\alpha \alpha z) = d(z\alpha[x,y]_\alpha)$ gives

$$[x,y]_\alpha \alpha d(z) = d(z)\alpha[x,y]_\alpha, \qquad (15)$$

for all $x,y,z \in I$ and $\alpha \in \Gamma$.

By condition $d(x\alpha y) = d(y\alpha x)$, for all $x,y \in I$ and $\alpha \in \Gamma$, we have $[d(x),y]_\alpha = [d(y),x]_\alpha$, for all $x,y \in I$ and $\alpha \in \Gamma$. In particular, $[d(x\alpha x),y]_\alpha = [d(y), x\alpha x]_\alpha$ and so

$$d(x)\alpha[x,y]_\alpha + [x,y]_\alpha \alpha d(x) = 0, \qquad (16)$$

for all $x,y \in I$ and $\alpha \in \Gamma$.

Since M is 2-torsion free, by (15) and (16), we have

$$[x,y]_\alpha \alpha d(x) = 0, \qquad (17)$$

for all $x,y \in I$ and $\alpha \in \Gamma$.

For any $m \in M$, we replace y by $m\beta y$ in (17) and use (*) to get $[x,m]_\alpha \alpha y \beta d(x) = 0$, for all $x,y \in I$ and $\alpha,\beta \in \Gamma$. Hence $[x,M]_\alpha \alpha I \beta d(x) = 0$, for all $x \in I$ and $\alpha,\beta \in \Gamma$ and by Lemma 2.3, we can arrive at the decision that M is commutative. □

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