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Reliability Analysis for n Non-Independent and Non-Identical Series Systems Using Masked Data

Fan Zhang and Yi-min Shi
Department of Applied Mathematics, Northwestern Polytechnical University , Xi'an 710072, P. R. China
ymshi@nwpu.edu.cn

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ABSTRACT

Based on the masked data, the reliability of n non-independent and non-identical series system subjected to $n+1$ sources of fatal shocks is investigated. We get the parameter estimations as well as reliability estimations by adopting Bayes approach. Also, a numerical simulation example is given to illustrate how one can utilize the method to tackle the practical problem

Keywords: masked data; non-independent system; Bayes approach; reliability analysis; shock model

1. Introduction

In reliability analysis, estimations of components reliabilities are often obtained through the analysis of system life data. Under ideal circumstances, this system life data contains the failure time along with information on the exact component causing the system failure. However, in some cases, the exact component responsible for the system failure can not be identified due to the cost of failure diagnosis and test, time constraints, the destructive nature of some component and so on. Instead, it is assumed that the component causes the system failure belongs to some subset of the components which considered potentially responsible for the failure. In this case, the cause of failure is masked.

Various studies used masked data to estimate the unknown parameters in a system. A. M. Sarhan [1] considered the maximum likelihood estimations(MLE) and Bayes estimations of exponential components, and he presented MLE of unknown parameters of Weibull failure rate components for the cases of two-component and three-component series systems[2]. A. M. Sarhan and Ahmed H. El-Bassiouny dicussed the case of parallel systems of complementary exponential components [3]. Most authors assumed that components in a system must be independent in order to construct models. However, in some cases, it is difficult to determine the independence. Moreover there exists some dependent and constrained relation between units very often. Thus, it is important and meaningful to discuss the reliability of non-independent series system using masked data.

Recently, several authors used shock model to estimate the parameters in non-independent series system. Such as, Grabski and Sarhan[5] and Sarhan [6]obtained estimations of some reliability measures for series and parallel systems with two non-independent and non-identical components; Awad El-Gohary[7] and Awad El-Gohary and Sarhan [9] deduced Bayes estimators for the parameters included in a two and three non-independent and non-identical component series system. Hongping Wu and Guofen Zhang [10] presented a Bayesian approach for estimating the unknown parameters in a n non-independent and non-identical series system subjected to $n+1$ sources of fatal shocks. However, there is very little statistical analysis of non-independent system under masked data at present.

In this paper, we discuss how to use the masked data of systems to analyze the system and component reliabilities of n non-independent and non-identical series system subjected to $n+1$ sources of fatal shocks which was proposed in [10]. Simulation studies are also done in order to explain how one can utilize the theoretical results obtained.

2. Likelihood function

Hongping Wu and Guofen Zhang [10] constructed the mathematical model of n non-independent and non-identical series system. The model is described as followings: The system consists of n components connected in series. There are $n+1$ independent sources of fatal shocks directed to the system. A shock from the ith source destroys the ith component, $i=1,2,...,n$, while the shock from source $n+1$ destroys all the components of the system. A shock from source i occurs at a random time, say U_i. The distribution function of U_i is the following form:

$$P(U_i \le t) = 1-\exp\{-(\alpha_i t + \frac{1}{2}\beta_i t^2)\}, \quad t \ge 0, \quad \alpha_i, \beta_i > 0, \quad i = 1,2,...,n+1.$$

Assuming T_i denotes the lifetime of component i, then $T_i = \min(U_i, U_{n+1})$, $i=1,2,...,n$. According to the independence of the shocks, the reliability function of component i is the following form:

$$R_i(t;\alpha_i,\beta_i,\alpha_{n+1},\beta_{n+1}) = P(T_i > t) = P(\min(U_i,U_{n+1}) > t) = P(U_i > t, U_{n+1} > t)$$

$$= P(U_i > t)P(U_{n+1} > t) = \exp\{-[(\alpha_i + \alpha_{n+1})t + \frac{1}{2}(\beta_i + \beta_{n+1})t^2]\}.$$

where $t \geq 0$, $\alpha_i, \beta_i, \alpha_{n+1}, \beta_{n+1} > 0$.

The system lifetime, say T_a, satisfies $T_a = \min(T_1, T_2, ..., T_n)$. Let $\underline{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_{n+1})$, $\underline{\beta} = (\beta_1, \beta_2, ..., \beta_{n+1})$, then the survival function of the system is given by

$$R_a(t; \underline{\alpha}, \underline{\beta}) = P(T_a > t) = \exp\{-\sum_{j=1}^{n+1}(\alpha_j t + \frac{1}{2}\beta_j t^2)\}, \quad t \geq 0, \alpha_i, \beta_i > 0, \quad i = 1, 2, ..., n+1$$

Assuming that m identical systems of the described type are put on the life test， and the test is terminated if all the systems have failed. Let S_i denote the set of components that possibly cause the system i to fail, and let s_i denote the realized set for S_i, then the observed data is $(t_1, s_1), ..., (t_m, s_m)$. It is explicit of the true cause of failure of system i when $S_i = \{j\}$, $j = 1, 2, ..., n+1$, where $S_i = \{n+1\}$ represents the all components are failure. On the other hand, the true cause of failure is masked when $S_i = \{1, 2, ..., n+1\} \triangleq S$. Let K_i denote the index of the component actually causing the ith system to fail. As for the ith system, the likelihood function of (t_i, s_i) is: $P(t_i, s_i) = \Sigma_{j=1}^{3} P(t_i, K_i = j) P(s_i \mid t_i, K_i = j)$, where $P(s_i \mid t_i, K_i = j)$ is the masking probability and it becomes 0 if $j \notin s_i$. Here we suppose that the masking probability is independent with the cause of failure, that is, for the fixed $j \in s_i$, $P(S_i = s_i \mid T_i = t_i, K_i = j) = P(S_i = s_i \mid T_i = t_i, K_i = j')$ for all $j' \in s_i$. Therefore, we get the simplified likelihood function:

$$L(\theta, data) = \prod_{i=1}^{m}[\sum_{j \in s_i} P(t_i, s_i)] = \prod_{i=1}^{m}[\sum_{j \in s_i}\sum_{j=1}^{n+1} P(t_i, K_i = j)],$$

where θ denotes the unknown parameters and $data$ denotes the observed data.

Let $x_{k,1}, x_{k,2}, ..., x_{k,n_k}$ be the observed system time to failure when $S_i = \{k\}$. It means that n_k is the number of the observations when $S_i = \{k\}$. Let $y_1, y_2, ..., y_N$ be the observed system time to failure when the causing system failure is masked. That is, N denotes the number of the observations when $S_i = S$. The likelihood function in this case reduces to

$$L = \prod_{k=1}^{n+1}\prod_{i=1|s_i=\{k\}}^{n_k} P(t_i, K_i = k) \prod_{i=1|s_i=S}^{N}\{P(t_i, K_i = 1) + ... + P(t_i, K_i = n+1)\}$$

Noted that $m=\sum_{i=1}^{n+1} n_i + N$ and

$$P(t_i, K_i=1)=P(t_i, t_i<U_i<t_i+dt_i, U_2>t_i, ..., U_{n+1}>t_i)=(\alpha_1+\beta_1 t_i)\exp\{-\sum_{j=1}^{n+1}(\alpha_j t_i+\frac{1}{2}\beta_j t_i^2)\} \quad .$$

Similarly, $P(t_i, K_i=k)=(\alpha_k+\beta_k t_i)\exp\{-\sum_{j=1}^{n+1}(\alpha_j t_i+\frac{1}{2}\beta_j t_i^2)\}$, $k=1,2,...,n+1$.

The likelihood function in this case becomes

$$L=\prod_{k=1}^{n+1}\prod_{i=1}^{n_k}(\alpha_k+\beta_k x_{k,i})\exp[-\sum_{j=1}^{n+1}(\alpha_j x_{k,i}+\frac{1}{2}\beta_j x_{k,i}^2)]\prod_{i=1}^{N}(\alpha'+\beta' y_i)\exp[-\sum_{j=1}^{N}(\alpha_j y_i+\frac{1}{2}\beta_j y_i^2)]$$

$$=\prod_{k=1}^{n+1}\prod_{i=1}^{n_k}(\alpha_k+\beta_k x_{k,i})\prod_{i=1}^{N}(\alpha'+\beta' y_i)\exp[-\sum_{j=1}^{n+1}(\alpha_j T+\beta_j \widetilde{T})] .$$

where $\alpha'=\alpha_1+...+\alpha_{n+1}$, $\beta'=\beta_1+...+\beta_{n+1}$, $T=\sum_{i=1}^{m} t_i$, $\widetilde{T}=\sum_{i=1}^{m} t_i^2 \Big/ 2$.

Applying the binomial expansion, $\prod_{i=1}^{n_j}(\alpha_j+\beta_j x_{j,i})$ can be written as: $\sum_{k_j=0}^{n_j} C_{n_j}^{k_j}\alpha_j^{n_j-k_j}\beta_j^{k_j}\tau_{k_j}$, where $\tau_{k_j}=\sum_{\substack{i_1,i_2,...,i_{k_j}\\ i_1\neq i_2\neq ...\neq i_{k_j}}} x_{j,i_1}\cdots x_{j,i_{k_j}}$. $\prod_{i=1}^{N}(\alpha'+\beta' y_i)$ can be written as:

$$\sum_{r=0}^{N}\sum_{r_1=0}^{N-r_1}\sum_{r_2=0}^{r_1}\cdots\sum_{r_n=0}^{r_{n-1}}\sum_{l_1=0}^{r}\sum_{l_2=0}^{l_1}\cdots\sum_{l_n=0}^{l_{n-1}}\tau_r C_N^{(r,l)}$$
$$\times\prod_{v=1}^{n+1}\alpha_v^{(N-r-r_1)\delta_{v1}+(r_1-r_2)\delta_{v2}+\cdots+(r_{n-1}-r_n)\delta_{vn}+r_n\delta_{v(n+1)}}$$
$$\times\prod_{v=1}^{n+1}\beta_v^{(r-l_1)\delta_{v1}+(l_1-l_2)\delta_{v2}+\cdots+(l_{n-1}-l_n)\delta_{vn}+l_n\delta_{v(n+1)}}$$

where $\tau_r=\sum_{\substack{i_1,i_2,...,i_r\\ i_1\neq i_2\neq ...\neq i_r}} y_{i_1}\cdots y_{i_r}$, $\delta_{vj}=\begin{cases}1 & v=j\\ 0 & v\neq j\end{cases}$,

and $C_N^{(r,l)}=\dfrac{N!}{(N-r-r_1)!(r_1-r_2)!...(r_{n-1}-r_n)!r_n!(r-l_1)!(l_1-l_2)!...(l_{n-1}-l_n)!l_n!}$.

Thus, the likelihood function can be written as in the following form:

$$L(t;\underline{\alpha},\underline{\beta})=\Sigma\cdots\Sigma\tau\cdot C_m^{(k,r,l)}\prod_{v=1}^{n+1}\alpha_v^{A_v}\prod_{v=1}^{n+1}\beta_v^{B_v}\exp[-\sum_{j=1}^{n+1}(\alpha_j T+\beta_j\widetilde{T})] .$$

where $\Sigma\cdots\Sigma \triangleq \sum_{k_1=0}^{n_1}\cdots\sum_{k_{n+1}=0}^{n_{n+1}}\sum_{r=0}^{N}\sum_{r_1=0}^{N-r}\sum_{r_2=0}^{r_1}\cdots\sum_{r_n=0}^{r_{n-1}}\sum_{l_1=0}^{r}\sum_{l_2=0}^{l_1}\cdots\sum_{l_n=0}^{l_{n-1}}$, $\tau=\tau_{k_1}...\tau_{k_{n+1}}\tau_r$,

$C_m^{(k,r,l)}=C_{n_1}^{k_1}\cdots C_{n_{n+1}}^{k_{n+1}}C_N^{(r,l)}$,

$A_1 = N - r - r_1 + n_1 - k_1$, $A_i = r_{i-1} - r_i + n_i - k_i$ $(i = 2,..,n)$, $A_{n+1} = r_n + n_{n+1} - k_{n+1}$,
and $B_1 = r - l_1 + k_1$, $B_i = l_{i-1} - l_i + k_i$ $(i = 2,..,n)$, $B_{n+1} = l_n + k_{n+1}$.

3. Bayes analysis

In this section, we will use Bayesian approach to estimate the unknown parameters and the reliability functions of components and system. The following assumptions will be considered.

A1. α_v, β_v, $v = 1,2,...,n+1$ are independent with each other.

A2. α_v, β_v, $v = 1,2,...,n+1$ have gamma prior distributions with known parameters g_v, h_v and c_v, d_v. It means that $\alpha_v \mid t \sim \Gamma(g_v, h_v)$, $\beta_v \mid t \sim \Gamma(c_v, d_v)$.

A3. The loss incurred when the vector of unknown parameters $\underline{\alpha}, \underline{\beta}$ are estimated by $\underline{\hat{\alpha}}, \underline{\hat{\beta}}$ are a quadratic. That is , the loss function is $L(\underline{\alpha}, \underline{\beta}; \hat{\alpha}, \hat{\beta}) = \sum_{v=1}^{n+1} k_{1v}(\hat{\alpha}_v - \alpha_v)^2 + \sum_{v=1}^{n+1} k_{2v}(\hat{\beta}_v - \beta_v)^2$.

Using the assumption 1 and 2, the joint prior probability density function (pdf) of $\underline{\alpha}, \underline{\beta}$, say $g(\underline{\alpha}, \underline{\beta})$, takes the following form:

$$g(\underline{\alpha}, \underline{\beta}) = \prod_{v=1}^{n+1} \frac{h_v^{g_v}}{\Gamma(g_v)} \alpha_v^{g_v - 1} e^{-h_v \alpha_v} \prod_{v=1}^{n+1} \frac{d_v^{c_v}}{\Gamma(c_v)} \beta_v^{c_v - 1} e^{-d_v \beta_v}.$$

Now we are ready to present a theorem that gives the joint posterior pdf of $(\underline{\alpha}, \underline{\beta})$ given the observed data.

Theorem 1 Given the observed data and under assumptions 1 and 2, the joint posterior pdf of $\underline{\alpha}, \underline{\beta}$ is

$$g(\underline{\alpha}, \underline{\beta} \mid data) = \frac{1}{\Phi(0)} \Sigma \cdots \Sigma \tau \cdot C_m^{(k,r,l)} \prod_{v=1}^{n+1} \alpha_v^{A_v + g_v - 1} \prod_{v=1}^{n+1} \beta_v^{B_v + c_v - 1} \prod_{v=1}^{n+1} \exp[-\alpha_v(h_v + T)] \exp[-\beta_v(d_v + \tilde{T})]$$

where $\Phi(0) = \Sigma \cdots \Sigma \tau \cdot C_m^{(k,r,l)} \prod_{v=1}^{n+1} \frac{\Gamma(A_v + g_v)}{(T + h_v)^{A_v + g_v}} \prod_{v=1}^{n+1} \frac{\Gamma(B_v + c_v)}{(\tilde{T} + d_v)^{B_v + c_v}}$.

Proof Using the Bayes theorem, the joint posterior of $(\underline{\alpha}, \underline{\beta})$ given the observed data is related to the following relation, $g(\underline{\alpha}, \underline{\beta} \mid data) = g(\underline{\alpha}, \underline{\beta}) L(data \mid \underline{\alpha}, \underline{\beta}) / \Phi(0)$, where

$$\Phi(0)=\int_{\Theta_{\alpha_1}}\cdots\int_{\Theta_{\alpha_{n+1}}}\int_{\Theta_{\beta_1}}\cdots\int_{\Theta_{\beta_{n+1}}} L(data\,|\,\underline{\alpha},\underline{\beta})g(\underline{\alpha},\underline{\beta})d\alpha_1\cdots d\alpha_{n+1}d\beta_1\cdots d\beta_{n+1}$$

$$=\Sigma\cdots\Sigma\tau\cdot C_m^{(k,r,l)}\prod_{v=1}^{n+1}\int_0^{+\infty}\alpha_v^{A_v+g_v-1}\exp[-\alpha_v(h_v+T)]d\alpha_v\prod_{v=1}^{n+1}\int_0^{+\infty}\beta_v^{B_v+c_v-1}\exp[-\beta_v(d_v+\widetilde{T})]d\beta_v$$

$$=\Sigma\cdots\Sigma\tau\cdot C_m^{(k,r,l)}\prod_{v=1}^{n+1}\frac{\Gamma(A_v+g_v)}{(T+h_v)^{A_v+g_v}}\prod_{v=1}^{n+1}\frac{\Gamma(B_v+c_v)}{(\widetilde{T}+d_v)^{B_v+c_v}}.\ \square$$

Corollary 1 The marginal posterior pdf of α_v,β_v, $v=1,2,...,n+1$, given the observed data is

$$g(\alpha_v\,|\,data)=\frac{1}{\Phi(0)}\Sigma\cdots\Sigma\tau\cdot C_m^{(k,r,l)}\times\alpha_v^{A_v+g_v-1}\exp[-\alpha_v(h_v+T)]\times\prod_{l\in L_v}\frac{\Gamma(A_l+g_l)}{(T+h_l)^{A_l+g_l}}\prod_{v=1}^{n+1}\frac{\Gamma(B_v+c_v)}{(\widetilde{T}+d_v)^{B_v+c_v}}$$

$$g(\beta_v\,|\,data)=\frac{1}{\Phi(0)}\Sigma\cdots\Sigma\tau\cdot C_m^{(k,r,l)}\times\beta_v^{B_v+c_v-1}\exp[-\beta_v(d_v+\widetilde{T})]\times\prod_{v=1}^{n+1}\frac{\Gamma(A_v+g_v)}{(T+h_v)^{A_v+g_v}}\prod_{l\in L_v}\frac{\Gamma(B_l+c_l)}{(\widetilde{T}+d_l)^{B_l+c_l}}$$

where $L_v=\{1,2,...,n+1\}/\{v\}$.

Proof The proof of this corollary can be reached by integrating the joint posterior pdf of $(\underline{\alpha},\underline{\beta})$ given the observed data over all variables α_l,β_l, $l\in L_v$ respectively. □

Lemma 1. The rth moment, $r=1,2,...$, of the marginal posterior pdf of α_v,β_v, $v=1,2,...,n+1$, are given in the following form:

$$\mu_{\alpha_v}^{(r)}=\Phi_{\alpha_v}(r)/\Phi(0),\quad \mu_{\beta_v}^{(r)}=\Phi_{\beta_v}(r)/\Phi(0),$$

where $\Phi_{\alpha_v}(r)=\Sigma\cdots\Sigma\tau\cdot C_m^{(k,r,l)}\times\frac{\Gamma(A_v+g_v+r)}{(T+h_v)^{A_v+g_v+r}}\times\prod_{l\in L_v}\frac{\Gamma(A_l+g_l)}{(T+h_l)^{A_l+g_l}}\prod_{v=1}^{n+1}\frac{\Gamma(B_v+c_v)}{(\widetilde{T}+d_v)^{B_v+c_v}}$

and $\Phi_{\beta_v}(r)=\Sigma\cdots\Sigma\tau\cdot C_m^{(k,r,l)}\times\frac{\Gamma(B_v+c_v+r)}{(\widetilde{T}+d_v)^{B_v+c_v+r}}\times\prod_{v=1}^{n+1}\frac{\Gamma(A_v+g_v)}{(T+h_v)^{A_v+g_v}}\prod_{l\in L_v}\frac{\Gamma(B_l+c_l)}{(\widetilde{T}+d_l)^{B_l+c_l}}$.

Proof . The proof of this lemma can be reached as follows:

The rth posterior moment of α_v,β_v are defined as the posterior expectation of α_v^r,β_v^r, that is $\mu_{\alpha_v}^{(r)}=\int\alpha_v^r g(\alpha_v\,|\,t)d\alpha_v$, $\mu_{\beta_v}^{(r)}=\int\beta_v^r g(\beta_v\,|\,t)d\beta_v$. Substituting the posterior pdf of α_v,β_v into these formulas and making some calculus arrangements, one can easily reach the proof. □

Theorem 2 Under the assumption A1- A3:

B1. The Bayes estimators of α_v,β_v are respectively $\hat{\alpha}_v=\mu_{\alpha_v}^{(1)}$, $\hat{\beta}_v=\mu_{\beta_v}^{(1)}$.

B2. The minimum posterior risk associated with the Bayes estimators $\hat{\alpha}_v, \hat{\beta}_v$ are

$$R_\pi(\alpha_v \mid data) = \Phi_{\alpha_v}(2)/\Phi(0) - \left(\Phi_{\alpha_v}(1)/\Phi(0)\right)^2 ,$$

$$R_\pi(\beta_v \mid data) = \Phi_{\beta_v}(2)/\Phi(0) - \left(\Phi_{\beta_v}(1)/\Phi(0)\right)^2 .$$

Proof The proof of this theorem depends on the assumption A3 .The Bayes estimators of α_v, β_v and the associated minimum posterior risk are defined respectively as the posterior expectation and posterior variance of α_v, β_v. Namely,

$$\hat{\alpha}_v = E(\alpha_v \mid data) = \mu^{(1)}_{\alpha_v}, \quad \hat{\beta}_v = E(\beta_v \mid data) = \mu^{(1)}_{\beta_v},$$

$$R_\pi(\alpha_v \mid data) = Var(\alpha_v \mid data) = \mu^{(2)}_{\alpha_v} - (\mu^{(1)}_{\alpha_v})^2 ,$$

$$R_\pi(\beta_v \mid data) = Var(\beta_v \mid data) = \mu^{(2)}_{\beta_v} - (\mu^{(1)}_{\beta_v})^2 .$$

Substituting the rth moment, $r = 1,2$, of the posterior pdf of α_v, β_v, one can complete the proof of theorem. □

The following theorems give the Bayes estimators and the associated minimum posterior risk for the value of reliability functions of components and system at a specified mission time t_0.

Theorem 3 Under the assumptions A1-A3:

C1. The Bayes estimator for the reliability function of ith component, $i = 1,2,...,n$, is

$$\hat{R}_i(t_0) = E(R_i(t_0) \mid data) = \Phi^{(1)}_{R_i} / \Phi(0) .$$

C2. The minimum posterior risk associated with the Bayes estimator $\hat{R}_i(t_0)$, $i = 1,2,...,n$,

is $\Psi_{\hat{R}_i(t_0)} = \Phi^{(2)}_{R_i} / \Phi(0) - [\Phi^{(1)}_{R_i} / \Phi(0)]^2$,

where

$$\Phi_{R_i}^{(w)} = \Sigma\cdots\Sigma\tau\cdot C_m^{(k,r,l)} \times \prod_{l\in M_i} \frac{\Gamma(A_l+g_l)}{(T+h_l)^{A_l+g_l}}\cdot\frac{\Gamma(B_l+c_l)}{(\tilde{T}+d_l)^{B_l+c_l}} \times \prod_{v=i,v=n+1} \frac{\Gamma(A_v+g_v)}{(T+h_v+wt_0)^{A_v+g_v}}\cdot\frac{\Gamma(B_v+c_v)}{(\tilde{T}+d_v+\frac{1}{2}wt_0^2)^{B_v+c_v}}\quad M_i=\{1,2,..,n\}\setminus\{i\},\ w=1,2.$$

Proof (1) Under the assumption A3, the Bayes estimator for the value of reliability of ith component defined as the expectation of $R_i(t_0)$, where

$R_i(t_0;\alpha_i,\beta_i,\alpha_{n+1},\beta_{n+1}) = \exp\{-[(\alpha_i+\alpha_{n+1})t_0]+(\beta_i+\beta_{n+1})t_0^2/2\}$.

Thus,

$$\hat{R}_i(t_0) = E(R_i(t_0)\,|\,data)$$

$$= \int_0^{+\infty}\cdots\int_0^{+\infty} R_i(t_0\,|\,\alpha_i,\beta_i,\alpha_{n+1},\beta_{n+1})\,g(\alpha_i,\beta_i,\alpha_{n+1},\beta_{n+1})\,d\alpha_i d\beta_i d\alpha_{n+1} d\beta_{n+1}$$

$$= \frac{1}{\Phi(0)}\Sigma\cdots\Sigma\tau\cdot C_m^{(k,r,l)}\int_0^{+\infty}\alpha_i^{A_i+g_i-1}\exp[-\alpha_i(h_i+T+t_0)]d\alpha_i$$
$$\times\int_0^{+\infty}\alpha_{n+1}^{A_{n+1}+g_{n+1}-1}\exp[-\alpha_{n+1}(h_{n+1}+T+t_0)]d\alpha_{n+1}\times\int_0^{+\infty}\beta_i^{B_i+c_i-1}\exp[-\beta_i(d_i+\tilde{T}+\frac{1}{2}t_0^2)]d\beta_i$$
$$\times\int_0^{+\infty}\beta_{n+1}^{B_{n+1}+c_{n+1}-1}\exp[-\beta_{n+1}(d_{n+1}+\tilde{T}+\frac{1}{2}t_0^2)]d\beta_{n+1}\times\prod_{l\in M_i}\frac{\Gamma(A_l+g_l)}{(T+h_l)^{A_l+g_l}}\cdot\frac{\Gamma(B_l+c_l)}{(\tilde{T}+d_l)^{B_l+c_l}}$$

$$= \frac{1}{\Phi(0)}\Sigma\cdots\Sigma\tau\cdot C_m^{(k,r,l)}\times\prod_{l\in M_i}\frac{\Gamma(A_l+g_l)}{(T+h_l)^{A_l+g_l}}\cdot\frac{\Gamma(B_l+c_l)}{(\tilde{T}+d_l)^{B_l+c_l}} \times\prod_{v=i,v=n+1}\frac{\Gamma(A_v+g_v)}{(T+h_v+t_0)^{A_v+g_v}}\cdot\frac{\Gamma(B_v+c_v)}{(\tilde{T}+d_v+\frac{1}{2}t_0^2)^{B_v+c_v}} \tag{2}$$

The minimum posterior risk associated with $\hat{R}_i(t_0)$, say $\Psi_{\hat{R}_i(t_0)}$, is defined as the posterior variance of $R_i(t_0)$. Namely, $\Psi_{\hat{R}_i(t_0)} = E[R_i^2(t_0)\,|\,data]-[E(R_i(t_0)\,|\,data)]^2$. Similar to $E(R_i(t_0)\,|\,data)$, we can easily obtain $E[R_i^2(t_0)\,|\,data]$. Thus, one can easily complete the proof. □

Theorem 4 Under the assumptions A1-A3:

D1. The Bayes estimator for the reliability function of system is

$\hat{R}_a(t_0) = E(R_a(t_0) \mid data) = \Phi^{(1)}_{R_a} / \Phi(0)$.

D2. The minimum posterior risk associated with the Bayes estimator $\hat{R}_a(t_0)$ is

$\Psi_{\hat{R}_a(t_0)} = \Phi^{(2)}_{R_a} / \Phi(0) - [\Phi^{(1)}_{R_a} / \Phi(0)]^2$,

where

$\Phi^{(r)}_{R_a} = \Sigma \cdots \Sigma \tau \cdot C_m^{(k,r,l)} \prod_{v=1}^{n+1} \Gamma(A_v + g_v) / (T + h_v + rt_0)^{A_v + g_v} \prod_{v=1}^{n+1} \Gamma(B_v + c_v) / (\widetilde{T} + d_v + \frac{1}{2} r t_0^2)^{B_v + c_v}$,

$r = 1,2$.

Proof The proof is similar to that of theorem 3. □

4. Numerical Simulation

We show in this section how one can apply the previous theoretical results obtained. This section is devoted to present numerical results based on a large simulation study. We make simulation of a two components connected in series system. It is assumed in the simulation that $\alpha_1 = 0.15$, $\alpha_2 = 0.2$, $\alpha_3 = 0.25$, $\beta_1 = 1.5$, $\beta_2 = 2$, $\beta_3 = 2.5$ and the prior distributions of α_i, β_i are $\Gamma(1,10)$ and $\Gamma(5,2)$ respectively.

It is assumed in this simulation that 10 systems were put on the life test. The masking level is 25%. Then the lifetime of each system and the set of components that may cause the system failure were observed. The simulated data are presented in Table 1. Based on the simulation data given in Table 1, the Bayes estimators of the parameters are computed. The specified mission time is $t_0 = 0.5$ while considering the reliability. Further, the percentage error associated with the estimators is computed. The percentage error associated with the estimator $\hat{\theta}$ of θ is given by $PE_{\hat{\theta}} = |\hat{\theta} - \theta| / \theta \times 100\%$ .The obtained results are presented in Table 2.

Table 1 Simulated system lifetime data ($n = 10$)

System	Observation		System	Observation	
i	t_i	S_i	i	t_i	S_i
1	0.5339	1,2,3	6	0.7050	3
2	0.5470	3	7	0.7936	2
3	0.3325	1	8	0.8583	2
4	0.4146	1,2,3	9	0.6276	1,2.3,
5	0.7191	1,2,3	10	0.8375	3

Table 2 Bayes estimators with percentage error

Parameter	α_1	α_2	α_3	β_1	β_2	β_3	R_1	R_2	R_a
True value	0.15	0.2	0.25	1.5	2	2.5	0.4966	0.4550	0.3499
Estimation	0.1520	0.1572	0.1772	1.8344	2.0840	2.2753	0.5774	0.5585	0.4385
PE	0.0134	0.2142	0.2913	0.2229	0.0420	0.0899	0.1627	0.2274	0.2530

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