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Lie and Jordan Structure in Simple Gamma Rings

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ABSTRACT

In this paper, we study Lie and Jordan structures in simple Γ-rings of characteristic not equal to two. Some properties of these Γ-rings are developed.

1. Introduction

The concepts of a Γ-ring was first introduced by Nobusausa [7] in 1964. He studied wedderburn's Theorem for Γ-ring with minimum one sided ideals. Now a day his Γ-ring is called a Γ-ring in the sense of Nobusausa. This Γ-ring is generalized by W.E. Barnes [1] in a broad sense that served now-a-day to call a Γ-ring.

J. Luh [3] studied on primitive Γ-rings with minimal one-sided ideals. Simple Γ-rings are also studied by him. S. Kyuno [5] worked on the structure of a Γ-ring with minimum condition. He obtained various results of the semi-prime Γ-rings.

In classical ring theories I. N. Herstein [4] studied the Lie and Jordan structure in simple rings.

In this paper, we generalized the results of I. N. Herstein [4] into Lie and Jordan structures in simple Γ-rings. We developed some characterizations of this Γ-rings.

2.1. Definitions

Gamma Ring. Let M and Γ be two additive abelian groups. Suppose that there is a mapping from $M \times \Gamma \times M \to M$ (sending (x, α, y) into $x\alpha y$) such that

i) $(x + y)\alpha z = x\alpha z + y\alpha z$
$x(\alpha + \beta)z = x\alpha z + x\beta z$
$x\alpha(y + z) = x\alpha y + x\alpha z$

ii) $(x\alpha y)\beta z = x\alpha(y\beta z)$,

where $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. Then M is called a Γ-ring.

Ideal of Γ-rings. A subset A of the Γ-ring M is a left (right) ideal of M if A is an additive subgroup of M and $M\Gamma A = \{c\alpha a \mid c\in M, \alpha\in\Gamma, a\in A\}(A\Gamma M)$ is contained in A. If A is both a left and a right ideal of M, then we say that A is an ideal or two sided ideal of M.

If A and B are both left (respectively right or two sided) ideals of M, then $A + B = \{a + b \mid a\in A, b\in B\}$ is clearly a left (respectively right or two sided) ideal, called the sum of A and B. We can say every finite sum of left (respectively right or two sided) ideal of a Γ-ring is also a left (respectively right or two sided) ideal.

Matrix Gamma Ring. Let M be a Γ-ring and let $M_{m,n}$ and $\Gamma_{n,m}$ denote, respectively, the sets of $m \times n$ matrices with entries from M and set of $n \times m$ matrices with entries from Γ, then $M_{m,n}$ is a $\Gamma_{n,m}$ ring and multiplication defined by $(a_{ij})(\gamma_{ij})(b_{ij}) = (c_{ij})$, where $c_{ij} = \sum_p \sum_q a_{ip}\gamma_{pq}b_{qj}$. If m = n, then M_n is a Γ_n-ring.

Identity element of a Γ-ring. Let M be a Γ-ring. M is called a Γ-ring with identity if there exists an element $e\in M$ such that $a\gamma e = e\gamma a = a$ for all $a\in M$ and some $\gamma\in\Gamma$.

We shall frequently denote e by 1 and when M is a Γ-ring with identity, we shall often write $1\in M$. Note that not all Γ-rings have an identity. When a Γ-ring has an identity, then the identity is unique.

Nilpotent element. Let M be a Γ-ring. An element x of M is called nilpotent if for some $\gamma\in\Gamma$, there exists a positive integer $n = n(\gamma)$ such that $(x\gamma)^n x = (x\gamma x\gamma ...\gamma x\gamma)x = 0$.

Nilpotent ideal. An ideal A of a Γ-ring M is called nilpotent if $(A\Gamma)^n A = (A\Gamma A\Gamma.. \ldots \ldots\Gamma A\Gamma)A = 0$, where n is the least positive integer.

Division gamma ring. Let M be a Γ-ring. Then M is called a division Γ-ring if it has an identity element and its only non zero ideal is itself.

Simple Γ-ring . A Γ-ring M is called a simple Γ-ring if $M\Gamma M \neq 0$ and its ideals are {0} and M.

Center of a Γ-ring. Let M be Γ-ring. The center of M, written as Z is the set of those elements in M that commute with every element in M, that is,

$$Z = \{m\in M \mid m\gamma x = x\gamma m \text{ for all } x\in M \text{ and } \gamma\in\Gamma\}.$$

3. Lie and Jordan structures

In this section we have developed some characterizations of Lie and Jordan structures in simple Γ-rings.

3.1 Theorem. Let M be a Γ-ring and $0 \neq P$ a right ideal of M. Suppose that, given $a\in P$, $(a\gamma)^n a = 0$, $\gamma\in\Gamma$ for fixed integer n; then M has a non-zero nilpotent ideal.

Proof. The argument we use is a variation of one given by Levitzki. We go by induction on n.

Let a ≠ 0 be in P satisfying aγa = 0; let A = aΓP. Suppose for the moment that A ≠ 0. If x∈M then $[(a+a\gamma x)\gamma]^n(a+a\gamma x)=0$, since it is in P, hence an expansion we get $[(a\gamma x)\gamma]^{n-1}(a\gamma x)\gamma a=0$. Thus $[(a\gamma x)\gamma]^{n-1}a\gamma x\,\Gamma A=0$. Let T = {x∈A| xΓA = 0}; of course, T is an ideal of A. Moreover, as we have just seen, y in A implies that $(y\gamma)^{n-1}y\in T$. Therefore $\overline{A}=A/T$ every element satisfies $(y\gamma)^{n-1}y=0$. By our induction hypothesis $\overline{A}$ has a nilpotent ideal $\overline{U}$ ≠ 0. Let U be its inverse image in A; since $(\overline{U}T)^k\overline{U}=0$, $(U\Gamma)^kU\subset T$, hence $(U\Gamma)^{k+1}U\subset T\Gamma A=0$. Also, since $\overline{U}\neq 0, U\not\subset T$ whence $U\supset U\Gamma A\neq 0$. But then $U\Gamma A=U\Gamma a\Gamma P\neq 0$ is a nilpotent right ideal of M.

Suppose then that a∈P, aγa = 0 implies that aΓP = 0. For any x∈P, since $(x\gamma)^n x=0$, we have $(x\gamma)^{n-1}x\gamma(x\gamma)^{n-1}x=0$ and so $(x\gamma)^{n-1}x\Gamma P=0$. Let; $W=\{x\in P|x\Gamma P=0\}$ W is an ideal of P. If W = P then PΓP = 0 and would provide us with a nilpotent right ideal. If W = P then in $\overline{P}=P/W$, $(\overline{x}\gamma)^n\overline{x}=0$; our induction gives us a nilpotent ideal $\overline{V}$ ≠ 0 in $\overline{P}$. If V is the inverse image of $\overline{V}$ in P then VΓP ≠ 0 ⊂ V and is nilpotent since V is. Again we have seen that M must have a non-zero nilpotent right ideal.

If M has a non-zero nilpotent right ideal it has (almost trivially) a non-zero nilpotent ideal. This proves the theorem.

Given any Γ-ring M we can induce on M, using its operations, two new structures, the Lie structure and the Jordan structure by defining the new products $[x, y]_\alpha = x\alpha y - y\alpha x$ and $(x, y)_\alpha = x\alpha y + y\alpha x$ for every, α∈Γ respectively. We propose to investigate the relationship between the associative structure of M and those induced Lie and Jordan structures.

We say that a subset A of M is a Lie sub-Γ-ring of M if A is an additive subgroup such that for a, b in A, aγb - bγa must also be in A for all γ∈Γ. Again a subset A of M is a Jordan sub-Γ-ring of M if A is an additive subgroup such that for a, b in A, aγb + bγa must also be in A for all γ∈Γ.

3.2 Definition. Let A be a Lie sub-Γ-ring of M. The additive subgroup U⊂A is to said to be a Lie ideal of A if whenever u∈U, a∈A, and α∈Γ then $[u, a]_\alpha = u\alpha a - a\alpha u$ is in U. Again, let A be a Jordan sub-Γ-ring of M. The additive subgroup U⊂A is to said to be a Jordan ideal of A if whenever u∈U, a∈A, and α∈Γ then $(u, a)_\alpha = u\alpha a + a\alpha u$ is in U.

Our first objective will be to determine the Lie and Jordan ideals of the Γ-ring M itself in the case when M is restricted to be a simple Γ-ring.

We begin with the Jordan ideals of M, which we a good ideal easier to characterize.

3.3 Theorem. If U is a Jordan ideal of M then for all a, b∈U, α∈Γ and x∈M, (aαb + bαa) αx - xα(aαb + bαa) ∈U.

Proof. Since a∈U, α∈Γ, for any x∈M, aα(xαb - bαx) + (xαb - bαx)α a in U. But aα(xαb - bαx) + (xαb - bαx)αa = {(aαx - xαa)αb +bα (aαx - xαa)} + {xα(aαb + bαa) - (aαb + bαa)αx}.
The left side and the first term on the right side are in U, hence xα(aαb + bαa) - (aαb + bαa)αx is also in U, proving the Theorem.

From this we now obtain the following Theorem :

3.4 Theorem. Let M be a Γ-ring in which 2x = 0 implies x = 0 and suppose further that M has no non-zero nilpotent ideals. Then any non-zero Jordan ideal of M contains a non-zero (associative) ideal of M.

Proof. Let U ≠ 0 be a Jordan ideal of M and suppose that a, b∈U. By Theorem 3.3, for any x∈M, α∈Γ xαc - cαx∈U where c = aαb + bαa. However, since c∈U, xαc + cαx∈U. Adding we get 2xαc∈U for all x, hence for y∈M, (2xαc)αy + yα(2xαc)∈U. Since 2yαxαc∈U we obtain 2xαcαy∈U, that is 2MΓcΓM⊂U. Now 2MΓcΓM is an ideal of M so we are done unless 2MΓcΓM = 0. If 2MΓcΓM = 0, by our assumptions MΓcΓM = 0 and so MΓcΓMΓc = 0. Since M has no nilpotent ideals this forces c = 0, that is, given a, b∈U then aαb + bαa = 0.

Let 0 ≠ a∈U; then for x∈M, α∈Γ, b = aαx + xαa∈U hence aα(aαx + xαa) + (aαx + xαa)αa = 0. That is, aαaαx + xαaαa + 2aαxαa = 0.Now for a∈U, 0 = aαa + aαa = 2aαa whence aαa = 0. The top relation the reduces to 2aαxαa = 0 for all x∈M, α∈Γ and so aΓMΓa = 0. But then aΓM ≠ 0 is a nilpotent right ideal of M, contrary to assumption. In other words, we have shown that U contains a non-zero ideal of M.

3.5 Corollary. If M is a simple Γ-ring of characteristic ≠ 2 then M is simple as a Jordan Γ-ring.

We now turn to the case of the Lie ideals of M.

3.6 Definition. If A, B are subsets of M then $[A, B]_\Gamma$ is the additive subgroup of M generated by all aαb - bαa with a, b∈B and α∈Γ.

3.7 Lemma. Let M be a Γ-ring with no non-zero nilpotent ideals in which 2x = 0 implies x = 0. Suppose that U ≠ 0 is both a Lie ideal and a sub-Γ-ring of M. Then either U⊂ Z or U contains a non-zero ideal of M.

Proof. Let us first suppose that U, as a Γ-ring, is not commutative. Then for some x, $y\in U$, $\gamma\in\Gamma$, $x\gamma y - y\gamma x \neq 0$. For any $m\in M$, all $\beta\in\Gamma$, $x\beta(y\gamma m) - (y\gamma m)\beta x$ is in U that is $(x\gamma y - y\gamma x)\beta m + y\beta(x\gamma m - m\gamma x)$ is in U. The second member of this is in U since both y and $x\gamma m - y\gamma x$ are in U (since U is both a Lie ideal and sub-Γ-ring). The net result of all this is that $(x\gamma y - y\gamma x)\Gamma M\subset U$. But then for m, $s\in M$ and $\alpha, \beta\in\Gamma$, $((x\gamma y - y\gamma x)\alpha m)\beta s - s\beta((x\gamma y - y\gamma x)\alpha m)\in U$ leading to $M\Gamma(x\gamma y - y\gamma x)\Gamma M\subset U$. We have now shown that the ideal $M\Gamma(x\gamma y - y\gamma x)\Gamma M$ is in U. If $M\Gamma(x\gamma y - y\gamma x)\Gamma M = 0$ then $M\Gamma(x\gamma y - y\gamma x)\Gamma M\Gamma(x\gamma y - y\gamma x)\Gamma M = 0$ contrary to assumption. We have shown that the result is correct if U as a sub-Γ-ring of M is not commutative.

So, suppose that U is commutative; we want to show that it lies in Z. Given $a\in U$, $x\in M$ then $a\gamma x - x\gamma a\in U$, so commutes with a. Now for x, $y\in M$, $a\gamma(a\gamma (x\gamma y) - (y\gamma x)\gamma a) = (a\gamma(x\gamma y) - (x\gamma y)\gamma a)\gamma a$. Expanding $a\gamma(x\gamma y) - (x\gamma y)\gamma a$ as $(a\gamma x - x\gamma a)\gamma y + x\gamma(a\gamma y - y\gamma a)$ and using that a commutes with this, with $a\gamma x - x\gamma a$ and with $a\gamma y - y\gamma a$ yields $2(a\gamma x - x\gamma a)\alpha\gamma (a\gamma y - y\gamma a) = 0$ for all x, $y\in M$ and $\alpha\in\Gamma$. Since $2m = 0$ forces $m = 0$ we obtain $(a\gamma x - x\gamma a)\alpha (a\gamma y - y\gamma a) = 0$. In this put $y = a\gamma x$, this results in $(a\gamma x - x\gamma a)\Gamma M\Gamma (a\gamma x - x\gamma a) = 0$. Since M has no nilpotent ideal we conclude that $a\gamma x - x\gamma a = 0$ and so, a must be in Z.

Note that in the latter part of the proof of Lemma 3.7 we have also proved the following sub-lemma:

3.8 Sub-lemma. Let M be a Γ-ring having no non-zero nilpotent ideals in which $2x = 0$ implies that $x = 0$.If $a\in M$ commutes with all $a\gamma x - x\gamma a$, $x\in M$ and $\gamma\in\Gamma$, then a is in Z.

Lemma 3.7 Immediately implies the following theorem :

3.9 Theorem. Let M be a simple Γ-ring of characteristic $\neq 2$. Then any Lie ideal of M which is also a sub-Γ-ring if M must either be M itself or contained in Z.

3.10 Definition. If U is a Lie ideal of M let $T(U) = \{x\in M \mid [x, M]_\Gamma \subset U\}$.

3.11 Lemma. For any Γ-ring M, if U is a Lie ideal of M, then T(U) is both a sub-Γ-ring and a Lie ideal of M; moreover $U\subset T(U)$.

Proof. Since U is a Lie ideal of M, $U\subset T(U)$; since $[T(U),M]_\Gamma \subset U\subset T(U)$, T(U) must certainly be a Lie ideal of M.

Now suppose that a, $b\in T(U)$, $m\in M$. Then $(a\gamma b)\gamma m - m\gamma(a\gamma b) =\{a\gamma(b\gamma m) - (b\gamma m)\gamma a + \{b\gamma(m\gamma a) - (m\gamma a)\gamma b\}$, so since a, $b\in T(U)$, the right side is in U. Therefore $[a\gamma b, M]_\Gamma \subset U$ that is $a\gamma b\in T(U)$.

We now prove the following theorem :

3.12 Theorem. Let M be a simple Γ-ring of characteristic ≠ 2 and let U be a Lie ideal of M. Then either U⊂Z or U⊃[M, M]$_\Gamma$.

Proof. By Theorem 3.9 and Lemma 3.11, since T(U) is both a sub-Γ-ring and a Lie ideal of M, either T(U)⊂Z or T(U) = M. If T(U) = M then by its very definition [M, M]$_\Gamma$ ⊂U; if T(U)⊂Z, since U⊂T(U), we obtain U ⊂ Z.

3.13 Corollary. If M is a non-commutative simple Γ-ring of characteristic ≠ 2 then the sub-Γ-ring generated by [M, M]$_\Gamma$ is M.

Proof. Any additive subgroup containing [M, M]$_\Gamma$ is, trivially, a Lie ideal of M. Hence the sub-Γ-ring generated by [M, M]$_\Gamma$ is a Lie ideal, thus by Theorem 3.9, it equals M or is in Z. If it is in Z then [M, M]$_\Gamma$ ⊂Z. Thus for a∈M, a commutes with all aγx - xγa, a∈M, γ∈Γ, by the sub-lemma 3.8, we get that a∈Z, that is, M⊂Z. Since we assume M to be non-commutative, that is ruled out; hence the corollary.

We now should like to settle the problem even when M has characteristic 2. Note that the characteristic of M has not entered into the discussion in the passage from Theorem 3.9 on. So we ask : when in characteristic 2, does Theorem 3.9 fail ?

If certainly fails in F_2, the matrix gamma ring of all 2 by 2 over F, a Γ-field of characteristic 2 for $U=\left\{\begin{pmatrix} a & b \\ b & a \end{pmatrix} \middle| a, b \in F\right\}$ is a Lie ideal and sub-Γ-ring of M which is neither in Z nor does it equal M. We aim to show that this is, effectively, the only counter-example.

Suppose that M is a simple Γ-ring of characteristic 2 and that U is a Lie ideal and sub-Γ-ring of M, U ≠ M and U ⊄ Z. As in the proof of Lemma 3.7 we obtain that U, as a sub-Γ-ring of M, must be commutative. That is, given u, v∈U then uγv + vγu = 0, γ∈Γ.

Let a∈U; then aγs + sγa∈U for all s∈M, γ∈Γ hence aγ(aγs + sγa) = (aγs + sγa)γa. This says that aγa∈Z. Since for any m∈M, aγm + mγa∈U we also have that (aγm + mγa)γ (aγm + mγa)∈Z.

If Z = 0 then aγa = 0, (aγm + mγa)γ (aγm + mγa) = 0 from which we get $\{(a\gamma m)\gamma\}^2$ (aγm) = 0. But then aΓM is a right ideal of M in which every element in the form $\{(a\gamma m)\gamma\}^2$ (aγm) is 0; by Theorem 3.1, M would have a nilpotent ideal, that is , M would be nilpotent, which is impossible for a simple Γ-ring.

Therefore we may assume that Z ≠ 0 and that there is an element a∈U, a∉Z such that aγa ≠ 0 ∈Z and (aγm + mγa)γ (aγm + mγa)∈Z for all m∈M, γ∈Γ.

To answer completely what the structure of M must be we prove a subsidiary Theorem :

3.14 Theorem. Let M be a simple Γ-ring of characteristic 2 and suppose that there exists an $a\in M$, $a\notin Z$ such that $a\gamma a\in Z$, $\gamma\in\Gamma$ and $[(a\gamma x+x\gamma a)\gamma]^3(a\gamma x+x\gamma a)\in Z$ for all $x\in M$ and $\gamma\in\Gamma$. Then M is 4-dimensional over Z.

Before proving the theorem we would like to point out that a more general theorem actually holds, namely : if M is a simple Γ-ring with an element $a\notin Z$ such that $[(a\gamma x-x\gamma a)\gamma]^{n-1}(a\gamma x-x\gamma a)\in Z$ for all $x\in M$ then M is 4-dimensional over Z.

Proof of theorem 3.14. If Z = 0 then both aγa = 0 and $[(a\gamma x+x\gamma a)\gamma]^3(a\gamma x+x\gamma a)=0$ hence $[(a\gamma x)\gamma]^4(a\gamma x) = a\gamma[(a\gamma x+x\gamma a)\gamma]^3(a\gamma x+x\gamma a)\gamma x=0$ for all $x\in M$. But then the right ideal aΓM satisfies $(u\gamma)^4u=0$ for all elements $u\in a\Gamma M$; by Theorem 3.1, this is not possible in a simple Γ-ring.

Suppose, then that Z ≠ 0, hence $1\in M$. If aγa = 0 then b = a + 1 satisfies bγb = 1 and $[(b\gamma x+x\gamma b)\gamma]^3(b\gamma x+x\gamma b)\in Z$ for all $x\in M$. Therefore we may assume that aγa = p ≠ 0 in Z. Let $Z'=Z(\sqrt{p})$ then $M'=M\otimes_Z\neq Z'$ is simple. Moreover, in M′ we have $[(a\gamma x'+x'\gamma a)\gamma]^3(a\gamma x'+x'\gamma a)\in Z'$ for all $x'\in M'$.

Since $\dim M'/Z'=\dim M/Z$, to prove the theorem it is enough to do so in M′. Also $b=a/q$ where $q\in Z'$, qγq = p satisfies bγb = 1 and $[(b\gamma x'+x'\gamma b)\gamma]^3(b\gamma x'+x'\gamma b)\in Z$. Hence, without loss of generality we may suppose that $a\in M$, $a\notin Z$, aγa = 1 and $[(a\gamma x+x\gamma a)\gamma]^3(a\gamma x+x\gamma a)\in Z$ for all $x\in M$.

Now M is a dense Γ-ring of linear Γ-transformations on a vector space V over a division Γ-ring Δ (since Z ≠ 0 and M is simple). Since (a + 1)γ(a + 1) = 0, a + 1 ≠ 0, V must be more than 1-dimensiononal over Δ. Since a ≠ 1 it is immediate that there is a $v\in V$ such that v, vγa are linearly Γ-independent over Δ.

If for some $w\in V$, v, vγa and wγ(1 + a) are linearly Γ-independent over Δ then the sub-Γ-space V_0 spanned by these is invariant under a and a induces the linear Γ-transformations $\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}$ on V_0. By density of M on V there is an $x\in M$ which induces $\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}$ on V_0 hence aγx + xγa induces $\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}$ on V_0.

But $[(a\gamma x + x\gamma a)\gamma]^3 (a\gamma x + x\gamma a) \in Z$ yet does not induce a scalar on V_0 since it induces $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Thus we have that for all w∈V, v, vγa and wγ(1 + a) are linearly Γ-dependent over Δ. If V is more than 2-dimensional over Δ, there is a w∈V such that v, vγa, w are linearly Γ-independent over Δ. By the above, wγa is in the sub-Γ-space V_1 they span. The matrix of a on V_1 is $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ p & q & r \end{pmatrix}$. By density there is an x∈M which induces $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ on V_1 ; but then aγx + xγa induces $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & p & 0 \end{pmatrix}$ where $[(a\gamma x + x\gamma a)\gamma]^3 (a\gamma x + x\gamma a)$ is not a scalar.

Thus we must have that V is 2-dimensional over Δ. All that remains is to show that Δ is commutative. Let $a = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$; then $a\Gamma_2 a = I_2$ where Γ_2 is the set of all 2 × 2 matrices gamma ring over Δ and I_2 is the identity matrix. Now we have $a\Gamma_2 a = I_2$.

Then $\begin{pmatrix} p & q \\ r & s \end{pmatrix}\begin{pmatrix} \gamma_{11} & \gamma_{11} \\ \gamma_{21} & \gamma_{22} \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Therefore $\begin{pmatrix} p\gamma_{11}p + q\gamma_{21}p + p\gamma_{12}r + q\gamma_{22}r & p\gamma_{11}q + q\gamma_{21}q + p\gamma_{12}s + q\gamma_{22}s \\ r\gamma_{11}p + s\gamma_{21}p + r\gamma_{12}r + s\gamma_{22}r & r\gamma_{11}p + s\gamma_{21}p + r\gamma_{12}q + s\gamma_{22}s \end{pmatrix}$
$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

It yields $p\gamma_{11}p + q\gamma_{21}p + p\gamma_{12}r + q\gamma_{22}r = 1$

$p\gamma_{11}q + q\gamma_{21}q + p\gamma_{12}s + q\gamma_{22}s = r\gamma_{11}p + s\gamma_{21}p + r\gamma_{12}r + s\gamma_{22}r = 0$

$r\gamma_{11}p + s\gamma_{21}p + r\gamma_{12}q + s\gamma_{22}s = 1$. In particular not both p = 0 and r = 0.

If t∈Δ then using $x = \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}$ and $[(a\Gamma_2 x + x\Gamma_2 a)\Gamma]^3 (a\Gamma_2 x + x\Gamma_2 a) \in Z$.

Now $a\Gamma_2 x + x\Gamma_2 a = \begin{pmatrix} p & q \\ r & s \end{pmatrix}\begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix}\begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix}$

$$= \begin{pmatrix} t\gamma_{11}p + t\gamma_{22}r & p\gamma_{11}t + q\gamma_{21}t + t\gamma_{12}q + t\gamma_{22}r \\ 0 & r\gamma_{11}t + s\gamma_{22}t \end{pmatrix}$$

Therefore $[(a\Gamma_2 x + x\Gamma_2 a)\Gamma_2]^3 (a\Gamma_2 x + x\Gamma_2 a) \in Z$. This gives for all t∈Δ, 4 times of $(t\gamma_{21}p + t\gamma_{22}r)$ and $(r\gamma_{11}t + s\gamma_{22}t)$ are in Z. If p ≠ 0, then $t\gamma_{21}p + t\gamma_{22}r$ runs through as t does, so every x∈Δ would satisfy $(x\Gamma_2)^3 x \in Z$. But a non-commutative division Γ-ring cannot be purely inseparable over its center. This p ≠ 0 implies Δ is commutative. Similarly r ≠ 0 implies Δ is commutative. Since one of these must hold we get that Δ is commutative and so M is 4-dimensional over Z.

Since the hypothesis of Theorem 3.14 is precisely the one lead to by the assumption that Theorem 3.9 (and so Theorem 3.12) was false we obtain.

3.15 Theorem. If M is a simple Γ-ring and if U is a Lie ideal of M then either U⊂Z or U⊃$[M, M]_\Gamma$ except if M is of characteristic 2 and is 4-dimensional over its center,

The theorem has as an immediate corollary the

3.16 Corollary. If M is a simple non-commutative Γ-ring then the sub-Γ-ring generated by $[M, M]_\Gamma$ is M.

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