Journal of Physical Sciences, Vol. 14, 2010, 173-180
ISSN: 0972-8791, www.vidyasagar.ac.in/journal
Published on December 24, 2010

Some Properties of Modular *n*-Ideals of a Lattice

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Received March 4, 2010; accepted October 25, 2010

ABSTRACT

An ideal M of a lattice L is called a modular ideal if for all ideals $I,\ J \in I(L)$ with $J \subseteq I$, the relation $I \cap (M \vee J) = (I \cap M) \vee J$ is satisfied. In this paper the authors have introduced the notion of modular n-ideals of a lattice. They have given several characterizations and properties of modular n-ideals when n is a neutral element in lattice L. They proved that the principal n-ideal $< s >_n$ is a modular n-ideal if and only if $s \wedge n$ and $s \vee n$ are modular elements in (n] and [n) respectively. Finally, they have characterized modular n-ideals with the help of relative n-annihilators.

Keywords: Modular n-ideal, Neutral element, Principal n-ideal, Relative annihilators, Relative n-annihilators

1. Introduction

Distributive, standard and neutral elements (ideals) of a lattice were studied extensively by Gratzer and Schmidt in [3], also see [2]. These elements are needed to study a larger class of non-distributive lattices. Again Talukder and Noor have introduced the notion of modular elements and ideals in [11] and [12] for directed below join semi lattices. On the other hand Noor and Latif have studied the standard n-ideals of a lattice in [9]. In a very recent paper [1] have studied the distributive n-ideals of a lattice. In this paper we have introduced the concept of modular n-ideals of a lattice and have included some of their characterizations.

An element m of a lattice L is called *modular* if for all $x, y \in L$ with $y \leq x$, $x \wedge (m \vee y) = (x \wedge m) \vee y$. On the other hand, Malliah and Bhatta in [5] have called an element m of a lattice modular if for all $x, y \in L$ with $x \leq y$,

$x \wedge m = y \wedge m$ and $x \vee m = y \vee m$ imply that $x = y$. It is easy to see that both the definitions are equivalent.

An ideal I of a lattice L is called *modular* if it is a modular element of the ideal lattice I(L). In [11] and [12] authors have given several characterizations of modular elements and modular ideals of a lattice.

By [2,3] an element s of a lattice L is called a *standard element* if $x \wedge (y \vee s) = (x \wedge y) \vee (x \wedge s)$ for all $x, y \in L$. It is called *neutral* if

(i) s is standard in L and

(ii) $s \wedge (x \vee y) = (s \wedge x) \vee (s \wedge y)$ for all $x, y \in L$.

s is called a *central element* if it is neutral and complemented in each interval containing it.

For a fixed element n of a lattice L, a convex sublattice containing n is called an n-*ideal*. The idea of n-ideals is a kind of generalizations of both ideals and filters of a lattice. The set of all n-ideals of a lattice L is denoted by $I_n(L)$, which is an algebraic lattice under set-inclusion. Moreover, $\{n\}$ and L are respectively the smallest and the largest elements of $I_n(L)$.

For any two n-ideals I and J of L, it is easy to check that $I \wedge J = I \cap J = \{x \in L : x = m(i, n, j)$ for some $i \in I, \ j \in J\}$, where $m(x, y, z) = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$ and $I \vee J = \{x \in L : i_1 \wedge j_1 \leq x \leq i_2 \vee j_2$, for some $i_1, \ i_2 \in I$ and $j_1, \ j_2 \in J\}$.

The n-ideal generated by a finite numbers of elements $a_1, a_2, ..., a_m$ is called a *finitely generated* n-*ideal*, denoted by $< a_1, a_2, ..., a_m >_n$. Moreover, $< a_1, a_2, ..., a_m >_n$ is the interval $[a_1 \wedge a_2 \wedge ... \wedge a_m \wedge n, \ a_1 \vee a_2 \vee ... \vee a_m \vee n]$. The n-ideal generated by a single element a is called a *principal* n-*ideal*, denoted by $< a >_n$ and $< a >_n = [a \wedge n, \ a \vee n]$.

The set of all principal n-ideals of a lattice L is denoted by $P_n(L)$. By [4] for a standard element $n \in L, P_n(L)$ is a meet semi lattice and $< a >_n \cap < b >_n = < m(a, \ n, \ b) >_n$. $P_n(L)$ is not necessarily a lattice. But if n is central, then $P_n(L)$ is a lattice. For detailed literature on n-ideals we refer the reader to consult [4], [8] and [9].

2. Modular n-Ideals of a Lattice

An n-ideal M of a lattice L is called a modular n-ideal if it is a modular element of the lattice $I_n(L)$. In other words M is called modular if for all $I, J \in I_n(L)$ with $J \subseteq I, I \cap (M \vee J) = (I \cap M) \vee J.$

We know from [11] that a lattice L is modular if and only if its every element is modular. Also from [4], we know that for a neutral element n of a lattice L, L is modular if and only if $I_n(L)$ is so. Thus, for a neutral element n, the lattice L is modular if and only if its every n-ideal is modular.

Following result gives a characterization of modular n-ideals of a lattice.

Theorem 2.1: $M \in I_n(L)$ *is modular if and only if for any* $a,b \in L$ *with* $<b>_n \subseteq <a>_n$, $<a>_n \cap (M \vee <b>_n) = (<a>_n \cap M) \vee <b>_n$.

Proof: Suppose M is modular. Then above relation obviously holds from the definition. Conversely, suppose $<a>_n \cap (M \vee <b>_n) = (<a>_n \cap M) \vee <b>_n$ for all $a,b \in L$ with $<b>_n \subseteq <a>_n$. Let $S, T \in I_n(L)$ with $T \subseteq S$. We need to show that $S \cap (M \vee T) = (S \cap M) \vee T$. Clearly $(S \cap M) \vee T \subseteq S \cap (M \vee T)$. To prove the reverse inclusion let $x \in S \cap (M \vee T)$. Then $x \in S$ and $x \in M \vee T$.
Then $m \wedge t \le x \le m_1 \vee t_1$ for some $m, m_1 \in M, t, t_1 \in T.$ Thus,
$x \vee n \le m_1 \vee t_1 \vee n$ which implies $x \vee n \in <m_1 \vee n>_n \vee <t_1 \vee n>_n$
$\subseteq M \vee <t_1 \vee n>_n$. Moreover, $x \vee n \in <x \vee t_1 \vee n>_n$ and
$<x \vee t_1 \vee n>_n \supseteq <t_1 \vee n>_n$. Hence by the given condition,
$x \vee n \in <x \vee t_1 \vee n>_n \cap (M \vee <t_1 \vee n>_n) =$
$(<x \vee t_1 \vee n>_n \cap M) \vee <t_1 \vee n>_n \subseteq (S \cap M) \vee T.$
By a dual proof of above we can easily see that $x \wedge n \in (S \cap M) \vee T$. Thus by convexity $x \in (S \cap M) \vee T$. Therefore, $S \cap (M \vee T) = (S \cap M) \vee T$, and so M is modular. □

Now we give another characterization of modular n-ideals when n is a neutral element in the lattice.

Therefore 2.2: *Suppose* n *is a neutral element of a lattice* L. *An* n-*ideal* M *is moudular if and only if for any* $x \in M \vee <y>_n$ *with* $<y>_n \subseteq <x>_n$, $x = (x \wedge m_1) \vee (x \wedge y) = (x \vee m_2) \wedge (x \vee y)$ *for some* $m_1, m_2 \in M$.

Proof: Suppose M is modular and $x \in M \vee <y>_n$. Then
$x \in <x>_n \cap (M \vee <y>_n) = (<x>_n \cap M) \vee <y>_n$. This implies
$p \wedge y \wedge n \le x \le q \vee y \vee n$ for some $p, q \in <x>_n \cap M$. By [6],
$q \in <x>_n \cap M$ implies that
$q = (x \wedge q) \vee (x \wedge n) \vee (q \wedge n) = (x \wedge (q \vee n)) \vee (q \wedge n)$. Thus,
$x \vee n \le (x \wedge (q \vee n)) \vee y \vee n \le x \vee n$, which implies
$x \vee n = (x \wedge (q \vee n)) \vee y \vee n = (x \wedge (q \vee n)) \vee (y \wedge (x \vee n)) \vee n =$
$(x \wedge (q \vee n)) \vee (x \wedge y) \vee n$, as n is neutral. Hence by the neutrality of n again,
$x = x \wedge (x \vee n) = x \wedge [(x \wedge (q \vee n)) \vee (x \wedge y) \vee n] =$
$(x \wedge [(x \wedge (q \vee n)) \vee (x \wedge y)]) \vee (x \wedge n) = (x \wedge (q \vee n)) \vee (x \wedge y) \vee (x \wedge n) =$
$(x \wedge (q \vee n)) \vee (x \wedge y)$, which is the first relation where $m_1 = q \vee n \in M$. A dual proof of above established the second relation.

Conversely, let $<y>_n \subseteq <x>_n$. By theorem 2.1, we need to show that $<x>_n \cap (M \vee <y>_n) = (<x>_n \cap M) \vee <y>_n$. Clearly R.H.S $\subseteq L.H.S$. To prove the reverse inclusion let $t \in <x>_n \cap (M \vee <y>_n)$. Then $t \in <x>_n$ and $t \in M \vee <y>_n$. Then $m \wedge y \wedge n \le t \le m_1 \vee y \vee n$ for some $m, m_1 \in M$.
Thus, $t \vee y \vee n \le m_1 \vee y \vee n$ and so $t \vee y \vee n \in M \vee <y \vee n>_n$ and
$<y \vee n>_n \subseteq <t \vee y \vee n>_n$. So by the given condition
$t \vee y \vee n = ((t \vee y \vee n) \wedge m') \vee (y \vee n)$ for some $m' \in M$. Since $t,\ y \in <x>_n$, so
$t \vee y \vee n \in <x>_n$. Moreover, by the neutrality of n,
$((t \vee y \vee n) \wedge m') \vee (y \vee n) = [(t \vee y \vee n) \wedge (m' \vee n)] \vee y =$
$m(t \vee y \vee n, n, m') \vee y \in (<x>_n \cap M) \vee <y>_n$. Therefore,
$t \vee y \vee n \in (<x>_n \cap M) \vee <y>_n$. By a dual proof we can show that
$t \wedge y \wedge n \in (<x>_n \cap M) \vee <y>_n$. Thus by the convexity,
$t \in (<x>_n \cap M) \vee <y>_n$. Therefore,
$<x>_n \cap (M \vee <y>_n) = (<x>_n \cap M) \vee <y>_n$ and so by theorem 2.1, M is modular. □

In [5], it has been proved that for a modular ideal M and an arbitrary ideal I if $I \vee M$ and $I \cap M$ are principal, then I is itself principal. Now we generalize this result for modular n-ideals. It should be mentioned that similar result on standard n-ideals has been proved by Noor and Latif in [10].

Theorem 2.3: *Let* n *be a neutral element of a lattice* L. *Suppose* M *is a modular* n-*ideal and* I *is any* n-*ideal of* L. *If* $M \vee I = <a>_n$ *and* $M \cap I = <b>_n$, *then* I *is principal*.

Proof: Here $M \vee I = <a>_n = [a \wedge n, a \vee n]$, then $a \vee n \leq m \vee i$ for some $m \in M$, $i \in I$. Since $m, i \leq a \vee n$, so $a \vee n = m \vee i$. Similarly $a \wedge n = m_1 \wedge i_1$ for some $m_1 \in M$ and $i_1 \in I$. Again,
$M \cap I = <b>_n$ implies $a \wedge n \leq b \leq a \vee n$. Thus,
$<a>_n = M \vee I \supseteq M \vee [b \wedge i_1 \wedge n, b \vee i \vee n] \supseteq [m_1 \wedge n, m \vee n] \vee [b \wedge i_1 \wedge n,$
$b \vee i \vee n] = [a \wedge n, a \vee n] = <a>_n$. This implies
$M \vee I = M \vee [b \wedge i_1 \wedge n, b \vee i \vee n]$. On the other hand,
$<b>_n = M \cap I \supseteq M \cap [b \wedge i_1 \wedge n, b \vee i \vee n] \supseteq M \cap <b>_n = <b>_n$ implies that
$M \cap I =$
$M \cap [b \wedge i_1 \wedge n, \ b \vee i \vee n]$. Since $[b \wedge i_1 \wedge n, \ b \vee i \vee n] \subseteq I$, So by the definition of modularity of M in [5],we have $I = [b \wedge i_1 \wedge n, \ b \vee i \vee n]$. Now by [4], we know that for a neutral element n, any finitely generated n-ideal contained in a principal n-ideal is principal. Since $[b \wedge i_1 \wedge n, \ b \vee i \vee n] \subseteq <a>_n$, so I is principal. □

Theorem 2.4: *If M is a modular* n-*ideal and* I *is any* n-*ideal of a lattice* L, *then* $I \cap M$ *is also modular in the sublattice* I.
Proof: Let *J, K* be any two n-ideals contained in I with $K \subseteq J$. Then
$J \cap [(I \cap M) \vee K] = J \cap [I \cap (M \vee K)]$, as M is modular and $K \subseteq I$. Thus,
$J \cap [(I \cap M) \vee K] = J \cap I \cap (M \vee K) = J \cap (M \vee K) = (J \cap M) \vee K$ (using the modularity of M again) = $(J \cap (I \cap M)) \vee K$. This implies $I \cap M$ is a modular n-ideal in I. □

Relative annihilators in lattices have been studied by many authors including Mandelker [6]. For $a, b \in L$, $<a, b> = \{x \in L : x \wedge a \leq b\}$ is known as *annihilator of* a *relative to* b, or simply a *relative annihilator*. In presence of distributivity, $<a, b>$ is an ideal of L.
Now we give a characterization of modular element of a lattice using relative annihilators.

Theorem 2.5 : *An element* $m \in L$ *is modular if and only if whenever* $b \leq a$, $x \in (b]$ *and* $m \in <a, b>$, *then* $x \vee m \in <a, b>$, $a, b, x \in L$.

Proof: Suppose m is modular. Since $m \in <a,b>$, so $a \wedge m \le b$. Also $x \le b \le a$. Thus by modularity of m, $a \wedge (m \vee x) = (a \wedge m) \vee x \le b$. This implies $m \vee x \in <a,b>$. Conversely, let the given condition holds. Suppose $x, z \in L$ with $z \le x$. Then $z \vee (m \wedge x) \le x$ and $z \in (z \vee (m \wedge x)]$. Also, $m \wedge x \le z \vee (m \wedge x)$ implies $m \in <x, z \vee (m \wedge x)>$. Then by the given condition, $z \vee m \in <x, z \vee (m \wedge x)>$. This implies $x \wedge (z \vee m) \le (m \wedge x) \vee z$. Since the reverse inequality is trivial, so m is a modular element. □

Theorem 2.6: *For an element* s *of a lattice* L, $<s>_n$ *is modular if and only if* $s \wedge n$ *and* $s \vee n$ *are*
modular in $(n]$ *and* $[n)$ *respectively.*

Proof: Let $s \wedge n$ and $s \vee n$ are modular in $(n]$ and $[n)$ respectively. Suppose $<b>_n \subseteq <a>_n$,
$a,b \in L$. Then $a \wedge n \le b \wedge n \le b \vee n \le a \vee n$. So, $<a>_n \cap (<s>_n \vee <b>_n) =$
$[a \wedge n, a \vee n] \cap [s \wedge b \wedge n, s \vee b \vee n] = (a \wedge n) \vee (s \wedge b \wedge n), (a \vee n) \wedge (s \wedge b \vee n) =$
$[(b \wedge n) \wedge ((s \wedge n) \vee (a \wedge n)), ((a \vee n) \wedge (s \vee n)) \vee (b \vee n)]$. Again,
$(<a>_n \cap <s>_n) \vee <b>_n =$
$[(a \wedge n) \vee (s \wedge n), (a \vee n) \wedge (s \vee n)] \vee [b \wedge n, b \vee n] =$
$[(b \wedge n) \wedge ((a \wedge n) \vee (s \wedge n)), ((a \vee n) \wedge (s \vee n)) \vee (b \vee n)]$. Thus
$<a>_n \cap (<s>_n \vee <b>_n) =$
$(<a>_n \cap <s>_n) \vee <b>_n$. Hence by Theorem 2.1, $<s>_n$ is modular.
Conversely let $<s>_n$ be modular. Suppose $n \le b \vee n \le a \vee n$. Then
$<b \vee n> \subseteq <a \vee n>$, and
$<a \vee n>_n \cap (<s>_n \vee <b \vee n>_n) = (<a \vee n>_n \cap <s>_n) \vee <b \vee n>_n$. Then by a routine calculation, $[n, (a \vee n) \wedge (s \vee b \vee n)] = [n, ((a \vee n) \wedge (s \vee n)) \vee (b \vee n)]$.
This implies $(a \vee n) \wedge ((s \vee n) \vee (b \vee n)) = ((a \vee n) \wedge (s \vee n)) \vee (b \vee n)$, and so $s \vee n$ is modular in $[n)$. Similarly $s \wedge n$ is also modular in $(n]$. □

In [7], Noor and Ayub has introduced the notion of relative n-annihilators. For $a,b \in L$ and a fixed element $n \in L$, $<a,b>^n = \{x \in L : m(a,n,x) \in <b>_n\} = \{x \in L : b \wedge n \le m(a,n,x) \le b \vee n\}$ is called *the annihilator of a relative to* b *around the element* n or simply a *relative* n-*annihilator.*

It is easy to see that for all $a,b \in L, <a,b>^n$ is always a convex subset containing n, but not necessarily an n-ideal. But in presence of distributivity of L,

$< a,b >^n$ is an n-ideal. Moreover $< a,b >^n = << a >_n, < b >_n >$,relative annihilator in $I_n(L)$.

We conclude the paper with the following characterization of modular n-ideals with the help of relative n-annihilators.

Theorem 2.7*: Let* n *be a neutral element in a lattice* L. *For an element* $s \in L$, *the following conditions are equivalent.*

i) $< s >_n$ *is modular,*

ii) *For* $< b >_n \subseteq < a >_n$ *and* $s \in < a,b >^n$ *implies* $s \wedge x, s \vee x \in < a,b >^n$ *for all* $x \in < b >_n$.

Proof: $(i) \Rightarrow (ii)$. Suppose (i) holds, $< b >_n \subseteq < a >_n$ and $s \in < a,b >^n$ .Then by Theorem 2.6, $s \vee n$ is modular in $[n)$. Also, $m(a,n,s) \in < b >_n$. Then $(a \wedge s) \vee (a \wedge n) \vee (s \wedge n) \le b \vee n$, which implies $a \wedge s \le b \vee n$. Thus, $m(a,n,s \vee b \vee n) = (a \vee n) \wedge (s \vee b \vee n) = (a \vee n) \wedge ((s \vee n) \vee (b \vee n)) = ((a \vee n) \wedge (s \vee n)) \vee (b \vee n) = (a \wedge s) \vee b \vee n = b \vee n$, as n is neutral. Hence $m(a,n,s \vee b \vee n) \in < b >_n$, and so $s \vee b \vee n \in < a,b >^n$. Again $s \wedge n$ is modular in $(n]$. So a similar proof shows that $s \wedge b \wedge n \in < a,b >^n$. Now for $x \in < b >_n$, $b \wedge n \le x \le b \vee n$. Then $s \wedge b \wedge n \le s \wedge x \le s \vee x \le s \vee b \vee n$ implies $s \wedge x, s \vee x \in < a,b >^n$, by convexity.

$(ii) \Rightarrow (i)$. Suppose (ii) holds and let $x, z \in [n)$ with $x \le z$. Then $x \vee ((s \vee n) \wedge z) \le z$, which implies $< x \vee ((s \vee n) \wedge z) >_n \subseteq < z >_n$. Now $x \le x \vee ((s \vee n) \wedge z)$ implies $x \in < x \vee ((s \vee n) \wedge z) >_n$. Again $(s \vee n) \wedge z) \le x \vee ((s \vee n) \wedge z)$ implies $m(s \vee n, n, z) = (s \vee n) \wedge z \in < x \vee ((s \vee n) \wedge z) >_n$. Hence $s \vee n \in < z, x \vee ((s \vee n) \wedge z) >^n$ .Thus by (ii), $s \vee n \vee x \in < z, x \vee ((s \vee n) \wedge z) >^n$ .That is, $(s \vee n \vee x) \wedge z \le x \vee ((s \vee n) \wedge z)$, which implies $s \vee n$ is modular in $[n)$. A dual proof of above shows that $s \wedge n$ is also modular in $(n]$. Hence by Theorem 2.6, $< s >_n$ is modular. □

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