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Near-Shortest Simple Paths on a Network with Imprecise Edge Weights

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ABSTRACT

We present an algorithm to obtain the near-shortest simple (cycle free) paths between a pair of specified vertices *s* and *t* on a network whose edge weights are of imprecise nature. In particular, we have considered the weights as interval numbers and triangular fuzzy numbers. Existing ideas on addition and comparison between two imprecise numbers of same types are introduced. Initially, a fuzzy optimal path is obtained to which the decision-maker always satisfies with different grades of satisfaction. On the basis of the obtained fuzzy shortest path, different sets of near-shortest simple paths are produced within specified tolerance limit. Retaining only the first *k* such paths according to their weights, the *k*-shortest simple paths problem is also solved.

1. Introduction.

Let $G = (V,E)$ be a graph with V as the set of vertices and E as the set of edges. A *path* between two vertices is an alternating sequence of vertices and edges starting and ending with two distinct vertices. A path is said to be *simple* if it contains no cycle. The *length* of a path is the sum of the weights of the edges on the path. There may exist more than one path between a pair of vertices. The *shortest path problem* is to find the path with minimum length between a specified pair of vertices. The *near-shortest simple paths problem* is to enumerate all the simple paths whose length is within a factor of $1 + \varepsilon$ of the shortest path for some specified positive number ε.

In classical graph theory, the weight of each edge is taken as a crisp real number. But, practically weight of each edge of the network may not be a fixed real number. It may come in an imprecise way like 'about 5 KM', 'between 10-15 yards', 'chance of lying the distance between 100 - 120 miles is 90%', 'mileage of the car is 20 KM/litre under standard conditions', etc. Besides those if we consider the time or cost in a transportation problem then, in a few cases it is possible to find the transportation time or cost as a deterministic value. 'About 5 KM' means the distance may be, say, 4.8 KM or 5.2 KM or it may take any value between 4.5 KM to 5.5 KM. This can be represented as an interval number or a triangular fuzzy number or a trapezoidal fuzzy number.

The shortest path problem involves addition and comparison of the edge weights. Since, the addition and comparison between two interval numbers or between two triangular fuzzy numbers are not alike those between two precise real numbers, it is necessary to introduce those at first. Various ranking methods for interval numbers and triangular fuzzy numbers are available in the literature [5, 6, 13, 14, 16]. Here, we have used the order relations given by Nayeem and Pal [13].

In the recent past, fuzzy shortest path problems are addressed by many researchers, namely, Klein [10], Lin and Chen [12], Li *et al*. [11], Okada and Gen [14, 15], Cherkassky *et al.* [3], Okada and Soper [16], Israeli and Wood [9], and many others. Nayeem and Pal [13] have developed an algorithm, based on Dijkstra's algorithm [4], which gives a unified approach to obtain a single fuzzy shortest path or a guideline to choose the best fuzzy shortest path on a network with interval valued or triangular fuzzy valued edge weights according to the decision-makers view. Near-shortest simple paths problem on a crisp network has been studied by Carlyle and Wood [2]. Using their algorithm to solve that problem, they have also solved the *K*-shortest simple paths problem, which has a long history in operations research and computer science. Eppstein [7] and Hadjiconstantinou and Christofides [8] provide excellent reviews, and Eppstein maintains an online bibliography at http://liinwww.ira.uka.de/bibliography/Theory/kpath.html on *K*-shortest paths problem.

In this paper, we have introduced the interval and triangular fuzzy arithmetic and the order relations among them, in the next section. Using those order relations, we have modified the Byers and Waterman's algorithm [1] to find the near-shortest simple paths on a network, each of whose edges are of imprecise weights of same kind between those under consideration. In Section 3, we have described the proposed algorithm.

2. Interval and Triangular Fuzzy Arithmetic and Some Order Relations.

In general, an interval number is defined as $A = [a_L, a_R] = \{a: a_L \le a \le a_R\}$ where, a_L and a_R are the real numbers called the left end point and the right end point of the interval A.

Another way to represent an interval number in terms of midpoint and width is $A = < m(A), w(A)>$, where $m(A)$ = midpoint of $A = (a_R + a_L)/2$ and $w(A)$ = half width of $A = (a_R - a_L)/2$.

The addition of two interval numbers $A = [a_L, a_R]$ and $B = [b_L, b_R]$ is given by $A \oplus B = [a_L + b_L, a_R + b_R]$. Alternately, in mean-width notations, if $A = < m_1, w_1 >$ and $B = < m_2, w_2 >$ then, $A \oplus B = < m_1 + m_2, w_1 + w_2 >$.

The order relation we used here is based on an index called the acceptability index ($\mathcal{A}$ -index). The acceptability index to the proposition 'A is inferior to B' is given by $\mathcal{A}(A \prec B) = (m_2 - m_1)/(w_1 + w_2)$. In connection with this 'acceptability index', we define 'total dominance' and 'partial dominance' of two interval numbers $A = < m_1, w_1 >$ and $B = < m_2, w_2 >$ one over another as follows:

Definition 1 If $\mathcal{A}$ $(A \prec B) \geq 1$ then, A is said to be 'totally dominating' over B in the sense of minimization and B is said to be 'totally dominating' over A in the sense of maximization. We denote this by $A \prec B$.

Definition 2 If $0 < \mathcal{A}$ $(A \prec B) < 1$ then A is said to be 'partially dominating' over B in the sense of minimization and B is said to be 'partially dominating' over A in the sense of maximization. This is denoted by $A \prec_P B$.

But, when $\mathcal{A}$ $(A \prec B) = 0$, i.e., $m_1 = m_2$ then we may not get an order relation from the above definitions. Then we emphasize on the widths of the interval numbers A and B. If $w_1 < w_2$ then the left end point of A is less than that of B and on finding a minimum distance, there is a chance that the distance may lie on A. But at the same time, since the right end point of A is greater than that of B, if one prefers A to B in minimization then in worst case, he may be looser than one who prefers B to A. Thus in such a situation an optimistic decision-maker would prefer A to B whereas a pessimistic decision-maker would do the converse.

A triangular fuzzy number is represented by a triplet $\widetilde{A} = \langle m, \alpha, \beta \rangle$ with the membership function

$$\mu(x) = \begin{cases} 0 & \text{for } x \leq m - \alpha \\ 1 - (m - x)/\alpha & \text{for } m - \alpha < x \leq m \\ 1 - (x - m)/\beta & \text{for } m < x \leq m + \beta \\ 0 & \text{for } x \geq m + \beta. \end{cases}$$

i.e., m is the point whose membership value is 1 and α and β are the left hand and right hand spreads respectively.

Let $\widetilde{a} = \langle m_1, \alpha_1, \beta_1 \rangle$ and $\widetilde{b} = \langle m_2, \alpha_2, \beta_2 \rangle$ be two triangular fuzzy numbers. Then the fuzzy sum of these two numbers is given by $\widetilde{a} \oplus \widetilde{b} = \langle m_1 + m_2, \alpha_1 + \alpha_2, \beta_1 + \beta_2 \rangle$. As in the case of interval numbers, we define the acceptability index to the proposition '$\widetilde{a}$ is inferior to $\widetilde{b}$' as $\mathcal{A}$ $(\widetilde{a} \prec \widetilde{b})$ $= (m_2 - m_1)/(\beta_1 + \alpha_2)$. With this index, the dominance relation among triangular fuzzy numbers can be defined as follows:

Definition 3 If $\mathcal{A}$ $(\widetilde{a} \prec \widetilde{b}) \geq 1$ then, $\widetilde{a}$ is said to be 'totally dominating' over $\widetilde{b}$ in the sense of minimization and $\widetilde{b}$ is said to be 'totally dominating' over $\widetilde{a}$ in the sense of maximization. We denote this by $\widetilde{a} \prec \widetilde{b}$.

Definition 4 If $0 < \mathcal{A}$ $(\widetilde{a} \prec \widetilde{b}) < 1$ then A is said to be 'partially dominating' over $\widetilde{b}$ in the sense of minimization and $\widetilde{b}$ is said to be 'partially dominating' over $\widetilde{a}$ in the sense of maximization. This is denoted by $\widetilde{a} \prec_P \widetilde{b}$.

Otherwise, a pessimistic decision-maker would prefer the number whose support is smaller than that of the other and an optimistic decision-maker would do the converse.

Two interval numbers $A = \langle m_1, w_1 \rangle$ and $B = \langle m_2, w_2 \rangle$ are said to be non-dominating if (i) $m_1 = m_2$ and (ii) $w_1 \neq w_2$.

Likewise, two triangular fuzzy numbers $A = \langle m_1, \alpha_1, \beta_1 \rangle$ and $B = \langle m_2, \alpha_2, \beta_2 \rangle$ are said to be non-dominating if (i) $m_1 = m_2$ and (ii) $\alpha_1 = \alpha_2$ or $\beta_1 = \beta_2$ but, not both simultaneously.
If two interval numbers (or triangular fuzzy numbers) are not non-dominating, then we may find a minimum between them using the following function.

Function MIN(A,B)
Input: Two not non-dominating interval numbers (or triangular fuzzy numbers) A and B.
Output: Minimum between A and B.
if (($A \prec B$) or ($A \prec_P B$)){minimum = A;}
else {minimum = B;}
return(minimum);
End MIN.

3. The Algorithm.

Let, $G = (V,E)$ be the network on which near-shortest simple paths between the vertices s and t are required to find. We describe the proposed algorithm in the following.

For all $v \in V$, find $d'(v)$, the shortest-path length from v to t. All of these values can be computed by the algorithm FUZZY-DIST [13], taking t as the source node and traversing edges backwards. It may be the case that there exists more than one shortest path from v to t with same mean, but different widths. If so happens, we choose a single shortest path from v to t, according to the decision maker's view.

Then we run a straightforward s-t path-enumeration algorithm, but do not allow cycles and extend an s-u sub-path to v along the edge $e = (u,v)$ if and only if $\text{MIN}(L(u) \oplus d_{uv} \oplus d'(v), L'_{\min}) = L'_{\min}$, where $L(u)$ is the length of the current s-u sub-path and $L'_{\min} = (1+\varepsilon)L_{\min}$, where $L_{\min} = d'(s)$. Whenever an s-t path is found using the rule, stop the process.

Algorithm FUZZY-NSSP(*T*,*s*,*t*,*ε*)
Input: The adjacency list T of the network G, the source vertex s, target vertex t and the tolerance $\varepsilon \geq 0$.

Output: The near-shortest paths between s and t whose length is less than $(1+\varepsilon)L_{\min}$.

Step 1: for (all $v \in V$)
{$d'(v) \leftarrow$ shortest path distance from v to t; }
$\bar{d} \leftarrow (1+\varepsilon)d'(s)$;
Step 2: for (all $v \in V$ and $\tau = 1,2,\cdots,T$)
{$nextEdge(v,t) \leftarrow firstEdge(v)$;}

theStack ← *s*; *L*(*s*) ← 0;
$\tau(s) \leftarrow 1$; for (all $v \in V - \{s\}$){ $\tau(v) \leftarrow 0$;}
/* τ (v) denotes the number of times the vertex *v* appears on the current sub-path. */
while (*theStack* is non-empty){
    u ← vertex at the top of *theStack*;
    if (*nextEdge*(u, τ (v)) ≠ φ){
        (*u,v*) ← the edge pointed by *nextEdge*(*u*);}
    increment *nextEdge*(u, τ (u));
    if ($\text{MIN}(L(u) \oplus d_{uv} \oplus d'(v), L'_{\min}) = L'_{\min}$){
        if (*v* = *t*){print (*theStack* ∪ *t*)}
        else {push *v* on *theStack*;
            τ (v) ← τ (v) + 1;
            $L(v) \leftarrow L(u) + d_{uv}$;}
    else {pop *u* from *theStack*;
        τ (u) ← τ (u) – 1;
        nextEdge(u, τ (u)) ← *firstEdge*(*u*);}
    }

End FUZZY-NSSP

To find the *k* shortest simple paths between *s* and *t*, we have to order the paths obtained above, by repeated use of the MIN function given in Section 2. Retaining only the first *k* such paths according to their weights, the *k* shortest simple paths problem is also solved.

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