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Multiple Regression of Fuzzy-Valued Variable

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ABSTRACT

In fuzzy domain, a variable (vague linguistic term) often depends not only on a single variable but on more then one variables. In such a situation multiple regression analysis is more appropriate than simple regression analysis involving one independent variable. This paper introduces fuzzy multiple regression equations of fuzzy sets those are treated as a variable with certain values assigned to them. The error analysis is done by using standard least square technique.

Keywords: Fuzzy set, Regression equation, Mean, Standard deviation.

1. Introduction

Regression analysis is quite a common tool in classical statistics. In fuzzy mathematics too, regression plays an important role in analyzing imprecise data. Mostly, simple regression equation involving a single dependent and a single independent fuzzy variable are used to analyze situation involving fuzzy data.

In Korner, Nather 16 and Wu 17 linear regression analysis for fuzzy input and output data has been discussed using fuzzy extension principle. In Kratschmer 19 too almost same approach has been used for vague concepts. In Ozelkan, Duckstein 20, several multi-objective fuzzy regression techniques are introduced to overcome the problem of fuzzy regression that does not allow all data points to influence the estimated parameter. Kim, Moskowitz, Koksalan 22 compared statistical linear regression and fuzzy linear regression from different perspective and it is shown that fuzzy regression can be used as a viable alternative when data are vague and/or model specification is poor.

Here, in this paper, we introduce the notion of multiple regression that involves one fuzzy dependent variable depending on more than one independent variables. In Bargiela et. al. 23 iterative algorithm for multiple regression with fuzzy data is used, where, regression problem is posed as a gradient-descent optimization. On the other-hand, we have built up our concept of fuzzy multiple regression as an extension of fuzzy simple regression described in section 2. We have also shown that some properties of simple regression are true for multiple regression too. Lastly, an example is cited to show the practical implication of the theory developed.

2. Fuzzy-Valued Variable, Simple Regression

Let $A=\{x_1, x_2, \ldots x_p\}$ be a finite universal set and F be the set of all normal, convex fuzzy sets defined over A. Let X ε F with membership grade function μ_X such that $\mu_X(x_k)$ ε [0,1], x_k ε A.

Now, we discuss something about fuzzy-valued variable and its simple regression equations.

A fuzzy-valued variable X is defined as a function X: F -> F $ where F is the set of normal, convex fuzzy sets defined over the universal set A.

As for example, we can take the vague term "Beauty" as a fuzzy-valued variable with comparatively less vague terms "Physical Beauty" and "Intelligence" as its values, all being defined on a set of people.

Obviously, a fuzzy-valued variable is quite different from fuzzy random variable which is nothing but a function that assigns fuzzy sets to elements of A [(Puri, Ralescu), (Kwakernaac), (Baudrit, Couso, Dubois)]. Moreover, the values of X are chosen to be fuzzy sets akin to X semantically. So, it can be assumed that values $x_1, x_2, \ldots, x_p$ of X have equal weights as far as intuitive similarity with X is concerned.

Similarly, 'marks in Mathematics' is a fuzzy-valued variable if its values are fuzzy sets like 'almost fail', 'round about 40', or 'about to pass' etc.

Suppose the result of ten students in a test of mathematics is represented by the following:

Three students 'nearly passed', four got 'average' and three scored 'high marks'. The fuzzy data table for fuzzy-valued variable 'marks in Mathematics' is given by Table-1.

X	Y
Nearly Passed	3
Average	4
High Marks	3

Table 1: Fuzzy data table

So we see that a fuzzy-valued variable is nothing but a fuzzy set. So why is this new approach? What is special about fuzzy-valued variable?

We know that there may be different ways to define a fuzzy set. But whatever be the approach, it is important to construct a proper membership grade function that captures the essence of the linguistic term represented by the fuzzy set. It would not be an exaggeration to say that comparatively less vague terms are easier to handle when question of construction of membership grade function arises.

In human conversation, it is quite common to express the meaning of an apparently vague linguistic term in terms of other less vague terms. This mechanism is easily employed by human brain which is so far the best known soft-computing machine. We have just tried to mimic this mechanism by our new approach of fuzzy-valued variable. In our research, every fuzzy set is treated as a fuzzy-valued variable with suitable and appropriate values.

Remark 1. That X is a function defined over F and not on A is another reason for not calling it fuzzy random variable.

Remark 2. These fuzzy sets can be considered as fuzzy extensions of class-intervals used in classical statistics.

Definition 1. Let X and Y be two fuzzy-valued variables. Let X takes values $X_1, X_2, \ldots, X_n$ and Y takes values $Y_1, Y_2, \ldots, Y_n$ where X_i and Y_i are the usual normal, convex fuzzy sets defined over the universal set A.

We define the fuzzy regression equations of X and Y as the system of equations

$$\begin{aligned} Y_1 &= A_{11} X_1 + A_{12}X_2 + \ldots + A_{1n}X_n + e_1 \\ Y_2 &= A_{21} X_1 + A_{22}X_2 + \ldots + A_{2n}X_n + e_2 \\ &\ldots \\ Y_n &= A_{n1} X_1 + A_{n2}X_2 + \ldots + A_{nn} X_n + e_n \end{aligned} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

subject to the condition that $\sum_k \mu^2{}_{ei}(x_k)$ is minimum for all i = 1, 2,…, n.

The fuzzy sets A_{ij}, i, j =1, 2,…, n defined over A are called the coefficients of fuzzy regression.

Let us define the fuzzy error sets e_i as

$$\mu_{ei}(x_k) = |-\mu_{yi}(x_k) + (1/n)(\mu_{Ai1}(x_k)\,\mu_{X1}(x_k) + \mu_{Ai2}(x_k)\,\mu_{X2}(x_k) + \ldots + \mu_{Ain}(x_k)\mu_{Xn}(x_k))| \qquad \ldots\ldots\ldots\ldots(2)$$

and

$$\mu_{Yi}(x_k) = |-\mu_{ei}(x_k) + (1/n)(\mu_{Ai1}(x_k)\,\mu_{X1}(x_k) + \mu_{Ai2}(x_k)\,\mu_{X2}(x_k) + \ldots + \mu_{Ain}(x_k)\mu_{Xn}(x_k))| \qquad \ldots\ldots\ldots(3)$$

for all *i=1, 2, ...,n.*

The product between fuzzy sets A_{ij} and X_j is considered to be understood in terms of their membership grades .

For

$$\begin{aligned} u &= \sum_k \mu^2{}_{ei}(x_k) \\ &= \sum_k [|-\mu_{yi}(x_k) + (1/n)(\mu_{Ai1}(x_k)\,\mu_{X1}(x_k) + \mu_{Ai2}(x_k)\,\mu_{X2}(x_k) + \ldots + \mu_{Ain}(x_k)\,\mu_{Xn}(x_k))|]^2 \end{aligned} \qquad \ldots\ldots\ldots(4)$$

to be minimum for all *i =1, 2, ..., n,* we differentiate u partially with respect to $\mu_{Ait}(x_k)$ for *t=1, 2, ..., n* and equate them to 0.

Thus for a fixed x_k and fixed *i*, we get,

$$\mu_{Yi}(x_k) = (1/n)(\mu_{Ai1}(x_k)\,\mu_{X1}(x_k) + \mu_{Ai2}(x_k)\,\mu_{X2}(x_k)+\ldots+\mu_{Ain}(x_k)\,\mu_{Xn}(x_k))$$

i.e., $n\,\mu_{Yi}(x_k) = \mu_{Ai1}(x_k)\,\mu_{X1}(x_k) + \mu_{Ai2}(x_k)\,\mu_{X2}(x_k)+ \ldots+\mu_{Ain}(x_k)\,\mu_{Xn}(x_k))$ ……… (5)

For a fixed x_k and fixed i, equation (5) is a single equation in n unknowns $\mu_{Ai1}(x_k)$, $\mu_{Ai2}(x_k)$, …, $\mu_{Ain}(x_k)$ and may possess an infinite number of solutions. But of these n unknowns, (n-1) must be independent and the remaining one depends on them.

We take that unknown as the dependent one whose coefficient is greatest among all the coefficients.

Let, $\mu_{Xr}(x_k) = \max(\mu_{X1}(x_k), \mu_{X2}(x_k), \ldots, \mu_{Xn}(x_k))$.

Therefore, we take $\mu_{Air}(x_k)$ as the dependent unknown and $\mu_{Ait}(x_k)$, t=1, 2, …, r-1, r+1,… ,n as the independent ones.

Let initial guess for independent unknowns be $\mu^0_{Ait}(x_k)$, *t=1, 2, …,r-1, r+1, … ,n.*

Hence, the dependent unknown is given by,

$$\begin{aligned}\mu^0_{Air}(x_k) = (1/\mu_{Xr}(x_k))\,(n\,\mu_{Yi}(x_k) - \mu^0_{Ai1}(x_k)\,\mu_{X1}(x_k)-\ldots\\ -\mu^0_{Air-1}(x_k)\,\mu_{Xr-1}(x_k) - \mu^0_{Air+1}(x_k)\,\mu_{Xr+1}(x_k)-\ldots\\ - \mu^0_{Ain}(x_k)\,\mu_{Xn}(x_k))\end{aligned} \quad \ldots\ldots\ldots.. (6)$$

Obviously, for $0 < \mu^0_{Ait}(x_k) < 1$, t=1, 2, … , r-1, r+1, …,n we get, $0 < \mu^0_{Air}(x_k) < 1$.

If $\mu^0_{Air}(x_k)$ is not less than 1, we apply the normality condition

$$\mu^0_{Air}(x_k) = \mu^0_{Air}(x_k) / ([\mu^0_{Air}(x_k)] + 1).$$

Thus we can find the regression coefficients A_{ij} for i,j=1, 2, … ,n.

Using equation (2) we can find e_i for i=1, 2, … ,n.

3. Multiple Regression of Fuzzy-valued variable

In the previous section, we have considered simple regression equations of fuzzy-valued variable. Now we extend our concept of simple regression to multiple regression with one dependent variable and more than one independent variable.

Let X be a fuzzy-valued variable with assigned values $X_1, X_2, \ldots, X_n$ as defined in definition 1.

Hereafter, the above fact will be denoted by X ($X_1, X_2, \ldots, X_n$).

With this notation in mind, let, $X^1 (X_1^1, X_2^1, \ldots, X_n^1)$, $X^2 (X_1^2, X_2^2, \ldots, X_n^2)$, …, $X^n (X_1^n, X_2^n, \ldots, X_n^n)$ be n independent fuzzy-valued variables and $Y(Y_1, Y_2, \ldots, Y_n)$ be the dependent variable depending on them.

Since values of X's and Y are chosen by an agent according to the need of a situation, it may so happen that the assigned values are not equal in numbers. In such cases, we shall make them equal by considering pseudo values. These pseudo values are fuzzy sets for which every element has membership 0.

Now, Y ($Y_1, Y_2, \ldots, Y_n$) depends on $X^1 (X_1^1, X_2^1, \ldots, X_n^1)$. Therefore, by applying the simple regression analysis as described in section 2, we get fuzzy regression coefficients A^1_{pq}, p, q = 1, 2, …, n and the error set $e^1 (e_1^1, e_2^1, \ldots, e_n^1)$ such that,

$$Y_1 = A_{11}^1 X_1^1 + A_{12}^1 X_2^1 + \ldots + A_{1n}^1 X_n^1 + e_1^1$$

$$Y_2 = A_{21}{}^1 X_1{}^1 + A_{22}{}^1 X_2{}^1 + \ldots + A_{2n}{}^1 X_n{}^1 + e_2{}^1$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$
$$Y_n = A_{n1}{}^1 X_1{}^1 + A_{n2}{}^1 X_2{}^1 + \ldots + A_{nn}{}^1 X_n{}^1 + e_n{}^1$$

Similarly, dependence of Y on each of X^1, X^2, …, X^n gives rise to (n-1) more such systems of n hyper-planes of dimension (n+1).
As a result, there are n systems of n, (n+1) dimensional hyper-planes given by

$$Y_1 = A_{11}{}^j X_1{}^j + A_{12}{}^j X_2{}^j + \ldots + A_{1n}{}^j X_n{}^j + e_1{}^j$$
$$Y_2 = A_{21}{}^j X_1{}^j + A_{22}{}^j X_2{}^j + \ldots + A_{2n}{}^j X_n{}^j + e_2{}^j$$
$$\ldots\ldots\ldots$$
$$Y_n = A_{n1}{}^j X_1{}^j + A_{n2}{}^j X_2{}^j + \ldots + A_{nn}{}^j X_n{}^j + e_n{}^j \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

for *j = 1, 2, …, n*.
Now, for a particular j ε {*1, 2, …, n*}, we get n^2 fuzzy simple regression coefficients $A_{pq}{}^j$, *p, q = 1, 2, …, n*, and *n* error sets $e_1{}^j$, $e_1{}^j$, …, $e_1{}^j$.
To find fuzzy regression coefficients for multiple regression, these simple regression coefficients should be coordinated in a comprehensive manner. We use the weighted aggregation operation to find the multiple regression coefficients $A_{pq}{}^{multiple}$ as follows:
For all x_k in A,

$$\mu_{Apq}{}^{multiple}(x_k) = \sum_{j=1}^{n} \mu_{Apq}{}^j (x_k)\, w_j \qquad \ldots\ldots\ldots\ldots (9)$$

where $w_1, w_2, \ldots, w_n$ are real number lying in [0,1] and represent intuitive dependence of Y on X^1, X^2, …, X^n respectively with the condition that $\sum w_j = 1$.

Replacing the simple regression coefficients $A_{pq}{}^j$ for *p, q, j = 1, 2, …, n* by corresponding multiple regression coefficients in equation (8) and calling $Y_1, Y_2, \ldots Y_n$ respectively $Y_1{}^j$, $Y_2{}^j$, …, $Y_n{}^j$ we get the equation for multiple regression of Y and X^j as given by,

$$Y_1{}^j = A_{11}{}^{multiple} X_1{}^j + A_{12}{}^{multiple} X_2{}^j + \ldots + A_{1n}{}^{multiple} X_n{}^j + e_1{}^j$$
$$Y_2{}^j = A_{21}{}^{multiple} X_1{}^j + A_{22}{}^{multiple} X_2{}^j + \ldots + A_{2n}{}^{multiple} X_n{}^j + e_2{}^j$$
$$\ldots\ldots\ldots$$
$$Y_n{}^j = A_{n1}{}^{multiple} X_1{}^j + A_{n2}{}^{multiple} X_2{}^j + \ldots + A_{nn}{}^{multiple} X_n{}^j + e_n{}^j \ \ldots\ldots (10)$$

for *j = 1, 2, …, n*.

Now, for a particular $j \varepsilon\{1, 2, \ldots n\}$, $Y_1^{\ j}$, $Y_2^{\ j}$, … $Y_n^{\ j}$ do not give required $Y_1, Y_2, \ldots Y_n$ respectively but together they must give $Y_1, Y_2, \ldots Y_n$ because each of $Y_1, Y_2, \ldots Y_n$ depend on all of $X^1, X^2, \ldots, X^n$.

So, $Y_i^1, Y_i^2, \ldots Y_i^n$ are to be combined to give a single Y_i and $e_i^1, e_i^2, \ldots e_i^n$ are to be combined to give e_i, $i=1, 2, \ldots, n$, the fuzzy sets constituting the error set e $(e_1, e_2, \ldots, e_n)$ of multiple regression. For the coordination we again use the weighted aggregation as in equation (9) and Y_i and e_i are given by, for all x_k in A,

$$\mu_{Yi}(x_k) = \sum \mu_{Yi}^{\ j}(x_k)\, w_j \qquad \ldots\ldots\ldots\ldots\ldots(11)$$

and

$$\mu_{ei}(x_k) = \sum \mu_{ei}^{\ j}(x_k)\, w_j \qquad \ldots\ldots\ldots\ldots\ldots\ldots(12)$$

Theorem 1. $\overline{\mu}_{Yi} = \sum_j \overline{\mu}_{Yi}^{\ j}\, w_j$ for *i= 1, 2, ..., n.*

Proof. We know, for all *i =1, 2, ..., n,*

$$\overline{\mu}_{Yi}^{\ 1} = \left(\sum_k \mu_{Yi}^{\ 1}(x_k)\right) / p$$

$$\overline{\mu}_{Yi}^{\ 2} = \left(\sum_k \mu_{Yi}^{\ 2}(x_k)\right) / p$$

$$\ldots\ldots\ldots\ldots\ldots$$

$$\overline{\mu}_{Yi}^{\ n} = \left(\sum_k \mu_{Yi}^{\ n}(x_k)\right) / p$$

Multiplying the above equations by $w_1, w_2, \ldots, w_n$ respectively and adding together we get, for all *i= 1, 2,…, n,*

$$\sum_j \overline{\mu}_{Yi}^{\ j}\, w_j = \sum_j\left(\sum_k \mu_{Yi}^{\ j}(x_k)\, w_j\right) / p$$

$$= \sum \overline{\mu}_{Yi}^{\ j}\, w_j$$

Theorem 2. $E(Y) \le E(e) + p$ provided the frequencies of Y_i's are same and frequencies of e_i' s also are same.

Proof. Here, we first recall the definition of frequency and expectation of a fuzzy-valued variable.

Frequency of a fuzzy set X is the number of elements of finite universal set A which have non-zero membership grade in X.

Expectation of fuzzy-valued variable Y is defined as

$$E(Y) = \left(\sum \overline{\mu}_{Yi}\, b_i\right) / \sum b_i, \text{ Where } b_i \text{ is the frequency of fuzzy set } Y_i.$$

Now, individual dependence of Y on $X^1, X^2, \ldots, X^n$ gives rise to equations (10).

We have,

$\overline{\mu}_{Yi}{}^{1} \leq \overline{\mu}_{ei}{}^{1} + p$

$\overline{\mu}_{Yi}{}^{2} \leq \overline{\mu}_{ei}{}^{2} + p$

…………..

$\overline{\mu}_{Yi}{}^{n} \leq \overline{\mu}_{ei}{}^{n} + p$ ....(13)

For all i = 1, 2, ... ,n.

Multiplying the above equations by w_j for j= 1, 2, …,n, respectively and adding together we get, E(Y) ≤ E(e) + p.
[Remembering that $\sum w_j = 1$]

4. Practical Application

Let us take the fuzzy-valued variable "Personality" as the dependent variable depending on fuzzy-valued variables "Intelligence" and "Beauty" which are taken to be independent.

Let "Personality (Y)" has two values, "Well-behaviour (Y_1)" and "Charm (Y_2)". "Intelligence (X^1)" has two values viz., "Knowledge ($X^1{}_1$)" and "Mental Sharpness ($X^1{}_2$)". "Beauty (X^2)" has two values, "Physical Beauty ($X^2{}_1$)" and "Noble-mind ($X^2{}_2$)".

Let set of people U = {x_1, x_2, x_3, x_4, x_5} be the universal set. All the above X and Y are defined on U.

Let,

	$X^1{}_1$	$X^1{}_2$	Y_1	Y_2
x_1	0.4	0.8	0.3	0.4
x_2	0.9	0.5	0.9	0.6
x_3	0.3	0.4	0.7	0.2
x_4	0.1	0.5	0.2	0.7
x_5	0.4	0.6	0.3	0.9

be the observed fuzzy sets. Using simple regression as defined in section 2 for Y and X^1 and using normality condition where necessary, we get the following regression coefficients for simple regression.

	$A^1{}_{11}$	$A^1{}_{12}$	$A^1{}_{21}$	$A^1{}_{22}$
x_1	0.3	0.6	0.5	0.75
x_2	0.8	0.7	0.6	0.2
x_3	0.6	0.75	0.9	0.32
x_4	0.3	0.74	0.4	0.9
x_5	0.7	0.53	0.2	0.96

From similar observations for “Beauty” and “Personality”, we get the simple regression coefficients as

	A^2_{11}	A^2_{12}	A^2_{21}	A^2_{22}
x_1	0.37	0.44	0.2	0.71
x_2	0.25	0.8	0.54	0.39
x_3	0.5	0.4	0.9	0.2
x_4	0.6	0.41	0.67	0.32
x_5	0.56	0.72	0.81	0.51

Let, the intuitive dependence of "Personality" on "Intelligence" be 0.6 and that on "Beauty" be 0.4, i.e., $w_1 = 0.6$ and $w_2 = 0.4$.

Therefore, we get the coefficients of multiple regression as

	$A^{multiple}_{11}$	$A^{multiple}_{12}$	$A^{multiple}_{21}$	$A^{multiple}_{22}$
x_1	0.33	0.54	0.38	0.73
x_2	0.15	0.74	0.58	0.28
x_3	0.56	0.61	0.9	0.27
x_4	0.42	0.6	0.5	0.67
x_5	0.64	0.6	0.44	0.78

We find error sets with the new found multiple regression coefficients $A^{multiple}_{pq}$ as follows.

	e^1_1	e^1_2	e^2_1	e^2_2
x_1	0.01	0.03	0.2	0.34
x_2	0.65	0.27	0.07	0.42
x_3	0.49	0.01	0.43	0.52
x_4	0.04	0.5	0.26	0.37
x_5	0.008	0.58	0.61	0.41

From above table, we get the coordinated error sets for fuzzy multiple regression as

	e_1	e_2
x_1	0.09	0.15
x_2	0.43	0.33
x_3	0.47	0.21
x_4	0.13	0.45
x_5	0.25	0.51

5. **Conclusion**

To find a relation between a dependent variable and several independent variables, a modular approach has been taken in this paper. First we have analyzed the relation of the dependent variable with each of the independent variables separately and then combined them to get the overall effect. This is simply the human thought process that we have followed in doing the regression analysis between fuzzy variables.

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