B.Sc./2nd Sem (H)/MTM/24(CBCS)

2024

2nd Semester Examination

MATHEMATICS (Honours)

Paper : GE 2-T

[Algebra]

[CBCS]

Full Marks : 60 Time : Three Hours

The figures in the margin indicate full marks. Candidates are required to give their answers in their own words as far as practicable.

1. Answer any ***ten*** questions : 2×10=20

(a) Find the principal value of the complex number $z = -1 - i$.

(b) Show that the equation $x^3 - 3x^2 - 9x + 27 = 0$ has a multiple root.

(c) Use Descartes's rule of sign to show that the equation $x^4 - x + 3 = 0$ has no real root.

(d) Show that if a function $f \circ g$ is injective and g is surjective function, then f is an injective function.

(e) Find two integers u and v satisfying $54u + 24v = 30$.

P.T.O.

(f) If x be real, prove that $i\log\frac{1+ix}{1-ix}=-2\tan^{-1}x$.

(g) Prove that the product of any three consecutive integers is divisible by 6.

(h) Find the value of k for which the set $S = \{(k,1,1), (1,k,1), (1,1,k)\}$ is linearly independent.

(i) Do the polynomials $x^3 - 2x^2 + 1$, $4x^3 - x + 3$ and $3x - 2$ generate $P_3(R)$? Justify.

(j) Let P be an orthogonal matrix with $\det(P) = -1$. Prove that -1 is an eigen value of P.

(k) Find the greatest value of b^3a^2 where a, b are positive real number such that $a + b = 10$. Determine the values of a, b for which the greatest value is attained.

(l) Using Euler's theorem, find the unit digit of 3^{100}.

(m) Find the rank of the matrix $A=\begin{bmatrix}0&0&2\\1&3&2\\2&6&2\end{bmatrix}$.

(n) Express the matrix A^{-1} in terms of A, where

$$A=\begin{bmatrix}1&2&1\\1&-1&1\\2&3&-1\end{bmatrix}.$$

(o) If a, b, c, d be positive real numbers, not all equal, prove that $(a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)>16$.

2. Answer any ***four*** questions : 5×4=20

(a) Solve the equation by Cardan's method :

$$x^3 - 15x^2 - 33x + 847 = 0.$$

(b) Find the characteristic equation of the matrix

$$A=\begin{bmatrix}4&3&1\\2&1&-2\\1&2&1\end{bmatrix}.$$ Hence find A^{-1}.

(c) Find the dimension of the subspace S of R^3 defined by :

$$S=\{(x,y,z)\in R^3 : x+2y=z;\ 2x+3z=y\}.$$

(d) If $\left(2+\sqrt{3}\right)^n = I + f$ where I and n are positive integers and $0 < f < 1$, show that I is an odd integer and $(1 - f)(I + f) = 1$.

(e) If α, β, γ be the roots of the equation $x^3+3x+1=0$, find the equation whose roots are

$$\frac{\alpha}{\beta}+\frac{\beta}{\alpha},\ \frac{\beta}{\gamma}+\frac{\gamma}{\beta},\ \frac{\gamma}{\alpha}+\frac{\alpha}{\gamma}.$$

(f) Show that the map $T:\mathbb{R}^3 \to \mathbb{R}^3$ defined by, $T(x,y,z) = (x+y+z,\ 2x+y+2z,\ x+2y+z)$, $\forall x,\ y,\ z \in \mathbb{R}$ is linear. Find the matrix representation $[T]_B^{B'}$ of T with respect to the ordered bases $B = \{(1,0,0),\ (1,1,0),\ (1,1,1)\}$ and $B' = \{(1,1,0),\ (1,0,1),\ (0,1,1)\}$. 2+3

3. Answer any ***two*** questions : 10×2=20

(a) (i) Show that $\gcd(a+2,\ a) = 1$ or 2 for every integer a.

(ii) Verify whether the set $\{3x-1,\ x^3+1,\ x-3\}$ is a basis for the vector space $P_3(\mathbb{R})$.

(iii) Prove that $n^4 + 4^n$ is a composite number for all $n > 1$. 3+3+4

(b) (i) Use principle of mathematical induction and establish the formula for all integers $n \geq 1$,

$$1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{4n^3 - n}{3}.$$

(ii) Show that the eigen vectors corresponding to the distinct eigen values are linearly independent. 6+4

(c) (i) Using Cayley-Hamilton theorem for the matrix A, compute A^{50}.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

(ii) Show that the eigen value of an orthogonal matrix has a unit modulus. 5+5

(d) (i) If $d = \gcd(a,m)$, prove that

$$ax \equiv ay \pmod{m} \Leftrightarrow x \equiv y \left(\bmod \frac{m}{d}\right).$$

(ii) Solve by Ferrari's method :

$x^4 + 4x^3 - 6x^2 + 20x + 8 = 0.$ 4+6