Journal of Physical Sciences, Vol. 13, 2009, 201-208
ISSN: 0972-8791 : www.vidyasagar.ac.in/journal

Generalized Weber Problem

***M.Thiyagarajan*[1] *and R.Manikandan*[2]**

[1]School of Computing,
SASTRA University,
Thanjavur-613402, INDIA.
m_thiyagarajan@yahoo.com

[2] TIFAC CORE,
SASTRA University,
Thanjavur-613402, INDIA.
manikandan75@core.sastra.edu

Received August 19, 2009; accepted October 22, 2009

ABSTRACT

We give a generalization random locational equilibrium problem for two dimensional case along its optimization.

Keywords: Bivariate normal distribution, Optimization problem, Numerical partial derivatives.

1. Introduction

The random locational equilibrium problem also known as Weber problem can be stated as follows. There are n locations (destination or sinks) in the two dimensional Euclidean space. It is required to find the location of a facility so that the sum o f the weighted distance from the location of the facility to the destination is minimized.

The problem has been considered by Cooper [1963] when the location of the destinations is deterministic. Katz and Cooper [1974] have presented a scheme numerical solution. Cooper [1974] has considered the problem when the location of the destination is stochastic in nature and the pair of coordinators follows a bivariate distribution. In this case it is recommended to minimise the sum of weighted average of linear distances of the destinations from the location facility. Although Cooper [1974] has presented the general algebraic expression of the equations which would lead to the optimum solution of the problem the numerical solution could be very involved. Wesolowsky[1970] has recommended the use of rectangular distance instead of linear distance and derived the optimum solution when the coordinates of the destination follow certain known forms of bivariate probability distribution.

Mustafi (1982) has given a simple solution of the problem when the coordinates of the destination follow a bivariate uniform distribution. We present our extension to bivariate normal distribution.

In VLSI theory we have AT^2 optimal problem which aims at minimizing computation time in the integer multiplication. This problem can be considered as a location optimization distance problem. This is studied by Mehlhorn.K and Preparata.F.P in early 1985 as a computational complexity problem.

2. A bivariate distribution and optimising equation

Suppose q with coordinates $(x_{0,}y_0)$ is the location of the new facility. Let P_j with coordinates $(x_{j,}y_j)$ have a bivariate uniform distribution given by

$$f(x_j, y_j) = 1/ab + 4\alpha_j / ab(x_j - b/2) \tag{1}$$

with 0< x_j < a , 0< y_j <b , -1<α_j ≤ 1, j=1,2,…n.

The j^{th} destination is randomly distributed in the rectangle bounded by two axes and the lines x=a and y=b. The quantity α_j is a parameter indicating the degree of dependence. If α_j = 0, x_j and y_j are independent.

The linear distance between p_j and q is given by

$$d_j = [(x_j - x_0)^2 + (y_j - y_0)^2]^{1/2} \tag{2}$$

suppose $w_j > 0$ (i=1,2,…n) is the weight assigned to the j^{th} destination. Then the coordinates (x_0 , y_0) of the new facility should be chosen in such a manner that

$$z = \sum_{j=1}^{n} w_j E(d_j) \tag{3}$$

is minimised.

The optimum values of x_0 and y_0 are solutions of the equations.

$$\frac{\partial z}{\partial x_0} = \sum_{j=1}^{n} w_j \int_0^a \int_0^b \frac{(x_0 - x_j)}{d_j} f_j(x_{j,}y_j) dx_j dyj = 0$$

$$\frac{\partial z}{\partial y_0} = \sum_{j=1}^{n} w_j \int_0^a \int_0^b \frac{(y_0 - y_j)}{d_j} f_j(x_{j,}y_j) dx_j dyj = 0 \tag{4}$$

It can be shown that for any bivariate density

$$\frac{\partial^2 z}{\partial x_0^2} > 0; \ \frac{\partial^2 z}{\partial y_0^2} > 0$$

$$\frac{\partial^2 z}{\partial x_0^2} . \frac{\partial^2 z}{\partial y_0^2} - (\frac{\partial^2 z}{\partial x_0 d y_0})^2 \ > \ 0 \tag{5}$$

The condition given in (5) ensure that z is minimised if x_0 and y_0 are chosen as a solution of the equation given in (4)

3. The optimum location of facility

Theorem 1. (Mustafi 1982), If the probability density function of the destination $(j = 1, 2, ...n)$ is a bivariate uniform distribution as given by (1) the optimum location of the new facility should be at $x_0 = \frac{a}{2}, y_0 = \frac{b}{2}$.

Proof: Using the form of $f_j\left(x_j, y_j\right)$ as given in (1) we obtain from (4)

$$\frac{\partial z}{\partial x_0} = \sum_{j=1}^{n} \int_0^a W_j \int_0^b \frac{\left(x_0 - x_j\right)}{\left\{\left(x_j - x_0\right)^2 + \left(y_j - y_0\right)^2\right\}} \frac{1}{2}\left[\frac{1}{ab} + \frac{4\alpha_j}{ab}\left(x_j - \frac{a}{2}\right)\left(y_j \frac{b}{2}\right)\right] dx_j dy_j \tag{6}$$

$$= T_1 + T_2$$

$$T_1 = \frac{1}{ab} \int_0^a \int_0^b \frac{\left(x_0 - x_j\right)}{\left[\left(x_j - x_0\right)^2 + \left(y_j - y_0\right)^2\right]^{\frac{1}{2}}} dx_j dy_j$$

$$= \frac{1}{ab} \int_0^b \left(\int_{-x}^{a - x_0} \frac{u_j}{\left(u_j{}^2 + \left(y_j - y_0\right)^2\right)^{\frac{1}{2}}} du_j \right) dy_j$$

At $x_0 = \frac{a}{2}, y_0 = \frac{b}{2}$

$$T_1=\frac{1}{ab}\int_0^b\left\{\left[\frac{a^2}{4}+\left(y_j-\frac{b}{2}\right)^2\right]^{\frac{1}{2}}-\left[\frac{a^2}{4}+\left(y_j-\frac{b}{2}\right)^2\right]^{\frac{1}{2}}\right\}dy_j=0...(7)$$

$$T_2=\frac{4\alpha_j}{ab}\int_0^a\int_0^b\frac{\left(x_j-\frac{a}{2}\right)\left(y_j-\frac{b}{2}\right)\left(x_0-x_j\right)dx_jdy_j}{\left[\left(x_j-x_0\right)^2+\left(y_j-y_0\right)^2\right]^{\frac{1}{2}}}$$

At $x_0=\frac{a}{2},y_0=\frac{b}{2}$

$$T_2=-\frac{4\alpha_j}{ab}\int_0^a\int_0^b\frac{\left(x_j-\frac{a}{2}\right)\left(y_j-\frac{b}{2}\right)}{\left[\left(x_j-\frac{a}{2}\right)^2+\left(y_j-\frac{b}{2}\right)^2\right]^{\frac{1}{2}}}dx_jdy_j$$

$$=-\frac{4\alpha_j}{ab}\int_{\frac{-b}{2}}^{\frac{b}{2}}\left\{\int_{\frac{-a}{2}}^{\frac{a}{2}}\frac{u_j^2}{\left(u_j^2+v_j^2\right)^{\frac{1}{2}}}du_j\right\}v_jdv_j$$

$$=-\frac{8\alpha_j}{ab}\int_{\frac{-b}{2}}^{\frac{b}{2}}\left\{\frac{a}{4}\sqrt{\frac{a^2}{4}+v_j^2}-\frac{v_j^2}{2}\log\left(\frac{a}{2}+\sqrt{\frac{a^2}{4}+v_j^2}\right)+\frac{v_j^2}{2}\log\sqrt{v_j^2}\right\}v_jdx$$

Since the integrand is an odd functioning, it vanishes. Hence

$$T_2=0 \tag{8}$$

In a similar way we can show $\frac{\partial z}{dy_0}=0$. This completes the proof of the theorem.

With the same notation of theorem 1, we states & prove the following result.

Theorem 2. The minimum value of z is given by

$$W\left[\frac{1}{6}\sqrt{a^2+b^2}+\frac{b^2}{12a}\log\left(\frac{a}{2}+\frac{1}{2}\sqrt{a^2+b^2}\right)+\frac{a^2}{12b}\log\left(\frac{b}{2}+\frac{1}{2}\sqrt{a^2+b^2}\right)-\frac{a^2}{12b}\log\frac{a}{2}-\frac{b^2}{12a}\log\frac{b}{2}\right].$$

where

$$W=\sum_{j=1}^{n}w_j \tag{9}$$

Proof: follows by using $x_0=\frac{a}{2}, y_0=\frac{b}{2}$ and the notation of the expectation of continuous random variable. We note that if $n=1, w_1=1, a=b,$ the minimum expected linear distance is given by

$$=1\left[\frac{1}{6}\sqrt{a^2+b^2}+\frac{a^2}{12a}\log\left(\frac{a}{2}+\frac{1}{2}\sqrt{a^2+b^2}\right)+\frac{a^2}{12a}\log\left(\frac{a}{2}+\frac{1}{2}\sqrt{2a^2}\right)-\frac{a^2}{12a}\log\frac{a}{2}-\frac{a^2}{12a}\log\frac{a}{2}\right]$$

$$=\frac{a}{6}\left(\sqrt{2}+\log\left(\sqrt{2}+1\right)\right)$$

=.383 a

4. An extension of the problem

Mustafi (1982) has considered the solution of the problem when the coordinates of the destination follow a bivariate uniform distribution. Here we consider the solution of the problem when coordinates of the destination follow a bivariate normal distribution.

5. Distribution and the optimising equations

Suppose Q with coordinates (x_0, y_0) is the location of the new facility. Let p_j with coordinates $(x_j y_j)$ be the location of the j^{th} destination $(j=1,2,......n)$. It is assumed $(x_j y_j)$ has a bivariate normal distribution given by

$$f(x_j y_j)=\frac{1}{2\pi\sqrt{1-p^2}}\exp\left(-\frac{1}{2(1-\rho^2)}\left(x_j^2-2\rho x_j y_j+y_j^2\right)\right)$$

$$-\infty < x_j < \infty$$

$$-\infty < y_j < \infty \text{...........(10)}$$

The linear distance between p_j and Q is given by

$$d_j = \left[\left(x_j - x_0\right)^2 + \left(y_j - y_0\right)^2\right]^{\frac{1}{2}} ....(11)$$

Suppose $w_j > 0 (j = 1, 2, ...n)$ is the weight assigned to the j^{th} destination. Then the coordinates $\left(x_0, y_0\right)$ of the new facility should be chosen in such a way that

$$z = \sum_{j=1}^{n} W_j E(d_j).......(12)$$

is minimized.

The optimum values of x_0 and y_0 are the solutions of the equations

$$\frac{dy}{dx_0} = 0 = \frac{dz}{dy_0} ....(13)$$

For any bivariate density function

$$\frac{\partial^2 z}{\partial x_0^2} > 0, \frac{\partial^2 z}{\partial y_0^2} > 0,$$

$$\frac{\partial^2 z}{\partial x_0^2}\frac{\partial^2 z}{\partial y_0^2} - \left(\frac{\partial^2 z}{\partial x_0 \partial y_0}\right)^2 > 0...........................(14)$$

The condition given in (14) ensures Z is minimized if x_0 and y_0 are chosen as solution of the equation given in (13).

6. The optimum location of the facility

We state and prove our result for the bivariate normal case **theorem 3**. If the probabilities density function of the destination (j = 1,2,..n) is bivariate normal distribution given by (10) the optimum location of the new facility should be at $x_0 = 0, y_0 = 0$

Proof: using the form of $f\,(x_j, y_j)$ as given in (10)

We obtain from (13)

$$\frac{\partial z}{\partial x_0} = \sum_{j=1}^{n} \int_{-\infty}^{\infty} w_j \int_{-\infty}^{\infty} \frac{x_0 - x_j}{[(x_j - x_0)^2 + (y_j - y_0)^2]^{1/2}} \times$$

$$\frac{1}{2\pi\sqrt{1-p^2}} \exp\left\{-\frac{1}{2(1-p^2}(x_j^2 - 2px_j y_j + y_j^2)\right\} dx_j .dy_j \qquad (15)$$

To evaluate the integral in (16), let us note

$$u_1^2 - 2\rho u_1 u_2 + u_2^2 = \left(1-\rho^2\right)u_1^2 + \left(u_2 - \rho u_1\right)^2$$

we may write

$$f(x_j, y_j) = \phi\left(x_j\right)\frac{1}{\sqrt{1-\rho^2}}\phi\left(\frac{x_j - \rho y_j}{\sqrt{1-\rho^2}}\right)$$

in which $\phi(u) = \dfrac{1}{1-\rho^2}\exp\left(\dfrac{-u^2}{2}\right)$

we thus determine

$$\frac{dz}{dx_0} = \sum_{j=1}^{n}\int_{-\infty}^{\infty} w_j \int_{-\infty}^{\infty} \frac{x_0 - x_j}{\left[\left(x_j - x_0\right)^2 + \left(y_j - y_0\right)^2\right]^{\frac{1}{2}}}\phi\left(x_j\right)\frac{1}{\sqrt{1-\rho^2}}\phi\left(\frac{x_j - hy_j}{\sqrt{1-\rho^2}}\right)dx_j dy_j$$

At $x_0 = 0, y_0 = 0$

$$\frac{dz}{dx_0} = -\sum w_j \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{x_j}{\left(x_j^2 + y_j^2\right)^{\frac{1}{2}}}\phi\left(x_j\right)\frac{1}{\sqrt{1-\rho^2}}\phi\left(\frac{x_j - \rho y_j}{\sqrt{1-\rho^2}}\right)dx_j dy_j = 0$$

In a similar way

$$\frac{\partial z}{\partial y_0} = 0$$

This complete the proof of the theorem.

With same notation of theorem 3, we state and prove the following result

Theorem 4. Minimum value of the expression for z is less than

$$w(1-p^2)^{1/2}/\sqrt{\pi} \quad where\ w = \sum_{j=1}^{n} w_j$$

Proof: The minimum value of z is

$$\sum_{j=1}^{n} w_j \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x_j^2 + y_j^2)^{1/2} f(x_j, y_j)\, dx_j\, dy_j$$

$$\leq w \times \mathrm{E}\,(\,\mathrm{Max}\,(x_j, y_j))$$

$$= w\,(1-p^{2)^{1/2}}\ /\sqrt{\pi}$$

from the known result of bivariate normal distribution [3].

7. Conclusion

We have generalized the problem studied by Mustafi to component normal distribution.

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