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Finitely Generated n-ideals Which Form Relatively Normal Lattices

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ABSTRACT

Here the author give several characterizations of Relatively normal lattices in terms of n-ideals. They introduce the notion of relative n-annihilators in a lattice. They show that the lattices of finitely generated n-ideals $F_n(L)$ is relatively normal if and only if for any two incompearable prlme n-ideals P and Q , $P \vee Q = L$.

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1. Introduction:

Relative annihilators in lattices and semilattices have been studied by many authors including Mandelker [5] and Varlet [8]. Cornish in [2] has used the annihilators in studying relative normal lattices.

For a, $b \in L$, $< a, b > = \{x \in L : x \wedge a \leq b\}$ is known as *annihilator of* a *relative* to b, or simply a *relative annihilator*. It is very easy to see that in presence of distributivity, $< a, \quad b >$ is an ideal of L. Again for a, $b \in L$ we define $< a, b >_d = \{x : x \vee a \geq b\}$, which we call a *dual annihilator of* a *relative to* b, or simply a *relative dual annihilator*. In presence of distributivity of L, $< a, b >_d$ is a dual ideal (filter). .For an element $n \in L$, a convex sublattice containing n is called an n-*ideal*. n-ideal generated by a finite number of elements a_1,........,a_r is called a finitely generated n-ideal, denoted by $\langle a_1, \ldots\ldots, a_r \rangle_n$. Set of all finitely generated n-ideals is a lattice, denoted by $F_n(L)$. n-ideal generated by a single element is called a principal n-ideal. Set of all principal n-ideals is denoted by $P_n(L)$. Moreover, $\langle a_1, \ldots\ldots., a_r \rangle_n = [a_1 \wedge \ldots\ldots. \wedge a_r \wedge n, \; a_1 \vee \ldots\ldots\ldots \vee a_r]$
$= \{x \in L \mid a_1 \wedge \ldots\ldots \wedge a_r \wedge n \leq x \leq a_1 \vee \ldots\ldots \vee a_r \vee n\}$ and $\langle a \rangle_n = [a \wedge n, a \vee n]$

For two finitly generated n-ideals $[a,b]$ and $[c,d]$, $[a,b]\wedge[c,d]=[a\vee c, b\wedge d]$ and $[a,b]\vee[c,d]=[a\wedge c, b\vee d]$. For a, $b\in L$ and a fixed element $n\in L$, we define
$<a,b>^n=\{x\in L: m(a,n,x)\in <b>_n\}=\{x\in L: b\wedge n\le m(a,n,x)\le b\vee n\}$.We call $<a,b>^n$ *the annihilator of* a *relative* to b around the element n or simply a *relative* n- *annihilator*. It is easy to see that for all $a,b\in L$, $<a,b>^n$ is always a convex subset containing n . In presence of distributivity, it can be easily seen that $<a,b>^n$ is an n-ideal. For two n-ideals A and B of a lattice L, $\langle A,B\rangle$ denotes $\{x\in L: m(a,n,x)\in B\}$ for all $a\in A$. In presence of distributivity, clearly $\langle A,B\rangle$ is an n-ideal. Moreover, we can easily show that
$<a,b>^n=<<a>_n, <b>_n>$. $<a,b>^n=<<a>_n,<b>_n>$. Recently [6] has studied relative n-annihilators extensively .

A distributive lattice with 0 is a *normal lattice* if its every prime ideal contains a unique minimal prime ideal. A distributive lattice L is called a *relatively normal lattice* if its every interval [a, b] is normal.

In this paper we characterize those $F_n(L)$ which are relatively normal in terms of n-ideals and relative n- annihilators. These results are generalizations of several results on relatively normal lattices. We show that $F_n(L)$ is relatively normal if and only if any two incomparable prime n-ideals of L are comaximal.

We start the paper with the following result on n-ideals due to [4].

Lemma 1.1: For $n\in L$, $F_n(L)\cong(n]^d\times[n)$. ■

Following result is also essential for the development of the paper, whch is due to [1,Theorem 2.1.12].

Lemma 1.2 : Let I and J be two n-ideals of a distriutive lattice. Then for any $x\in I\vee J$, $x\vee n=i_1\vee j_1$ and $x\wedge n=i_2\wedge j_2$ for some $i_1,i_2\in I$, $j_1,j_2\in J$, with $i_1,j_1\ge n$, and $i_2,j_2\le n$, ■

Now we include the following result which is due to [6] and is a generalization of [2, lemma 3.6].

Theorem 1.3 *Let* L *be a distributive lattice. Then the following hold*

(i) $\langle\langle x\rangle_n\vee\langle y\rangle_n,\langle x\rangle_n\rangle=\langle\langle y\rangle_n,\langle x\rangle_n\rangle$;

(ii) $\langle\langle x\rangle_n,J\rangle=\bigvee_{y\in J}\langle\langle x\rangle_n,\langle y\rangle_n\rangle$, *the supremum is taken in the lattice of n-ideals of* L, *for any* $x\in L$ *and any* n-*ideal* J. ■

Following lemmas will be needed for further development of this paper. Lemma 1.4 is the dual of [2,lemma 3.6]. .Moreover, lemma 1.5 and lemma 1.6 are due to
[6].We prefer to omit the proof as they are easy to prove.

Lemma 1.4 *Let* L *be a distributive lattice. Then the following hold.*

(i) $\langle x \wedge y, x\rangle_d = \langle y, x\rangle_d$;

(ii) $\langle [x), F\rangle_d = \bigvee_{y \in F} \langle x, y\rangle_d$, *where* F *is a filter of* L .

(iii) $\{\langle x, a\rangle_d \vee \langle y, a\rangle_d\} \cap [a, b]$
$= \{\langle x, a\rangle_d \cap [a, b]\} \vee \{\langle y, a\rangle_d \cap [a, b]\}$. ■

Lemma 1.5 *Let* L *be a distributive lattice with* $n \in L$, *Suppose* a, b, c $\in$ L.

i) *If* a, b, c $\geq n$, *then* $\langle\langle m\,(a, n, b)\rangle_n, \langle c\rangle_n\rangle$
$= \langle\langle a\rangle_n, \langle c\rangle_n\rangle \vee \langle\langle b\rangle_n, \langle c\rangle_n\rangle$ *is equivalent to*
$\langle a \wedge b, c\rangle = \langle a, c\rangle \vee \langle b, c\rangle$;

ii) *If* $a, b, c \leq n$ *then*
$\langle\langle m\,(a,n,b)\rangle_n, \langle c\rangle_n\rangle = \langle\langle a\rangle_n, \langle c\rangle_n > \vee \langle\langle < b\rangle_n, \rangle \langle c\rangle_n\rangle$ *is equivalent to*
$\langle a \vee b, c\rangle_d = \langle a, c\rangle_d \vee \langle b, c\rangle_d$ ■

Lemma 1.6 *Let* L *be a distributive lattice with* $n \in L$, *Suppose* a, b, c, $\in L$.

(i) *For* $a, b, c \geq n$,
$<<c>_n, <a>_n \vee <b>_n> = <<c>_n, <a>_n> \vee <<c>_n, <b>_n>$

is equivalent to $\langle c, a \vee b\rangle = \langle c, a\rangle \vee \langle c, b\rangle$;

(ii) *For* $a, b, c \leq n$, $\langle\langle c\rangle_n, \langle a\rangle_n \vee \langle b\rangle_n\rangle$
$= \langle\langle c\rangle_n, \langle a\rangle_n\rangle \vee \langle\langle c\rangle_n, \langle b\rangle_n\rangle$ *is equivalent to*
$\langle c, a \wedge b\rangle_d = \langle c, a\rangle_d \vee \langle c, b\rangle_d$. ■

A distributive lattice L with 1 is called a dual normal lattice if L^d is a normal lattice. In other words, a distributive lattice L with 1 is called *dual normal* if every prime filter of L is contained in a unique ultrafilter (maximal and proper) of L.

In fact, this condition in a lattice is self-dual. Thus for a bounded distributive lattice, the concept of normality and dual normality coincides.

Following the technique of the proof of [2, Theorem 2.4], we can similarly prove the following result, which gives some characterizations of dual normal lattices.

Let x be any element of a lattice L with 0. We denote $(x]^{*} = \{y \in L : y \wedge x = 0\}$. In presence of distributivity this is an ideal. Similarly for a lattice L with 1, we denote
$[x)^{*d} = \{y \in L : y \vee x = 1\}$.

Theorem 1.7 *Let* L *be a distributive lattice with* 1. *Then the following conditions are equivalent.*

(i) L *is dual normal ;*

(ii) *Each prime filter of* L *is contained in a unique ultrafilter (maximal and proper) ;*

(iii) *For each* $x, y \in L, [x \vee y)^{*d} = [x)^{*d} \vee [y)^{*d}$;

If $x \vee y = 1, x, y \in L$, *Then* $[x)^{*d} \vee [y)^{*d} = L$ ■

Corollary 1.8 L *be a bounded distributive lattice. Then the following conditions are equivalent.*

(i) L *is normal*

(ii) *For each* $x, y \in L, (x \wedge y]^{*} = (x]^{*} \vee (y]^{*}$

(iii) *If* $x \wedge y = 0$, *Then* $(x]^{*} \vee (y]^{*} = L$

(iv) *For each* $x, y \in L,\ [x \vee y)^{*d} = [x)^{*d} \vee [y)^{*d}$

(v) *If* $x \vee y = 1$, *then* $[x)^{*d} \vee [y)^{*d} = L$.

Recall that a distributive lattice L is *relatively normal* if each interval [x, y] with x < y $(x, y \in L)$ is a normal lattice.

Since for a bounded distributive lattice the concept of normality and dual normality coincides, so the concept of relative normality is self-dual in any distributive lattice.

Following result is due to [2, Theorem 3.7].

Theorem 1.9 *Let* L *be a distributive lattice. Let* $a, b, c \in L$ *be arbitrary elements and* A, B *be arbitrary ideals. Then the following are equivalent*.

i) L *is relatively normal*,

ii) $\langle a,b\rangle \vee \langle b,a\rangle = L$,

iii) $\langle c, a \vee b\rangle = \langle c,a\rangle \vee \langle c,b\rangle$,

iv) $\langle (c], A \vee B\rangle = \langle (c], A\rangle \vee \langle (c], B\rangle$,

v) $\langle a \wedge b, c\rangle = \langle a,c\rangle \vee \langle b,c\rangle$.

Now we include the following result due to [6] whose technique of proof is dual to [2, Theorem 3.7].

Theorem 1.10 *Let* L *be a distributive lattice. Let* $a,b,c \in L$ *be arbitrary elements and A, B arbitrary filters. Then the following are equivalent:*

(i) L *is relatively normal*

(ii) $\langle a, b\rangle_d \vee \langle b, a\rangle_d = L$;

(iii) $\langle c, a \wedge b\rangle_d = \langle c, a\rangle_d \vee \langle c, b\rangle_d$;

(iv) $<[c), A \vee B>_d = <[c), A>_d \vee <[c), B>_d$;

(v) $\langle a \vee b, c\rangle_d = \langle a, c\rangle_d \vee \langle b, c\rangle_d$. ■

Now we prove our main results of this paper which are generalizations of [2, Theorem 3.7] and [5, Theorem 5] . These give characterizations of those $F_n(L)$ and $P_n(L)$ which are relatively normal.

Theorem 1.11. *Let* $F_n(L)$ *be distributive lattice and* A *and* B *be two n-ideals of* L, *Then for all* $a,b,c \in L,$ *the following conditions are equivalent.* $F_n(L)$ *is relatively normal.*

(i) $\langle\langle a\rangle_n, \langle b\rangle_n\rangle \vee \langle\langle b\rangle_n, \langle a\rangle_n\rangle = L$;

(ii) $\langle\langle c\rangle_n, \langle a\rangle_n \vee \langle b\rangle_n\rangle = \langle\langle c\rangle_n, \langle a\rangle_n\rangle \vee \langle\langle c\rangle_n, \langle b\rangle_n\rangle$;

(iii) $\langle\langle c\rangle_n, A \vee B\rangle = \langle\langle c\rangle_n, A\rangle \vee \langle\langle c\rangle_n, B\rangle$;

(iv) $\langle\langle m(a,n,b)\rangle_n, \langle c\rangle_n\rangle = \langle\langle a\rangle_n, \langle c\rangle_n\rangle \vee \langle\langle b\rangle_n, \langle c\rangle_n\rangle$;

Proof: (i) $\Rightarrow$ (ii). Let $z \in L$, consider the interval $I = \lfloor \langle a\rangle_n \cap \langle b\rangle_n \cap \langle z\rangle_n, \langle z\rangle_n \rfloor$ in $F_n(L)$. Then $\langle a\rangle_n \cap \langle b\rangle_n \cap \langle z\rangle_n$ is the smallest element of the interval I. By (i), I is normal, then by [2,Theorem 2.4] there exists finitely generated -ideals [p, q] , [r, s] $\in I$ such that. $\langle a\rangle_n \cap \langle z\rangle_n \cap [p, q]$ $= \langle a\rangle_n \cap \langle b\rangle_n \cap \langle z\rangle_n = \langle b\rangle_n \cap \langle z\rangle_n \cap [r, s]$ and $<z>_n = [p, q] \vee [r, s]$

Now,

$<a>_n \cap [p, q] = <a>_n \cap [p, q] \cap <z>_n$

$= <a>_n \cap <b>_n \cap <z>_n \subseteq <b>_n$ implies

$[p, q] \subseteq << a>_n, <b>_n>$. Also $<b>_n \cap [r, s] = <b>_n \cap <z>_n \cap [r, s]$

$= <a>_n \cap <b>_n \cap <z>_n \subseteq <a>_n$ implies $[r, s] \subseteq << b>_n, <a>_n>$.Thus

$\langle z\rangle_n \subseteq \langle\langle a\rangle_n, \langle b\rangle_n\rangle \vee \langle\langle b\rangle_n, \langle a\rangle_n\rangle$, and so $z \in \langle\langle a\rangle_n, \langle b\rangle_n\rangle \vee \langle\langle b\rangle_n, \langle a\rangle_n\rangle$

Hence $\langle\langle a\rangle_n, \langle b\rangle_n\rangle \vee \langle\langle b\rangle_n, \langle a\rangle_n\rangle = L$.

(ii) $\Rightarrow$ (iii). Suppose (ii) holds. For (iii), R. H. S. $\subseteq$ L. H. S. is obvious. Now, let $z \in \langle \langle c \rangle_n , \langle a \rangle_n \vee \langle b \rangle_n \rangle$. Then $z \vee n \in \langle \langle c \rangle_n , \langle a \rangle_n \vee \langle b \rangle \rangle$, and so $m(z \vee n, n, c) \in [a \wedge b \wedge n, a \vee b \vee n]$. That is, $(z \vee n) \wedge (c \vee n) \leq a \vee b \vee n$. Now by (ii), $z \vee n \in \langle \langle a \rangle_n , \langle b \rangle_n \rangle \vee \langle \langle b \rangle_n , \langle a \rangle_n \rangle$.

So $z \vee n \leq (p \vee n) \vee (q \vee n)$ for some $p \vee n \in \langle \langle a \rangle_n , \langle b \rangle_n \rangle$ and $q \vee n \in << b >_n , < a >_n >$

Hence, $z \vee n = ((z \vee n) \wedge (p \vee n)) \vee ((z \vee n) \wedge (q \vee n)) = r \vee s$ (say)

Now, $m(p \vee n, n, a) = (p \vee n) \wedge (a \vee n) \leq b \vee n$. So

$(b \wedge n) \leq r \wedge (a \vee n) \leq b \vee n$.

Hence, $r \wedge (c \vee n) = r \wedge (z \vee n) \wedge (c \vee n) \leq r \wedge (a \vee b \vee n)$

$= (r \wedge (a \vee n)) \vee (r \wedge (b \vee n)) \leq b \vee n$.

This implies $r \in \langle \langle c \rangle_n , \langle b \rangle_n \rangle$, similarly, $s \in \langle \langle c \rangle_n , \langle a \rangle_n \rangle$.

Hence $z \vee n \in \langle \langle c \rangle_n , \langle a \rangle_n \rangle \vee \langle \langle c \rangle_n , \langle b \rangle_n \rangle$.

Again $z \in \langle \langle c \rangle_n , \langle a \rangle_n \vee \langle b \rangle_n \rangle$ implies

$z \wedge n \in \langle \langle c \rangle_n , \langle a \rangle_n \vee \langle b \rangle_n \rangle$ Then a dual calculation of above shows that $z \wedge n \in \langle \langle c \rangle_n , \langle a \rangle_n \rangle \vee \langle \langle c \rangle_n , \langle b \rangle_n \rangle$.

Thus by convexity, $z \in \langle \langle c \rangle_n , \langle a \rangle_n \rangle \vee \langle \langle c \rangle_n , \langle b \rangle_n \rangle$ and so (iii) holds.

(iii) $\Rightarrow$ (iv). Suppose (iii) holds. In (iv), R. H. S. $\subseteq$ L. H. S is obvious. Now let $x \in \langle \langle c \rangle_n , A \vee B \rangle$. Then $x \vee n \in \langle \langle c \rangle_n , A \vee B \rangle$.

Thus $m(x \vee n, n, c) \in A \vee B$.

Now $m(x \vee n, n, c) = (x \vee n) \wedge (n \vee c) \geq n$ implies $m(x \vee n, n, c) \in (A \vee B) \cap [n)$.

Hence by Theorem 1.3 (ii), $x \vee n \in \langle \langle c \rangle_n , (A \cap [n)) \vee (B \cap [n)) \rangle$

$= \bigvee_{r \in (A \cap [n)) \vee (B \cap [n)))} \langle \langle c \rangle_n , \langle r \rangle_n \rangle$.

But by Theorem 1.2, $r \in (A \cap [n)) \vee (B \cap [n))$ *implies*

$r = s \vee t$ *for some* $s \in A, t \in B$ *and* $s, t \geq n$.

They by (iii), $\langle \langle c \rangle_n , \langle r \rangle_n \rangle = \langle \langle c \rangle_n , \langle s \vee t \rangle_n \rangle$

$= \langle \langle c \rangle_n , \langle s \rangle_n \vee \langle t \rangle_n \rangle = \langle \langle c \rangle_n , \langle s \rangle_n \rangle \vee \langle \langle c \rangle_n , \langle t \rangle_n \rangle$

$\subseteq \langle \langle c \rangle_n , A \rangle \vee \langle \langle c \rangle_n , B \rangle$.

Hence $x \vee n \in \langle \langle c \rangle_n , A \rangle \vee \langle \langle c \rangle_n , B \rangle$.

Also $x \in \langle\langle c\rangle_n, A \vee B\rangle$ implies $x \wedge n \in \langle\langle c\rangle_n, A \vee B\rangle$.
Since $m(x \wedge n, \; n, c) = (x \wedge n) \vee (n \wedge c) \leq n$, So $x \wedge n \in << c >_n, (A \vee B) \cap (n] >$ Then by Theorem 1.3 (ii), $x \wedge n \in << c >_n, (A \cap (n]) \vee (B \cap (n]) >$

$$= \bigvee_{t \in (A \cap (n]) \vee (B \cap (n])} << c >_n, < t >_n > .$$

Using Theorem 1.2 again, we se that $t = p \wedge q$ where $p \in A, q \in B, p, q \leq n$.
Then by (iii), $\langle\langle c\rangle_n, \langle t\rangle_n\rangle = \langle\langle c\rangle_n, \langle p \wedge q\rangle_n\rangle$

$$= \langle\langle c\rangle_n, \langle p\rangle_n \vee \langle q\rangle_n\rangle$$

$$= \langle\langle c\rangle_n, \langle p\rangle_n\rangle \vee \langle\langle c\rangle_n, \langle q\rangle_n\rangle$$

$$\subseteq \langle\langle c\rangle_n, A\rangle \vee \langle\langle c\rangle_n, B\rangle$$

Hence $x \wedge n \in \langle\langle c\rangle_n, A\rangle \vee \langle\langle c\rangle_n, B\rangle$. Therefore by Convexity, $x \in \langle\langle c\rangle_n, A\rangle \vee \langle\langle c\rangle_n, B\rangle$. and so (iv) holds.

(iv) $\Rightarrow$ (iii) is trivial.

(ii) $\Rightarrow$ (v). In (v) R. H. S. $\subseteq$ L. H. S. is obvious. Let $z \in$ L. H. S. Then $z \in \langle\langle m(a, n, b)\rangle_n, \langle c\rangle_n\rangle$, which implies $z \vee n \in \langle\langle m(a, n, b)\rangle_n, \langle c\rangle_n\rangle$. By (ii), $z \vee n \in \langle\langle a\rangle_n, \langle b\rangle_n\rangle \vee \langle\langle b\rangle_n, \langle a\rangle_n\rangle$. Then by Theorem 1.2, $z \vee n = x \vee y$ for some $x \in \langle\langle a\rangle_n, \langle b\rangle_n\rangle$ and $y \in \langle\langle b\rangle_n, \langle a\rangle_n\rangle$ and $x, y \geq n$. Thus, $\langle x\rangle_n \cap \langle a\rangle_n \subseteq \langle b\rangle_n$, and so

$$\langle x\rangle_n \cap \langle a\rangle_n = \langle x\rangle_n \cap \langle a\rangle_n \cap \langle b\rangle_n \subseteq \langle z \vee n\rangle_n \cap \langle a\rangle_n \cap \langle b\rangle_n$$

$= \langle z \vee n\rangle_n \cap \langle m(a, n, b)\rangle_n \subseteq \langle c\rangle_n$. This implies $x \in \langle\langle a\rangle_n, \langle c\rangle_n\rangle$ Similarly $y \in \langle\langle b\rangle_n, \langle c\rangle_n\rangle$, and so $z \vee n \in \langle\langle a\rangle_n, \langle c\rangle_n\rangle \vee \langle\langle b\rangle_n, \langle c\rangle_n\rangle$.

Similarly, a dual calculation of above shows that $z \wedge n \in \langle\langle a\rangle_n, \langle c\rangle_n\rangle \vee \langle\langle b\rangle_n, \langle c\rangle_n\rangle$.

Thus by convexity, $z \in \langle\langle a\rangle_n, \langle c\rangle_n\rangle \vee \langle\langle b\rangle_n, \langle c\rangle_n\rangle$ and so (v) holds.

(v) $\Rightarrow$ (i). Suppose (v) holds, Let a, b, c $\geq n$. By (v), $\langle\langle m(a,n,b)\rangle_n, \langle c\rangle_n\rangle = \langle\langle a\rangle_n, \langle c\rangle_n\rangle \vee \langle\langle b\rangle_n, \langle c\rangle_n\rangle$. But by Lemma 1.5 (i), this is equivalent to $\langle a \wedge b, c\rangle = \langle a, c\rangle \vee \langle b, c\rangle$.Then by [2, Theorem 3.7], this shows that $[n)$ is a relatively normal lattice. Similarly, for a, b, c $\leq n$, using the Lemma 1.5 (ii) and Theorem1.10, we find that $(n]$ is relatively normal.

Therefore $F_n(L)$ is relatively normal by Lemma1.1.

Finally we need to prove (iii) $\Rightarrow$ (i). Suppose (iii) holds. Let a, b, c $\in (n]$.

By (iii), $\langle\langle c\rangle_n, \langle a\rangle_n \vee \langle b\rangle_n\rangle = \langle\langle c\rangle_n, \langle a\rangle_n\rangle \vee \langle\langle c\rangle_n, \langle b\rangle_n\rangle$.

But by Lemma 1.6(i), this is equivalent to $\langle c, a \vee b\rangle = \langle c, a\rangle \vee \langle c, b\rangle$ which says by [2, Theorem 3.7] $[n)$ is relatively normal.

Similarly for a, b, c $\leq n$, using the Lemma 1.6 (ii) and Theorem 1.10, we find that $(n]$ is relatively normal. Hence by Lemma 1.1 , $F_n(L)$ is relatively normal.

Following result is due to [2, Lemma 3.4].

Theorem 1.12 A distributive lattice is relatively normal if and only if any two incomparable prime ideals are comaximal. ■

Now we generalize the above result.

Theorem 1.12 *Let* L *be a distributive lattice. Then the following conditions are equivalent:*

(i) *$F_n(L)$ is relatively normal.*

(ii) *Any two incomparable prime* n*-ideals* P *and* Q *are comaximal, that is $P \vee Q = L$.*

Proof : Suppose (i) holds. Let P, Q be two incomparable prime n-ideals of L. Then there exist a, b $\in L$ such that $a \in P - Q$ and $b \in Q - P$. Then $\langle a\rangle_n \subseteq P - Q, \langle b\rangle_n \subseteq Q - P$. Since $F_n(L)$ is relatively normal, so by Theorem 1.11

$\langle\langle a\rangle_n, \langle b\rangle_n\rangle \vee \langle b\rangle_n, \langle a\rangle_n > = L$. But as P, Q are prime, so it is easy to see that, $\langle\langle a\rangle_n, \langle b\rangle_n\rangle \subseteq Q$ and $\langle\langle b\rangle_n, \langle a\rangle_n\rangle \subseteq P$, Therefore $L \subseteq P \vee Q$ and so $P \vee Q = L$. That is, (ii) holds.

Conversely, suppose (ii) holds. Let P_1 and Q_1 be two incomparable prime ideals of $[n)$. Then by [2, Lemma 3.4] there exist incomparable prime ideals P and Q of L such that $P_1 = P \cap [n)$ and $Q_1 = Q \cap [n)$. Since $n \in P_1$ and $n \in Q_1$, so by Lemma 1.7 P, Q are in fact two incomparable prime n-ideals of L. Then by (ii),

$P \vee Q = L$. Therefore, $P_1 \vee Q_1 = (P \vee Q) \cap [n) = [n)$. Thus by [2, Theorem 3.5], $[n)$ is relatively normal.

Similarly, considering two prime filters of $(n]$ and proceeding as above and using the dual result of [2, Theorem 3.5] we find that $(n]$ is relatively normal. Therefore by Lemma 1.1, $F_n(L)$ is relatively normal.

By [3] an element $n \in L$ is called neutral if for all $x, y \in L$ $x \wedge (y \vee n) = (x \wedge y) \vee (x \wedge n)$ and $n \wedge (x \vee y) = (n \wedge n) \vee (x \wedge y)$.
n is called a central element if it is neutral and complemented in each interval containing it . We know that $P_n(L) = F_n(L)$ when n is a central element of L. So we conclude the paper with the following result.

Corollary 1.14 *Let* n *be a central element of a distributive lattice* L. *Then the following conditions are equivalent.*

(i) $P_n(L)$ *is a relatively normal lattice*

(ii) *For all* $a, b, c \in L$

(iii) $<<a>_n, <b>_n> \vee <<b>_n, <a>_n> = L$

(iii) For all $a, b, c \in L, <<c>_n, <a>_n \vee <b>_n> = <<c>_n, <a>_n> \vee <<c>_n, <b>_n>$.

(iv) Any two incomparable prime n-ideals P and Q are comaximal; that is $P \vee Q = L$ ■

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