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On Some Particular Connected Sums of Spaces

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ABSTRACT

Given two spaces *X* and *Y*, where both *X* and *Y* are either metric spaces or topological spaces, a connected sum *Z* of *X* and *Y* has been constructed. Some properties and several features of the constructed connected sums have been studied in this paper.

Keywords. Sum, connected sum, common extension, compatible spaces, locally compact, locally connected, path connected etc.

1. Introduction

Sums and connected sums are special operations to obtain some particular extensions of spaces and surfaces. Therefore, these are very interesting and important tools to construct new spaces and surfaces. Many researchers studied spaces and surfaces obtained by making a number of sums and connected sums. A sum of topological spaces has been defined and studied by S. Majumdar and M. Assaduzzaman in [5]. A number of sums of topological spaces have been defined and studied also in N. Bourbaki [1] and J. Dugundji [2]. Certain connected sums of spaces and surfaces have been studied in W. Massey [6] and M. A. Hossain [4]. In this paper, we have constructed connected sums for metric spaces and arbitrary topological spaces and studied some of their properties. Some terminology and notations of [4], [5], [6] are used here in general.

We now recall some definitions and results for convenience. If (X, T_1) and (X, T_2) are two topological spaces, their *sum* is the space $(X \cup Y = Z, T)$, where T is the topology in Z generated by $T_1 \cup T_2$. If both *X* and *Y* are open subspaces of *Z*, *Z* is called *usual extension* of *X* and *Y*. A usual extension exists if (i) $X \cap Y$ is open in both *X* and *Y*, and (ii) the class of all intersections of *X* with the open sets in *Y* is identical with the class of intersections of *Y* with the open sets in *X*, i.e., if $T_1 \cap Y = T_2 \cap X$. Two topological spaces (X, T_1) and (X, T_2) are called *compatible for sum* with each other if they satisfy (i) and (ii). A topological space X is said to be *locally compact* at a point x if there is some open set U containing x whose closure $\overline{U}$ is compact. The space is locally compact if it is locally compact at each of its points. A space X is said to be *locally connected* at a point x if for every open set U containing x there is a connected open set V containing x and contained in U. The space is locally connected if it is locally connected at each of its points. For two points x, y in a topological space X, a path joining x and y is a continuous map $f:[0,1]\to X$ such that $f(0)=x$, $f(1)=y$. The space X is path connected if any two points of X can be joined by a path.

2. Connected Sum of Metric Spaces

Let (X, d_1) and (Y, d_2) be two disjoint metric spaces such that the closed spheres $S = \overline{S_{r_1}(x_0)} = \{x \in X \,/\, d_1(x, x_0) \le r_1\}$ and $S' = \overline{S_{r_2}(y_0)} = \{y \in Y \,/\, d_2(y, y_0) \le r_2\}$ are homeomorphic. Let $f: S \to S'$ be a homeomorphism. If C and C' denote the boundary of S and S' respectively, then $C = \{x \in X \,/\, d_1(x, x_0) = r_1\}$ and $C' = \{y \in Y \,/\, d_2(y, y_0) = r_2\}$. Also suppose that f restricted to C, is not only a homeomorphism but also an isometry of C onto C', i.e., $d_1(x_1, x_2) = d_2(f(x_1), f(x_2))$ for every pair of points x_1, x_2 on C. Let $Z = (X - IntS) \cup (Y - IntS')$. Define a relation R on Z as follows:

i) for each $z \in Z - (C \cup C')$, $z\,R\,z$

ii) for each $z \in C$, $z\,R\,z$ and $z\,R\,f(z)$

iii) for each $z' \in C'$, $z'\,R\,z'$ and $z'\,R\,f^{-1}(z')$.

Then R is an equivalence relation on Z. Under identification topology $\frac{Z}{R}\ (=\overline{Z})$ is termed as the *connected sum* of X and Y, written $\overline{Z} = X\#Y$. We can regard $\overline{Z}$ as $X \cup Y - (Int\,S \cup Int\,S')$ under the identification of x with $f(x)$ for all $x \in C$. So, from now we will write $C = C' = (X - Int\,S) \cap (Y - Int\,S')$.

We shall now investigate whether $\overline{Z}$ inherits a metric from the metrics d_1 and d_2, or not. Before going to this investigation, we will prove the following theorem:

Theorem 1. *Let* (X, d_1) *and* (Y, d_2) *be two metric spaces such that* $X \cap Y \neq \Phi$ *and let* $d_1\big|_{(X\cap Y)\times(X\cap Y)} = d_2\big|_{(X\cap Y)\times(X\cap Y)}$. *Let us now define* $d:(X\cup Y)\times(X\cup Y)\to \boldsymbol{R}$ *by*

$$d(z,z') = \begin{cases} d_1(z,z') & if\ z, z' \in X \\ d_2(z,z') & if\ z, z' \in Y \\ \underset{c\in X\cap Y}{Inf}\ \{d_1(z,c)+d_2(c,z')\} & if\ z\in X\ and\ z'\in Y. \end{cases}$$

Then d is a metric in $X\cup Y$.

Proof. Since $d_1(z,z') = d_2(z,z')$ for all $(z,z')\in(X\cap Y)\times(X\cap Y)$, d is well defined. Since $d_1 \geq 0$ and $d_2 \geq 0$, so $d \geq 0$. Also, it is clear from the definition of d that $d(t,t) = 0$ for all $t\in X\cup Y$.

For any t_1, t_2 in $X\cup Y$,

$$d(z,z') = \begin{cases} d_1(t_1,t_2) = d_1(t_2,t_1)\ if\ t_1, t_2 \in X \\ d_2(t_1,t_2) = d_2(t_2,t_1)\ if\ t_1, t_2 \in Y \\ \underset{c\in X\cap Y}{Inf}\ \{d_1(t_1,c)+d_2(c,t_2)\}\ if\ t_1\in X\ and\ t_2\in Y \\ = \underset{c\in X\cap Y}{Inf}\ \{d_2(t_2,c)+d_1(c,t_1)\} \end{cases}$$

$$= d(t_2,t_1).$$

If for any three elements t_1, t_2 and t_3 in $X\cup Y$, $t_i\in X$ or $t_i\in Y$ $(i=1,2,3)$, then clearly $d(t_1,t_2)+d(t_2,t_3)\geq d(t_1,t_3)$. We prove the triangle property for the case when any two of the t_i's are in X or Y. Let $t_1,t_2\in X$ and $t_3\in Y$. Then

$$\begin{aligned} d(t_1,t_2)+d(t_2,t_3) &= d_1(t_1,t_2) + \underset{c\in X\cap Y}{Inf}\ \{d_1(t_2,c)+d_2(c,t_3)\} \\ &= \underset{c\in X\cap Y}{Inf}\ \{d_1(t_1,t_2)+d_1(t_2,c)+d_2(c,t_3)\} \\ &\geq \underset{c\in X\cap Y}{Inf}\ \{d_1(t_1,c)+d_{21}(c,t_3)\} \\ &= d(t_1,t_3). \end{aligned}$$

$$\text{i.e., } d(t_1,t_2)+d(t_2,t_3)\geq d(t_1,t_3).$$

Therefore d is a metric in $(X\cup Y)$ i.e., $(X\cup Y)$ is a metric space.

The metric on $\overline{Z}$: Let us now define $\overline{d}$ on $\overline{Z}$ by

$$\overline{d}(z_1,z_2)=\begin{cases} d_1(z_1,z_2) \ \ if \ z_1,z_2\in X \\ d_2(z_1,z_2) \ \ if \ z_1,z_2\in Y \\ \underset{c\in C}{Inf}\{d_1(z_1,c)+d_2(c,z_2)\} \ \ if \ z_1\in X \ and \ z_2\in Y, \end{cases}$$

C being as defined above.

Then $\overline{d}$ is well-defined, since $d_1\big|_{C\times C}=d_2\big|_{C\times C}$. Now by the Theorem 1, it follows clearly that $\overline{d}$ is a metric space on $\overline{Z}$. Therefore the connected sum *X#Y* of metric spaces *X* and *Y* is a metric space.

Theorem 2. *Let* (X, d_1) *and* (Y, d_2) *be two compact metric spaces, then the connected sum X#Y is compact.*

Proof. Let $\{G_\alpha\}$ be any open cover of *X#Y*. Then $\{G_\alpha\}$ is also an open cover of *X* and of *Y*. Since *X* and *Y* are compact, there exist finite sub-covers $\{G_{\alpha_i}\}$ and $\{G_{\alpha_j}\}$ of *X* and *Y* respectively. Then clearly $\{G_{\alpha_i}\}\bigcup\{G_{\alpha_j}\}$ is a finite sub-cover of *X#Y*. Therefore, *X#Y* is compact.

3. Connected Sum of Topological Spaces

Let (X, T) and (Y,T') be two topological spaces such that $X\cap Y\neq\Phi$. Suppose that there exists non-empty closed sets F and F' of *X* and *Y* respectively such that *F* is homeomorphic to F'. Let $f:F\to F'$ be a homeomorphism. Let $\overline{f}=f\big|_{b(F)}$ where $b(F)$ is the boundary of *F*. Then $\overline{f}$ is a homeomorphism $\overline{f}:b(F)\,(=B)\to b(F')\,(=B')$. Let $B=F-Int(F)$, $B'=F'-Int(F')$, and $Z=(X-Int(F))\cup(Y-Int(F'))$ where Z has the topoogy of a sum. Define a relation *R* on *Z* as follows:

i) for each $z\in Z-(B\cup B')$, $z\ R\ z$;

ii) for each $z\in B$, *z R z* and *z R f*(*z*);

iii) for each $z'\in B'$, $z'\ R\ z'$ and $z'\ R\ f^{-1}(z')$.

Then *R* is an equivalence relation on *Z*. Under identification topology, $\frac{Z}{R}\ (=\overline{Z})$ is thus a topological spaces and is termed the *connected sum* of *X* and *Y*. We can regard $\overline{Z}$ as $(X\cup Y)-(Int(F)\cup Int(F'))$ under the identification of *x* on *B* with *f*(*x*) on B'. Here, we regard

$B = B' = (X - Int(F)) \cap (Y - Int(F'))$. We shall denote the connected sum (in this case) of two topological spaces *X* and *Y* by $X\#_F Y$ or $X\#_{F'} Y$.

Theorem 3. *The connected sum* $X\#_{F'} Y$ *is connected if and only if both* $X - Int(F)$ *and* $Y - Int(F')$ *are connected.*

To establish the above theorem we first prove the following lemma, where *X* and *Y* are topological spaces such that $X \cap Y \neq \Phi$ so that $X \cup Y$ is a topological space and the topology is $\{G \cup H /\quad G \text{ open in } X, H \text{ open in } Y\}$.

Lemma 4. $X \cup Y$ *is connected if and only if both X and Y are connected and* $X \cap Y \neq \Phi$.

Proof. The result follows from a standard theorem since *X* and *Y* are subspaces of $X \cup Y$.

Proof of Theorem 3. Since $X\#_F Y$ is $(X - Int(F)) \cup (Y - Int(F'))$ and $b(F) = b(F') = (X - Int(F)) \cap (Y - Int(F'))$ is a non-empty subspace of $X\#_F Y$, the theorem follows from the direct consequence of the above lemma 4.

Lemma 5. *If X is locally compact and Y, a closed subspace of X then Y is locally compact.*

Proof. Let $y \in Y$. Since $y \in X$ and *X* is locally compact, there exists an open set *V* in *X* such that $y \in V$ and $\overline{V}$ is compact in *X*. Then $V \cap Y$ is open in Y and $y \in V \cap Y$. Let $\{W_\alpha\}$ be an open cover of $\overline{(V \cap Y)_Y}$ in *Y*. Then, for all α, $W_\alpha = U_\alpha \cap Y$, for some U_α which is open in *X*. Thus $\{U_\alpha\}$ is an open cover of $\overline{(V \cap Y)_Y}$ in *X*. Since $\overline{(V \cap Y)_X}$ is a closed subset of $\overline{V}$ in *X*, $\overline{(V \cap Y)_X}$ is compact in *X*. Hence there exist U_{α_1} ...... U_{α_n} such that $\overline{(V \cap Y)_Y} \subseteq U_{\alpha_1} \cup \cdots\cdots \cup U_{\alpha_n}$. Hence $\overline{(V \cap Y)_Y} \subseteq W_{\alpha_1} \cup \cdots\cdots \cup W_{\alpha_n}$. Therefore *Y* is locally compact.

Theorem 6. *If X and Y are locally compact then* $X\#_F Y$ *is locally compact.*

Proof. Let *X* and *Y* be locally compact spaces. Since $X\#_F Y$ can be written as $(X - Y) \cup C \cup (Y - X)$ where *C* = *b*(*F*), and since (*X*-*Y*), *C* and (*Y*-*X*), being subspaces of locally compact spaces X and Y, are locally compact by the above lemma 6. Hence $X\#_F Y$ is locally compact.

Lemma 7. *If X and Y are locally connected then the sum* $X \cup Y$ *is locally connected.*

Proof. Let $z \in X \cup Y$. If W is an open set in $X \cup Y$ with $z \in W$, then $W = U \cup V$ with U open in X and V open in Y. If $z \in U$, there exists connected open set U' in X with $z \in U'$ and if $z \in V$, there exists connected open set V' in Y with $z \in V'$. Since U' and V' are open in $X \cup Y$, so $X \cup Y$ is locally connected.

Lemma 8. *If X is locally connected and R is an equivalence relation on X, then the quotient space* $\frac{X}{R}$ *is locally connected.*

Proof. Let $\pi : X \to \frac{X}{R}$ denote the mapping given by $\pi(x) = cls\, x$ .Then π is continuous, open and onto. Let $x \in X$, and let $\overline{U}$ be an open set in $\frac{X}{R}$ such that $cls\, x \in \overline{U}$. Then $\overline{U} = \pi(U)$, for some open set U in X such that $x \in U$. Since X is locally connected, there exists a connected open set U' in X such that $x \in U'$. Then, $\pi(U')$ is a connected open set in $\frac{X}{R}$ and $cls\, x \in \pi(U')$. Hence $\frac{X}{R}$ is locally connected.

Let X and Y be locally connected spaces. We know that $X\#_F Y = \frac{X \cup Y}{R}$, where R is an equivalence relation defined as in the definition of $X\#_F Y$. Thus by the lemma 7 and lemma 8, it is clear that $X\#_F Y$ is locally connected. Hence we have the following result:

Theorem 9. *If X and Y are locally connected, then* $X\#_F Y$ *is also locally connected.*

Theorem 10. (i) *If X and Y are path connected, then* $X\#_F Y$ *is path connected.*

(ii) *If* $X\#_F Y$ *is path connected, then X- Int(F) and* $Y - Int(F')$ *are path connected.*

Proof. (i) Let X and Y be path connected. Let $z_1, z_2 \in X\#_F Y$. Since $X\#_F Y = (X - Int(Y)) \cup (Y - Int(F'))$, if z_1, z_2 both belong to $X - Int(F)$ or $Y - Int(F')$ then there always exists a path from z_1 to z_2. So, we consider the case when $z_1 \in X - Int(F)$ and $z_2 \in Y - Int(F')$. Since $b(F) \neq \Phi$, we can take a point $z \in b(F)$. Then $z \in X \cap Y$. So there are paths f from z_1 to z and g from z to z_2. Then $g * f$ is a path from z_1 to z_2, where $g * f : [0,1] \to (X - Int(F)) \cup (Y - Int(F'))$ is given by $(g * f)(t) = f(2t)$ whenever $0 \leq t \leq \frac{1}{2}$ and $(g * f)(t) = g(2t - 1)$ whenever $\frac{1}{2} \leq t \leq 1$. Hence $X\#_F Y$ is path connected.

(ii) It is obvious from the fact that $X\#_F Y = (X - Int(F)) \cup (Y - Int(F'))$.

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