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Unsteady MHD Flow Between two Eccentric Rotating Disks

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ABSTRACT

The unsteady MHD flow of a viscous incompressible electrically conducting fluid between two parallel disks, rotating with uniform angular velocity Ω about two different axes has been studied. An exact solution of the governing equations has been obtained by using Laplace transform. The solution for the velocity distributions as well as shear stresses have been obtained for small times as well as for large times. It is found that both the primary and secondary velocities increase on the left of the axis of rotation with increase in rotation parameter K^2 and the result is reversed on the right of the axis of rotation. It is also found that with increase in Hartman number, the primary velocity increases whereas the secondary velocity decreases on the left of the axis of rotation, while the reversed result observed on the right of the axis of rotation.

Keywords: MHD flow, eccentric rotation, non-coaxial, magnetic parameter.

1. Introduction

The incompressible viscous flow between eccentric rotating disks has been studied by a numbers of researchers. Beker [1] has considered the flow between two disks rotating with same angular velocity. Three dimensional flow between parallel plates which are rotating about a common axis or about distinct axes has been studied by Lai et al. [2]. Knight [3] has studied the inertia effects of the non-Newtonian flow between eccentric disks rotating at different speeds. Rajagopal[4] has considered the flow of a second order fluid between two rotating parallel plates. Hydromagnetic flow between eccentric rotating disks with the same angular velocity has been studied by Mohanty [5]. Rao and Kasiviswanathan [6] have considered the flow of an incompressible viscous fluid between two eccentric rotating disks. Hall effects on the hydromagnetic flow between two rotating disks with non-coincident parallel axes of rotation have been studied by Kanch and Jana [7]. Erdogan [8] has studied the unsteady viscous flow between eccentric rotating disks. Unsteady flow due to concentric rotation of eccentric rotating disks has been studied by Ersoy [9]. Guria et al. [10] have studied the unsteady MHD flow between two eccentric disks.

Our present paper is devoted to study the effect of rotation on the unsteady MHD flow between two disks, rotating with same angular velocity about two different axes at a distant l apart. An external uniform magnetic field is applied perpendicular to the disks. We assumed that the magnetic Reynolds number is small so that the induced magnetic field is neglected. It is found that both the primary and secondary velocities increase on the left of the axis of rotation with increase in rotation parameter K^2 and the result is reversed on the right of the axis of rotation.

2. Mathematical Formulation and its solution

Consider the viscous incompre-ssible conducting fluid occupying the space between two non-coaxial disks. Choose a cartesian system with $z-$ axis perpendicular to the plane of the disks. The upper and lower disks situated at $z = \pm h$ are rotating about the axes through the points $P(0, l, h)$ and $Q(0, -l - h)$ respectively. The middle point of PQ is taken as the origin. Initially, the disks are rotating about $z-$axis with uniform angular velocity Ω. At time $t > 0$, the upper and lower disks suddenly start to ratate about z' and z'' [see Fig.1] axes re--spectively with the same angular velocity Ω. A uniform magnetic field B_0 is applied perpe-ndicular to the disks.

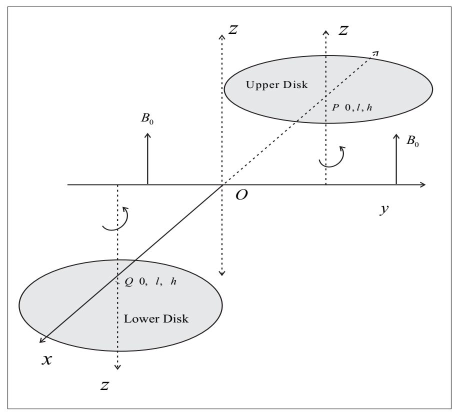

Fig.1 Geometry of the problem

The initial and boundary conditions of the problem are

$$
\begin{aligned}
&u = -\Omega y, && v = \Omega x, w = 0 \quad \text{at } t = 0 \quad \text{for } -h < z < h,\\
&u = -\Omega(y - l), && v = \Omega x, w = 0 \quad \text{at } z = h \quad \text{for } t > 0, \qquad (1)\\
&u = -\Omega(y + l), && v = \Omega x, w = 0 \quad \text{at } z = -h \text{ for } t > 0,
\end{aligned}
$$

where (u, v, w) are the velocity components along x, y and z directions respectively.

The geometry of the problem suggests a solution of the form

$$u = -\Omega y + f(z,t),\ v = \Omega x + g(z,t),\ w = 0. \qquad (2)$$

We shall assume that the magnetic Reynolds number for the flow is very small so that the induced magnetic field may be neglected. Assuming, the magnetic field $\vec{B} \equiv (B_x, B_y, B_z)$, the solenoidal relation $\nabla \cdot \vec{B} = 0$ gives $B_y =$ constant $= B_0$

everywhere in the flow. If (j_x, j_y, j_z) be the components of the electric current density $\vec{j}$, the equation of the conservation of charge $\nabla \cdot \vec{j} = 0$ gives $j_y =$ constant, this constant is zero since disks are non-conducting. Hence, $j_y = 0$ everywhere in the flow. Since the induced magnetic field is neglected, the Maxwell's equation $\nabla \times \vec{E} = 0$ gives $\frac{\partial E_x}{\partial z} = 0$ and $\frac{\partial E_y}{\partial z} = 0$. This implied that $E_x =$ constant and $E_y =$ constant everywhere in the flow.

Subject to the above conditions and on using (2), the equations of motion along x - and y - direction are

$$\nu \frac{\partial^2 f}{\partial z^2} - \frac{\partial f}{\partial t} - \frac{\sigma B_0^2}{\rho} f = \frac{1}{\rho}\frac{\partial p}{\partial x} - \Omega^2 x - \Omega g, \tag{3}$$

$$\nu \frac{\partial^2 g}{\partial z^2} - \frac{\partial g}{\partial t} - \frac{\sigma B_0^2}{\rho} g = \frac{1}{\rho}\frac{\partial p}{\partial y} - \Omega^2 y + \Omega f. \tag{4}$$

The initial and boundary conditions become

$$f(z,0) = 0,\ g(z,0) = 0 \ \text{ for } -h < z < h,$$

$$f(\pm h, t) = \Omega l,\ g(\pm h, t) = 0 \ \text{ for } t > 0. \tag{5}$$

Differentiating (3) with respect to x and (4) with respect to y and adding, we get

$$\nabla^2 p = 2\rho\Omega^2, \tag{6}$$

whose solution is

$$p = \frac{1}{2}\rho\Omega^2\left(x^2 + y^2\right) + A_1 x + B_1 y + C_1, \tag{7}$$

where A_1 and B_1 are unknown functions and C_1 is a constant.

On the use of (7), the equations (3) and (4) become

$$\nu \frac{\partial^2 f}{\partial z^2} - \frac{\partial f}{\partial t} - \frac{\sigma B_0^2}{\rho} f + \Omega g = \frac{A_1}{\rho}, \tag{8}$$

$$\nu \frac{\partial^2 g}{\partial z^2} - \frac{\partial g}{\partial t} - \frac{\sigma B_0^2}{\rho} g - \Omega f = \frac{B_1}{\rho}. \tag{9}$$

Combining (8) and (9), we get

$$\nu \frac{\partial^2 F}{\partial z^2} - \frac{\partial F}{\partial t} - \left(\frac{\sigma B_0^2}{\rho} + i\Omega\right) F = C_2, \tag{10}$$

where $F = f + ig$ and $C_2 = \frac{1}{\rho}\left(A_1 + iB_1\right)$. (11)

As the flow is symmetry, we may choose the constant $C_2 = 0$, i.e. $A_1 = B_1 = 0$ and hence the equation (10) becomes

$$\nu \frac{\partial^2 F}{\partial z^2} - \frac{\partial F}{\partial t} - \left(\frac{\sigma B_0^2}{\rho} + i\Omega\right) F = 0. \tag{12}$$

Introducing non-dimensional variables

$$F_1 = \frac{F}{\Omega l}, \eta = \frac{z}{h}, \tag{13}$$

above equation (12) becomes

$$\frac{\partial^2 F_1}{\partial \eta^2} - \frac{\partial F_1}{\partial \tau} - \left(M^2 + iK^2\right) F_1 = 0, \tag{14}$$

where $K^2 = \frac{\Omega h^2}{\nu}$ is the rotation parameter and $M^2 = \frac{\sigma B_0^2 h^2}{\rho \nu}$ the magnetic parameter.

The corresponding initial and boundary conditions become

$$F_1(\eta, 0) = 0 \text{ for } -1 < \eta < 1,$$

$$F_1(\pm 1, \tau) = \pm 1, F_1(0, \tau) = 0 \text{ [symmetric condition]}. \tag{15}$$

To solve the equation (14), we substitute

$$F_1(\eta, \tau) = H(\eta, \tau) e^{-\left(M^2 + iK^2\right)\tau}, \tag{16}$$

and we get

$$\frac{\partial H}{\partial \tau} = \frac{\partial^2 H}{\partial \eta^2}. \tag{17}$$

The initial and boundary conditions (15) become

$$H(\eta, 0) = 0 \text{ for } -1 < \eta < 1,$$

$$H(\pm 1, \tau) = \pm e^{\left(M^2 + iK^2\right)\tau}, H(0, \tau) = 0. \tag{18}$$

Taking Laplace's transform of (18), we get

$$\frac{d^2 \bar{H}}{d\eta^2} = s\bar{H}, \tag{19}$$

where $\bar{H} = \int_0^\infty H e^{-s\tau} d\tau.$ (20)

The corresponding boundary conditions for $\bar{H}(\eta, s)$ are

$$\bar{H}(\pm 1, s) = \pm \frac{1}{s - \left(M^2 + iK^2\right)}. \tag{21}$$

The solution of the equation (19) subject to the boundary conditions (21) is

$$\bar{H}(\eta, s) = \frac{1}{s - \left(M^2 + iK^2\right)} \left(\frac{e^{\sqrt{s}\eta} - e^{-\sqrt{s}\eta}}{e^{\sqrt{s}} - e^{-\sqrt{s}}} \right). \tag{22}$$

Taking inverse Laplace's transform of the equation (22), we have

$$H(\eta,\tau)=\frac{\sinh\left(\sqrt{M^2+iK^2}\ \eta\right)}{\sinh\sqrt{M^2+iK^2}}+\sum_{n=1}^{\infty}\frac{2n\pi(-1)^n\sin(n\pi\eta)}{n^2\pi^2+M^2+iK^2}\,e^{-\left(n^2\pi^2+M^2+iK^2\right)\tau}. \qquad (23)$$

On the use of (16), we have

$$F_1(\eta,\tau)=\frac{\sinh(\alpha+i\beta)\eta}{\sinh(\alpha+i\beta)}+\sum_{n=1}^{\infty}\frac{2n\pi(-1)^n}{n^2\pi^2+M^2+iK^2}\sin(n\pi\eta)\,e^{-\lambda_n^2\tau}, \qquad (24)$$

where $\lambda_n^2=n^2\pi^2+M^2+iK^2,\ \alpha,\beta=\frac{1}{\sqrt{2}}\left[\left(M^4+K^4\right)^{\frac{1}{2}}\pm M^2\right]^{1/2}.$ (25)

On separating into a real and imaginary parts, we get

$$\frac{f}{\Omega l}=\frac{S(\alpha\eta)S(\alpha)+C(\alpha\eta)C(\alpha)}{S^2(\alpha)+C^2(\alpha)}+2\sum_{n=1}^{\infty}\frac{n\pi(-1)^n\sin(n\pi\eta)}{\left(n^2\pi^2+M^2\right)^2+K^4}\times\left[\left(n^2\pi^2+M^2\right)\cos K^2\tau-K^2\sin K^2\tau\right]e^{-\left(n^2\pi^2+M^2\right)\tau}, \qquad (26)$$

$$\frac{g}{\Omega l}=\frac{C(\alpha\eta)S(\alpha)-S(\alpha\eta)C(\alpha)}{S^2(\alpha)+C^2(\alpha)}-2\sum_{n=1}^{\infty}\frac{n\pi(-1)^n\sin(n\pi\eta)}{\left(n^2\pi^2+M^2\right)^2+K^4}\times\left[K^2\cos K^2\tau+\left(n^2\pi^2+M^2\right)\sin K^2\tau\right]e^{-\left(n^2\pi^2+M^2\right)\tau}, \qquad (27)$$

where $S(\alpha\eta)=\sinh(\alpha\eta)\cos(\alpha\eta),\quad C(\alpha\eta)=\cosh(\alpha\eta)\sin(\alpha\eta),$

$S(\alpha)=\sinh(\alpha)\cos(\alpha),\quad C(\alpha)=\cosh(\alpha)\sin(\alpha).$ (28)

If $M^2=0$, then the equations (26) and (27) coincide with equations (3.5) and (3.6) of the Erdogan [2].

For small time $(\tau \ll 1)$ which correspond to large s, the equation (22) can be rewritten as

$$\bar{H}(\eta,s)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\left(M^2+iK^2\right)^n}{s^{n+1}}\left[e^{-(1+2m-\eta)\sqrt{s}}-e^{-(1+2m+\eta)\sqrt{s}}\right] \qquad (29)$$

The inverse transform of the above equation (24) and on using (16), yields

$$F_1(\eta,\tau)=e^{-\left(M^2+iK^2\right)\tau}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left\{\left(M^2+iK^2\right)^n(4\tau)^n\times\left[i^{2n}\mathrm{erfc}\left(\frac{2m+1-\eta}{2\sqrt{\tau}}\right)-i^{2n}\mathrm{erfc}\left(\frac{2m+1+\eta}{2\sqrt{\tau}}\right)\right]\right\}, \qquad (30)$$

where $i^n\mathrm{erfc}(x)=\int_x^{\infty}i^{n-1}\mathrm{erfc}(\xi)\,d\xi,$

$$i^{-1}\operatorname{erfc}(x)=\frac{2}{\sqrt{\pi}}e^{-x^2},\ i^0\operatorname{erfc}(x)=\operatorname{erfc}(x). \tag{31}$$

If $M^2=0$, then the equation (24) coincides with the equation (26) of Erdogan [2].

On the use of (11) and (13) and separating into a real and imaginary parts, equation (30) yields

$$\frac{f}{\Omega l}=e^{-M^2\tau}\left[A(\eta,\tau)\cos K^2\tau+B(\eta,\tau)\sin K^2\tau\right],$$

$$\frac{g}{\Omega l}=e^{-M^2\tau}\left[B(\eta,\tau)\cos K^2\tau-A(\eta,\tau)\sin K^2\tau\right], \tag{32}$$

where $A(\eta,\tau)=T_0+M^2(4\tau)T_2+\left(M^4-K^4\right)(4\tau)^2T_4$

$$+\left(M^6-3M^2K^4\right)(4\tau)^3T_6+\cdots, \tag{33}$$

$$B(\eta,\tau)=K^2(4\tau)T_2+2M^2K^2(4\tau)^2T_4+3\left(M^2K^2-K^6\right)(4\tau)^3T_6+\cdots, \tag{34}$$

with $$T_r=\sum_{m=0}^{\infty}\left[i^r\operatorname{erfc}\left(\frac{2m+1-\eta}{2\sqrt{\tau}}\right)-i^r\operatorname{erfc}\left(\frac{2m+1+\eta}{2\sqrt{\tau}}\right)\right], r=0,2,4,6,\cdots. \tag{35}$$

3. Results and discussion

To study the effect of rotation parameter K^2 and the magnetic parameter M^2, the stream wise velocity profiles for the primary velocity $f/\Omega l$ and secondary velocity $g/\Omega l$ are depicted graphically against η for different values of K^2, τ and M^2 in the Figs.2-6. Fig.2 shows that both the primary velocity and the secondary velocity increase on the left of the axis of rotation with increase in rotation parameter K^2 and the result is reversed on the right of the axis of rotation. It is observed from Fig.3 that the primary velocity increases whereas the magnitudes of the secondary velocity decreases on the left of the axis of rotation with increase in M^2 and the reversed result observed on the right of the axis of rotation.

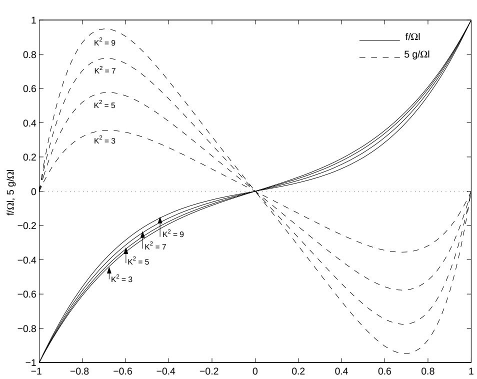

Fig.2 Variations of $f/\Omega l$ and $g/\Omega l$ for $M^2=5, \tau=0.2$.

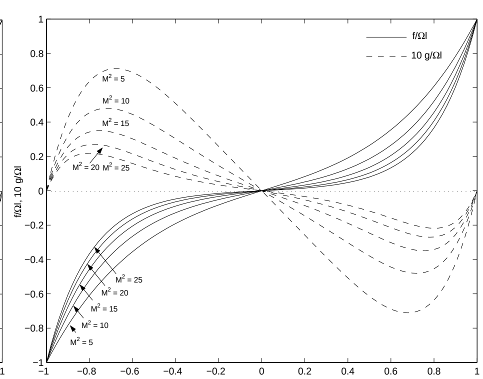

Fig.3 Variations of $f/\Omega l$ and $g/\Omega l$ for $M^2=3, \tau=0.2$

It is seen from Fig.4 that the primary velocity decreases whereas the secondary velocity increases on the left of the axis of rotation with increase in time τ. On the other hand on the left of the axis of rotation of the disk the primary velocity increases while the secondary velocity decreases with increases in τ. For small times, the velocity distributions are shown in the Figs.5 and 6. It is observed from Fig.5 that the primary velocity $f/\Omega l$ increases whereas the secondary velocity $g/\Omega l$ decreases on the left of the axis of rotation with increase in magnetic parameter M^2 and the result is reversed on the right of the axis of rotation. Fig.6 shows that the primary velocity $f/\Omega l$ decreases whereas the secondary velocity $g/\Omega l$ increases on the left of the axis of rotation with increase in time τ and the reversed result shows on the right of the axis of rotation.

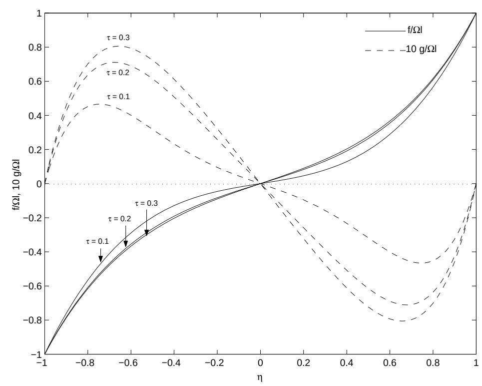

Fig.4 Variations of $f/\Omega l$ and $g/\Omega l$ for $M^2 = 5, K^2 = 3$.

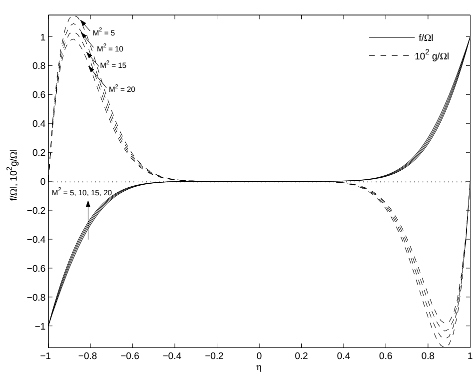

Fig.5 Variations of $f/\Omega l$ and $g/\Omega l$ for small time solution with $K^2 = 5, \tau = 0.002$.

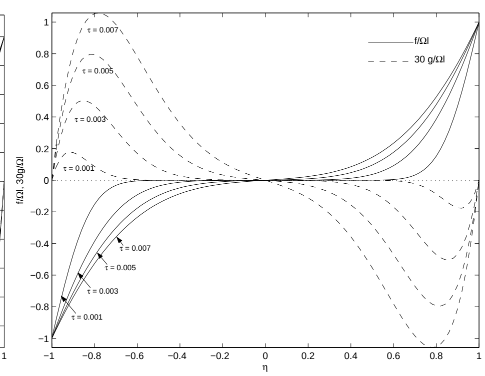

Fig.6 Variations of $f/\Omega l$ and $g/\Omega l$ for the small time solution with $M^2 = 5, K^2 = 3$.

The non-dimensional shear stress components, for general solution, at the disks $\eta = \pm 1$ are given by

$$\tau_x\Big)_{\eta=\pm 1} = \frac{\alpha\sinh 2\alpha + \beta\sin 2\beta}{\cosh 2\alpha - \cos 2\beta} + 2\sum_{n=1}^{\infty}\frac{n^2\pi^2}{\left(n^2\pi^2+M^2\right)^2+K^4}$$
$$\times\left[\left(n^2\pi^2+M^2\right)\cos K^2\tau - K^2\sin K^2\tau\right]e^{-\left(n^2\pi^2+M^2\right)\tau}, \tag{36}$$
$$\tau_y\Big)_{\eta=\pm 1} = \frac{\beta\sinh 2\alpha - \alpha\sin 2\beta}{\cosh 2\alpha - \cos 2\beta} - 2\sum_{n=1}^{\infty}\frac{n^2\pi^2}{\left(n^2\pi^2+M^2\right)^2+K^4}$$
$$\times\left[K^2\cos K^2\tau + \left(n^2\pi^2+M^2\right)\sin K^2\tau\right]e^{-\left(n^2\pi^2+M^2\right)\tau}, \tag{37}$$

For small times, the non-dimensional shear stresses at the disks $\eta = 1$ and $\eta = -1$ are given by

$$\tau_x = \frac{e^{M^2\tau}}{2\sqrt{\tau}}\left[C(\pm 1,\tau)\cos K^2\tau + D(\pm 1,\tau)\sin K^2\tau\right], \tag{38}$$
$$\tau_y = \frac{e^{M^2\tau}}{2\sqrt{\tau}}\left[D(\pm 1,\tau)\cos K^2\tau - C(\pm 1,\tau)\sin K^2\tau\right], \tag{39}$$

where $C(\eta,\tau) = Y_{-1} - M^2(4\tau)Y_1 + \left(M^4-K^4\right)(4\tau)^2 Y_3$
$$+\left(M^6 - 3M^2K^4\right)(4\tau)^3 Y_5 + \cdots, \tag{40}$$
$$D(\eta,\tau) = K^2\left[(4\tau)Y_1 + 2M^2(4\tau)^3 Y_3 + \left(3M^4 - K^4\right)(4\tau)^5 Y_5 + \cdots\right], \tag{41}$$

with
$$Y_{r-1} = \sum_{m=0}^{\infty}\left[i^{r-1}\operatorname{erfc}\left(\frac{2m+1-\eta}{2\sqrt{\tau}}\right) + i^{r-1}\operatorname{erfc}\left(\frac{2m+1+\eta}{2\sqrt{\tau}}\right)\right],$$
$$r = 0, 2, 4, 6, \cdots. \tag{42}$$

Table-I

Shear stress due to primary flow for $M^2 = 5$

$K^2 \setminus \tau$	$-\tau_x$ (For General solution)			$-\tau_x$ (Solution for small times)		
	0.005	0.010	0.015	0.005	0.010	0.015
0	5.896093	3.644242	2.673381	5.896093	3.644238	2.673358
4	6.032860	3.765285	2.782767	6.032872	3.765358	2.782959
8	6.290388	3.998794	2.998242	6.290404	3.998886	2.998493
12	6.515985	4.208713	3.196242	6.515995	4.208762	3.196378

Table-II

Shear stress due to secondary flow for $M^2 = 5$

$K^2 \setminus \tau$	τ_y (For General Solution)			τ_y (Solution for small times)		
	0.005	0.010	0.015	0.005	0.010	0.015
0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
4	0.393570	0.361540	0.337486	0.393615	0.361790	0.338174
8	0.595888	0.555407	0.524776	0.595908	0.555517	0.525087
12	0.660252	0.622620	0.594005	0.660258	0.622650	0.594089

The numerical values of the shear stress components calculated from equations (36), (37), (38) and (39) are given in Tables-I and -II for different values of K^2 and τ. It is observed that for small times the shear stresses calculated from the equations (38) and (39) give better result than that calculated from equations (36) and (37). Hence, we conclude for small times shear stress components should be evaluated from equations (38) and (39) instead of equations (36) and (37).

Conclusion

Unsteady MHD flow of a viscous incompressible electrically conducting fluid between two disks, rotating with same angular velocity about two different axes for small as well as large times τ is studied. It is found that both the primary and secondary velocities increase on the left of the axis of rotation with increase in rotation parameter K^2 and the result is reversed on the right of the axis of rotation. It is also found that with increase in Hartman number, the primary velocity increases whereas the secondary velocity decreases on the left of the axis of rotation, while the reversed result observed on the right of the axis of rotation.

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