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Edge Partition of the Boolean Graph $BG_1(G)$

T. N. Janakiraman[1], M. Bhanumathi[2] and S. Muthammai[2]
[1]National Institute of Technology, Tiruchirapalli-620015, India.
tnjraman2000@yahoo.com; janaki@nitt.edu
[2]Government Arts College for Women, Pudukkottai-622001, India.

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ABSTRACT

Let G be a simple (p, q) graph with vertex set V(G) and edge set E(G). $B_{G,\,NINC,\,\bar{K}_q}(G)$ is a graph with vertex set V(G) ∪ E(G) and two vertices are adjacent if and only if they correspond to two adjacent vertices of G or to a vertex and an edge not incident to it in G. For simplicity, denote this graph by $BG_1(G)$, Boolean graph of G-first kind. In this paper, partitions of edges of $BG_1(G)$ are studied.

Keywords: *Boolean graph $BG_1(G)$, Edge Partition.*

1. Introduction

Let G be a finite, simple, undirected (p, q) graph with vertex set V(G) and edge set E(G). For graph theoretic terminology refer to Harary [11], Buckley and Harary [10].

A graph G is called *Hamiltonian* if it has a spanning cycle. Any spanning cycle of G is called *Hamilton cycle*. A *Hamiltonian path* in G is a path, which contains every vertex of G. Clearly, every Hamiltonian graph is 2-connected.

A *decomposition* of a graph G is a collection of subgraphs of G, whose edge sets partition the edge set of G. The subgraphs of the decomposition are called the *parts of the decomposition*.

A graph G is said to be *F-decomposable or F-packable* if G has a decomposition in which all of its parts are isomorphic to the graph F. A graph G can be decomposed into *Hamilton cycles (paths)* if the edge set of G can be partitioned into Hamilton cycles (paths).

Following theorem is used in the partition of $BG_1(G)$

Theorem 1.1 [9](Bermond)
(i) If p is even, K_p can be decomposed into p/2 Hamiltonian paths.
(ii) If p is odd, K_p can be decomposed into (p−1)/2 Hamiltonian cycles.

If p is even, K_p can be decomposed into (p−2)/2 cycles of length n and $(p/2)K_2$′s.

A *path (cycle) partition* of a graph G is a collection of paths (cycles) in G such that every edge of G lies in exactly one path (cycle).

<u>Motivation:</u> The Line graphs [7], Middle graphs [1,2], Total graphs [5,6] and Quasi-total graphs [18] are very much useful in the construction of various related networks from the underlying graph of networks. In analogous to line graph, total graph, middle graph and quasi-total graph, thirty two graphs can be defined using different adjacency relations. Out of these operations, eight were already studied. Among the remaining twenty-four graph operations, two are defined and analyzed in [13] and [14]. All the others have been defined and various structural properties are studied [4], [13], [14], [15] and [16]. This is illustrated below.

G/ $\overline{G}$/K_p/ $\overline{K}_p$	Incident (INC)/ Non-incident (NINC)	L(G)/ $\overline{L}$(G)/K_q/ $\overline{K}_q$

Defining a new graph from a given graph by using the adjacency relation between two vertices or two edges and incident relationship between vertices and edges is known as Boolean operation. It defines new structure from the given graph either by adding extra information or extracting new information from the substructure or by mixing the above cases of the original graph.

In Management and in social networks, the incident and non-incident relations of vertices and edges are used to define various networks. So these are very much applicable in socio-economical problems. In some cases, it is not possible to retrieve the original graph from the Boolean graphs in polynomial time. So these graph operations may be used in graph coding or coding of some grouped signal. Also, it is possible to study the structure of original graphs through these Boolean graph operations. This motivates the study for the exploration of various Boolean operations and study of their structural properties.

In [4], the Boolean graph $BG_1(G)$ of a graph G is defined as follows. Let G be a simple (p, q) graph with vertex set V(G) and edge set E(G). BG, NINC, $\overline{K}_q(G)$ is a graph with vertex set $V(G) \cup E(G)$ and two vertices are adjacent if and only if they correspond to two adjacent vertices of G or to a vertex and an edge not incident to it in G. For simplicity, denote this graph by $BG_1(G)$, Boolean graph of G-first kind.

$BG_1(G)$ has p+q vertices, p point vertices with degree q and q line vertices with degree p–2. $BG_1(G)$ is always bi-regular and is regular if and only if q = p–2; clearly, in this case G is disconnected. It is easy to determine that $BG_1(G)$ has q(p–1) edges and $\overline{BG}_1(G)$ has (q(q+1)/2)+(p(p–1)/2) edges.

In this paper, partitions of edges of $BG_1(G)$ are studied.

2. Partition of edges of $BG_1(G)$

First, partition of edges of $BG_1(G)$ into edges of G and stars for a given graph G is studied.

Proposition 2.1 Let G be a (p, q) graph. Then the edges of $BG_1(G)$ can be partitioned into E(G) and q times $E(K_{1,p-2})$.

Proof: In $BG_1(G)$, degree of a line vertex is p–2 and degree of a point vertex is q, each line vertex is adjacent to exactly (p–2) point vertices only. Therefore, corresponding to q line vertices there are q times $K_{1,p-2}$ and the remaining edges are edges of G only. Hence the proposition is proved.

Next theorem gives the partition of edges of $BG_1(G)$ into edges of G, a regular graph and stars, when G is a m regular Hamiltonian graph.

Theorem 2.1 Let G be a m regular Hamiltonian graph, with n vertices. Then the edges of $BG_1(G)$ can be partitioned into G, (n–2) regular graph on 2n vertices and ((mn/2)–n) times $K_{1,n-2}$.

Proof: Number of edges in G = mn/2. Therefore, $BG_1(G)$ has n point vertices and mn/2 line vertices. Degree of a line vertex in $BG_1(G)$ is n–2 and degree of a point vertex is mn/2.

Consider the n point vertices and any n line vertices, which form a cycle in G.

Let H be the induced subgraph of $BG_1(G)$ formed by these 2n vertices. In H, degree of a point vertex is m+(n–2) and degree of a line vertex is n–2. Therefore, the edges of this induced subgraph can be partitioned into edges of G and a (n–2) regular graph on these 2n vertices. Number of remaining line vertices = (mn/2)–n, which are adjacent to exactly (n–2) point vertices. Therefore, the edges of $BG_1(G)$ can be partitioned into edges of G and a (n–2)- regular graph on 2n vertices and ((mn/2)–n) times $K_{1,n-2}$.

Corollary 2.1 Let G be a cycle C_n. Then the edges of $BG_1(G)$ can be partitioned into G and a (n–2) regular graph on 2n vertices.

Proof: In Theorem 2.1, take m = 2. Number of edges of G = n. Number of vertices of $BG_1(G)$ is 2n and degree of a point vertex is n and degree of a line vertex is n–2.

Edges of $BG_1(G)$ can be partitioned into G = C_n, and (n–2) regular graph on 2n vertices.

Next, we give the partition of edges of $BG_1(G)$, when G = C_n, P_n, K_n, $K_{n,n}$, $K_{1,n}$, nK_2.

Proposition 2.2 H = $BG_1(G)$, where G = K_p. Edges of H can be partitioned into edges of K_p and p(p+1)/2 subgraphs $K_{1,p-2}$, in such a way that, K_p and $K_{1,p-2}$ have p–2 common vertices.

Proof: H has G as induced subgraph and each line vertex is adjacent to p–2 point vertices. Hence the result follows.

Proposition 2.3 H = $BG_1(G)$, where G = $K_{1,n}$. Edges of H can be partitioned into n+1 stars, n times $K_{1,n-1}$ and one $K_{1,n}$ such that the center of all stars form a maximum independent set and any two stars of first type have n–2 common points and each $K_{1,n-1}$ and $K_{1,n}$ have n–1 common points.

Proof: In H, the central vertex of G is not adjacent to any line vertices and each line vertex is adjacent to exactly n-1 point vertices. Hence edges of H can be partitioned into n+1 stars, n times $K_{1,n-1}$ and one $K_{1,n.}$

Now consider point vertex of G, which is not the central vertex. In H, this vertex has degree n and exactly n such vertices. Hence taking these point vertices as center, edges of H can be partitioned into n stars $K_{1,n}$.

Following theorem deals with the partition of edges of $BG_1(C_n)$ into cycles of different lengths.

Theorem 2.2 When G = C_n, the edges of $BG_1(G)$ can be partitioned into (a) C_n, ((n–3)/2) C_{2n} and nK_2, if n is odd. (b) C_n and ((n–2)/2)C_{2n}, if n is even.
Proof: Let G = C_n. $BG_1(G)$ has n point vertices with degree n and n line vertices with degree n–2.

Case 1: n is even.
Let v_1, v_2, ..., v_n be the point vertices, e_{12}, e_{23}, e_{34}, …, e_{n-1n} be the line vertices. In $BG_1(G)$, each v_j is adjacent to (n–2) line vertices e_{12}, e_{23}, …, $e_{(j-2)(j-1)}$, $e_{(j+1)(j+2)}$, …, $e_{(n-1)n}$, e_{n1}. Combine these line vertices in two's as n is even. There are (n–2)/2 such collections, which are adjacent to v_j, j = 1, 2, …, n.

Consider the point vertex v_1. v_1 is adjacent to e_{23}, e_{34}, ..., e_{n-1n}. Now, combine these into (e_{23}, e_{34}); (e_{45}, e_{56}); ($e_{(n-2)(n+1)}$, $e_{(n-1)n}$). Get the (n–2)/2 cycles of length 2n in $BG_1(G)$ as follows:

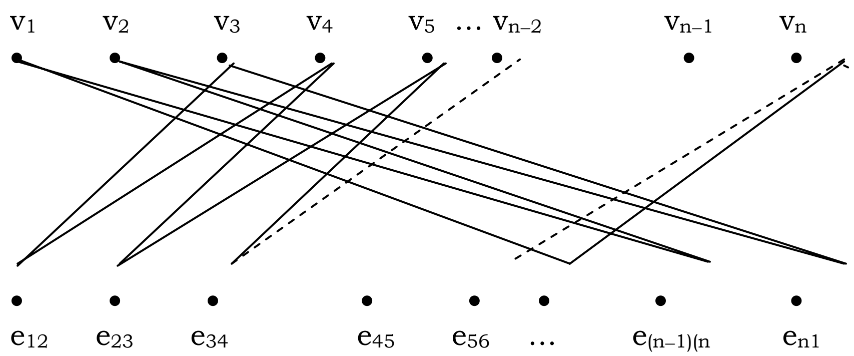

Fig: 2.1

1. $v_1\ e_{(n-1)n}\ v_2\ e_{n1}\ v_3\ e_{12}\ v_4 \ldots v_n\ e_{(n-2)(n-1)}\ v_1$
2. $v_1\ e_{(n-3)(n-2)}\ v_2\ e_{(n-2)(n-1)}\ v_3\ e_{(n-1)n}\ v_4\ e_{n1}\ v_5\ e_{12} \ldots v_n\ e_{(n-4)(n-3)}\ v_1$
3. $v_1\ e_{(n-5)(n-4)}\ v_2\ e_{(n-4)(n-3)}\ v_3\ e_{(n-3)(n-2)}\ v_4\ e_{(n-2)(n-1)} \ldots v_n\ e_{(n-6)(n-5)}\ v_1$

.

(n–2)/2. $v_1\ e_{34}\ v_2\ e_{45}\ v_3\ e_{56}\ v_4 \ldots e_{12} v_n\ e_{23}\ v_1$

From this construction and from the definition of $BG_1(G)$, it follows that edges of $BG_1(G)$ are union of edges of G and these cycles.

Case 2: n is odd.
There are n point vertices v_1, v_2, …, v_n and n line vertices e_{12}, e_{23}, …, e_{n1}. Combine (n–1) line vertices into two's. v_1 is adjacent to e_{23}, e_{34}, …, $e_{(n-1)n}$. Leaving $e_{(n-1)n}$, combine these into (e_{23}, e_{34}); (e_{45}, e_{56}); ($e_{n-3\ n-2}$, $e_{n-2\ n-1}$). Similarly, v_2 is adjacent to (e_{34}, e_{45}); (e_{56}, e_{67}); …; ($e_{n-2\ n-1}$, $e_{n-1\ n}$), and e_n. As in case (1) there are ((n–1)–2)/2 cycles of length 2n and nK_2′s $v_1e_{(n-1)n}$; v_2e_{n1}; v_3e_{12}; …; $v_ne_{(n-2)(n-1)}$. Therefore, edges of $BG_1(G)$ can be partitioned into C_n, ((n–3)/2)C_{2n} and nK_2, when n is odd. Hence the proof of the theorem follows.

In the following theorem, partition of edges of $BG_1(P_n)$ into paths of different lengths is studied.

Theorem 2.3 If G = P_n, then edges of $BG_1(G)$ can be partitioned into
1. P_n, $((n-3)/2)P_{2n-1}$ and $(n-1)K_2$, if n is odd.
2. P_n and $((n-2)/2)P_{2n-1}$, if n is even.

Proof: Similar to Theorem 2.2.

Next we give the partition of $BG_1(K_{1,n})$ into cycles and stars.

Theorem 2.4 If G = $K_{1,n}$, the edges of $BG_1(G)$ can be partitioned into
1. $K_{1,n}$; $((n-1)/2)C_{2n}$, if n is odd.
2. $K_{1,n}$, $((n-2)/2)C_{2n}$; nK_2, if n is even.

Proof: Let G = $K_{1,n}$. Then $BG_1(G)$ has n+1 point vertices with degree n and n line vertices with degree n–1. Let $v_1, v_2, \ldots, v_n$, v be the point vertices, $e_1 = vv_1$, $e_2 = vv_2, \ldots, e_n = vv_n \in E(G)$ be the n line vertices, where v is the central node of G.

<u>Case 1:</u> <u>n is odd.</u>
In $BG_1(G)$, each v_j is adjacent to (n–1) line vertices $e_1, e_2, \ldots, e_{j-1}, e_{j+1}, \ldots, e_n$. Combine these vertices into two by two. Thus, there are (n–1)/2 such collections, which are adjacent to v_j, j = 1, 2, …, n. Consider v_1, combine the edges as (e_2, e_3); (e_4, e_5); …; (e_{n-1}, e_n). The n–1 cycles of length 2n in $BG_1(G)$ can be obtained as follows:

(1) $v_1\, e_3\, v_2\, e_4\, v_3\, e_5\, v_4\, e_6 \ldots e_{n-1}\, v_{n-2}\, e_n\, v_{n-1}\, e_1\, v_n\, e_2\, v_1$.

(2) $v_1\, e_5\, v_2\, e_6\, v_3\, e_7\, v_4\, e_8 \ldots e_n\, v_{n-3}\, e_1\, v_{n-2}\, e_2\, v_{n-1}\, e_3\, v_n\, e_4\, v_1$.

((n–1)/2) $v_1\, e_n\, v_2\, e_1\, v_3\, e_2\, v_4\, e_3 \ldots v_{n-2}\, e_{n-3}\, v_{n-1}\, e_{n-2}\, v_n\, e_{n-1}\, v_1$.

From this construction and from the definition of $BG_1(G)$, it follows that edges of $BG_1(G)$ can be grouped into edges of G = $K_{1,n}$ and these (n–1)/2 cycles of length 2n.

<u>Case 2:</u> <u>n is even.</u>
There are n+1 point vertices, $v_1, v_2, \ldots, v_n$, v and n line vertices $e_1, e_2, \ldots, e_n$ in $BG_1(G)$, $e_j = vv_j \in E(G)$. By the definition of $BG_1(G)$ each v_j is adjacent to (n–1) line vertices, $e_1, e_2, e_3, \ldots, e_{j-1}, e_{j+1}, \ldots, e_n$. Among this, (n–2) line vertices can be grouped into pairs.

Consider v_1. v_1 is adjacent to (n–1) line vertices, $e_2, e_3, \ldots, e_{n-1}, e_n$; leaving e_n, get (n–2)/2 pairs $(e_2, e_3), (e_4, e_5), \ldots, (e_{n-2}, e_{n-1})$. Similarly, v_2 is adjacent to $e_1, e_3, \ldots, e_{n-1}, e_n$. Leaving e_1, get (n–2)/2 pairs $(e_3, e_4), (e_5, e_6), \ldots, (e_{n+1}, e_n), \ldots$

As in case1, there are ((n–1)–1)/2 cycles of length n and nK_2′s given by $v_1e_n, v_2e_1, v_3e_2, \ldots, v_ne_{n-1}$. Therefore, edges of $BG_1(G)$ can be partitioned into $K_{1,n}$, $((n-2)/2)C_{2n}$ and nK_2 when G = $K_{1,n}$, where n is even.

In the next two theorems, we give the partition of $BG_1(K_n)$ into cycles and stars or regular graphs.

Theorem 2.5 If G = K_n, then edges of $BG_1(G)$ can be partitioned into
(1) $((n-2)/2)C_{2n}$, $((n-2)^2/4)C_{2n}$, $(n/2)K_{1,n-2}$, $(n/2)K_2$, when n is even.
(2) $((n-1)/2)C_n$, $((n-1)/2)((n-3)/2)C_{2n}$ and $(n(n-1)/2)K_2$, when n is odd.

Proof:G = K_n. $BG_1(G)$ has n point vertices and n(n–1)/2 line vertices. In $BG_1(G)$, degree of each point vertex is n(n–1)/2 and degree of each line vertex is n–2.

Case 1: n is even.
When n is even, edges of K_n can be partitioned into ((n–2)/2)C_n and (n/2)K_2′s. Therefore, by Theorem 2.2 edges of $BG_1(K_n)$ can be partitioned into ((n–2)/2)C_n, ((n–2)/2)((n–2)/2)C_{2n}, (n/2)$K_{1,n-2}$ and (n/2)K_2′s.

Case 2: n is odd.
When n is odd, edges of K_n can be partitioned into ((n–2)/2)C_n. Hence, by Theorem 2.2 edges of $BG_1(K_n)$ can be partitioned into ((n–1)/2)C_n, ((n–1)/2)((n–3)/2)C_{2n}, ((n–1)/2)nK_2.

Theorem 2.6 If G = K_n with n > 3, the edges of $BG_1(G)$ can be partitioned into (1) (n–1)/2 times (n–2) regular graph with 2n vertices and K_n when n is odd, such that the n point vertices are in all of these regular graphs. (2) (n–2)/2 times (n–2) regular graph with 2n vertices, K_n and (n/2)$K_{1,n-2}$ when n is even such that the n point vertices are in all of these regular graphs and each of the point vertices are in exactly (n/2–1) $K_{1,n-2}$′s.
Proof: Let G = K_n. Then $BG_1(G)$ has n point vertices and n(n–1)/2 line vertices. In $BG_1(G)$, degree of each point vertex is n(n–1)/2 and degree of each line vertex is n–2.

Case 1: n is odd.
In this case, edges of K_n can be partitioned into (n–1)/2 cycles of length n. In $BG_1(G)$, consider n line vertices which form a n-cycle in G and the n point vertices. In $BG_1(G)$, these 2n vertices form a (n–2) regular graph on 2n vertices and K_n. Corresponding to the (n–1)/2 cycles of length n in G, there exist (n–1)/2 times (n–2) regular graph on 2n vertices and K_n, but this K_n formed by n point vertices is common. Hence, when n is odd, edges of G can be partitioned into K_n and (n–1)/2 times (n–2) regular graph on 2n vertices such that the n point vertices are in all of these regular graphs.

Case 2: n is even.
Edges of K_n can be partitioned into (n–2)/2 times cycles of length n and (n/2)K_2′s. In $BG_1(G)$, consider the n line vertices which form a n-cycle in G. These n line vertices with the n point vertices form a (n–2) regular graph on 2n vertices and K_n. Corresponding to the (n–2)/2 cycles of length n in G, there exist (n–2)/2 times (n–2) regular graph on 2n vertices and K_n and this K_n formed by point vertices is common.

Now, consider the (n/2)K_2′s on G. In $BG_1(G)$, these n/2 line vertices are adjacent to exactly n–2 point vertices. Therefore, in $BG_1(G)$ there are (n/2)$K_{1,n-2}$ and (n/2)K_2, which is in K_n and each of the n point vertices are in exactly ((n/2)–1)$K_{1,n-2}$′s. This proves the theorem.

Theorem 2.7 Edges of $BG_1(nK_2)$ can be partitioned into n times K_2+(n–1)K_1.

Proof: $G = nK_2$, let $V(G) = \{v_1, v_2, v_3, \ldots, v_n, u_1, u_2, \ldots, u_n\}$ and $e_1 = u_1v_1, \ldots, e_n = u_nv_n \in E(G)$.

Consider e_1. In $BG_1(G)$, v_1 is adjacent to $e_2, e_3, ..., e_n$ and u_1 is adjacent to $e_2, e_3, \ldots, e_n$. Also, v_1 and u_1 are adjacent. Hence, they form $K_2+(n-1)K_1$. Similarly, for other $e_2, \ldots, e_n$.

Hence, the edges of $BG_1(nK_2)$ can be partitioned into n times $K_2+(n-1)K_1$.

Theorem 2.8 If $G = K_{n,n}$, edges of $BG_1(G)$ can be partitioned into (1) $(n/2)C_n$, $((n/2)(n-1))C_{4n}$, if n is even. (2) $((n-1)/2)C_{2n}$, $((n-1)^2/2)C_{4n}$, $nK_{1,n-2}$ and nK_2, if n is odd.

Proof: **<u>Case 1:</u>** <u>n is even.</u>

The edges of $K_{n,n}$ can be partitioned into $(n/2)C_{2n}$, By Theorem 2.2, edges of $BG_1(C_{2n})$ can be partitioned into C_{2n}, $((2n-2)/2)C_{4n}$. Therefore, $BG_1(K_{n,n})$ can be partitioned into $(n/2)C_{2n}$, $((n/2)(2n-2)/2)C_{4n}$.

<u>Case 2:</u> <u>n is odd.</u>

Edges of $K_{n,n}$ can be partitioned into $((n-1)/2)C_{2n}$, nK_2. Therefore, edges of $BG_1(K_{n,n})$ can be partitioned into $((n-1)/2)C_{2n}$, $(((n-1)/2)(2n-2)/2)C_{4n}$, $nK_{1,2n-2}$ and nK_2. That is, $((n-1)/2)C_{2n}$, $((n-1)^2/2)C_{4n}$, $nK_{1,2n-2}$ and nK_2.

3. Path and cycle partition of edges of $BG_1(G)$

In this section, we study the partition of edges of $BG_1(G)$ into paths or cycles, when $G = C_n, P_n, K_{1,n}, K_n, K_{n,n}$ and nK_2.

First we study partition of $BG_1(C_n)$ into paths of length $(n-1)$.

Theorem 3.1 Let $G = C_n$. Then the edges of $BG_1(G)$ can be partitioned into n paths of length $(n-1)$, each path contains exactly one edge from $G = C_n$. That is $BG_1(G)$ is P_{n-1}- packable.

Proof: Let $v_1, v_2, ..., v_n \in V(G)$. Let $e_{12} = v_1v_2$, $e_{23} = v_2v_3$, ..., $e_{(n-1)n} = v_{m-1}v_m$, $e_{n1} = v_nv_1 \in E(G)$.

<u>Case 1:</u> <u>n is even.</u>

Edge set of $BG_1(G)$ can be partitioned as follows:

(1) $v_2\ v_1\ e_{23}\ v_4\ e_{12}\ v_5\ e_{n1}\ v_6\ e_{(n-1)n}\ v_7 \ldots v_{(n/2)+1}\ e_{((n/2)+4)((n/2)+5)}\ v_{(n/2)+2}$.

(2) $v_3\ v_2\ e_{34}\ v_5\ e_{23}\ v_6 \ldots v_{(n/2)+2}\ e_{((n/2)+5)((n/2)+6)}\ v_{(n/2)+3}$.

(3) $v_4\ v_3\ e_{45}\ v_6\ e_{34}\ v_7 \ldots v_{(n/2)+3}\ e_{((n/2)+6)((n/2)+7)}\ v_{(n/2)+4}$..

(n/2) $v_{(n/2)+1}\ v_{n/2}\ e_{((n/2)+1)((n/2)+2)}\ v_{(n/2)+3}\ e_{(n/2)((n/2)+1)} \ldots e_{34}\ v_1$..

(n) $v_1\ v_n\ e_{12}\ v_3\ e_{n1}\ v_4\ e_{n-1n} \ldots e_{((n/2)+3)((n/2)+4)}\ v_{(n/2)+1}$.

Thus, the edges of $BG_1(G)$ can be partitioned into n paths of length n–1 when n is even.

<u>Case 2:</u> <u>n is odd.</u>

Here $\lceil n/2 \rceil = (n+1)/2$. Edges of $BG_1(G)$ can be partitioned as follows.

(1) $v_2\ v_1\ e_{23}\ v_4\ e_{12}\ v_5\ e_{n1}\ v_6\ e_{(n-1)n}\ v_7 \ldots v_{\lceil n/2 \rceil+1}\ e_{(\lceil n/2 \rceil+3)(\lceil n/2 \rceil+4)}$.

(2) $v_3\ v_2\ e_{34}\ v_5\ e_{23}\ v_6 \ldots v_{\lceil n/2 \rceil+2}\ e_{(\lceil n/2 \rceil+4)(\lceil n/2 \rceil+5)}$.

(3) $v_4\ v_3\ e_{45}\ v_6\ e_{34}\ v_7 \ldots v_{\lceil n/2\rceil+3}\ e_{(\lceil n/2\rceil+5)(\lceil n/2\rceil+6)}$..
$\lceil n/2\rceil$ $v_{\lceil n/2\rceil+1}\ v_{\lceil n/2\rceil}\ e_{(\lceil n/2\rceil+1)(\lceil n/2\rceil+2)}\ v_{\lceil n/2\rceil+3}\ e_{(\lceil n/2\rceil-1)\lceil n/2\rceil} \ldots v_1\ e_{34}$. .
(n) $v_1\ v_n\ e_{12}\ v_3\ e_{n1}\ v_4\ e_{(n-1)n} \ldots v_{\lceil n/2\rceil}\ e_{(\lceil n/2\rceil+2)(\lceil n/2\rceil+3)}$.
Here, the subscripts are all taken modulo n. This proves the theorem.

Next, we prove the edges of $BG_1(K_{1,n})$ can be partitioned into paths of length n.

Theorem 3.2 When $G = K_{1,n}$, edges of $BG_1(G)$ can be partitioned into n paths of length n, each path containing exactly one edge from $G = K_{1,n}$. That is $BG_1(G)$ is P_n-packable.
Proof: **<u>Case 1:</u>** <u>n is even</u>.
Let v be the central node of G and $v_1, v_2, \ldots, v_n$ be other vertices of G and $vv_i = e_i$
Now, the edges of $BG_1(G)$ can be partitioned as follows:
(1) $v\ v_1\ e_2\ v_3\ e_1\ v_4\ e_n\ v_5\ e_{n-1}\ v_6 \ldots v_{(n/2)+1}\ e_{(n/2)+3}$.
(2) $v\ v_2\ e_3\ v_4\ e_2\ v_5\ e_1\ v_6\ e_n\ v_7 \ldots v_{(n/2)+2}\ e_{(n/2)+4}$. .
(n/2) $v\ v_{n/2}\ e_{(n/2)+1}\ v_{(n/2)+2}\ e_{(n/2)}\ v_{(n/2)+3}\ e_{(n/2)-1} \ldots v_n\ e_2$. .
(n) $v\ v_n\ e_1\ v_2\ e_n\ v_3\ e_{n-1}\ v_4\ e_{n-2}\ v_5 \ldots v_{(n/2)}\ e_{(n/2)+2}$.

<u>Case 2:</u> <u>n is odd.</u>
Edges of $BG_1(G)$ can be partitioned into,
(1) $v\ v_1\ e_2\ v_3\ e_1\ v_4\ e_n\ v_5\ e_{n-1}\ v_6 \ldots e_{(n-1/2)+4}\ v_{(n-1/2)+2}$.
(2) $v\ v_2\ e_3\ v_4\ e_2\ v_5\ e_1\ v_6\ e_n\ v_7 \ldots e_{(n-1/2)+5}\ v_{(n-1/2)+3}$..
(n) $v\ v_n\ e_1\ v_2\ e_n\ v_3\ e_{n-1}\ v_4\ e_{n-2}\ v_5 \ldots e_{(n-1/2)+3}\ v_{(n-1/2)+1}$.
This proves the theorem.

In the next theorem, we partition edges of $BG_1(K_n)$ into paths of length n−1.

Theorem 3.3 When $G = K_n$, edges of $BG_1(G)$ can be partitioned into n(n−1)/2 paths of length n−1. That is $BG_1(G)$ is P_{n-1} packable.
Proof: Using Theorems 3.1 and 3.2, we can prove this theorem. Take $G = K_n$. Let $v_1, v_2, \ldots, v_n \in V(G)$.

<u>Case 1:</u> <u>n is odd.</u>
Clearly, when n is odd, edges of K_n can be partitioned into (n−1)/2 cycles of length n. By Theorem 3.1, for each cycle C_n, $BG_1(C_n)$ can be partitioned into n paths of length n−1. Hence, $BG_1(K_n)$ can be partitioned into n(n−1)/2 paths of length n−1.

<u>Case 2:</u> <u>n is even.</u>
Now, consider K_n as $K_{n-1}+K_1$, where K_{n-1} is the complete graph with vertices $v_1, v_2, \ldots, v_{n-1}$ and K_1 is v_n. Edges of $BG_1(K_n)$ are edges of $BG_1(K_{n-1})$, edges of $BG_1(K_{1,n-1})$, and the edges joining the line vertices of $BG_1(K_{n-1})$ to v_n.

Now by case 1, edges of $BG_1(K_{n-1})$ can be partitioned into (n−1)(n−2)/2 paths of length n−2 and by Theorem 3.2, edges of $BG_1(K_{1,n-1})$ can be partitioned into (n−1) paths of length (n−1). Now, consider the (n−1)(n−2)/2 paths of length (n−2). To each of this path join one edge joining a line vertex to v_n in $BG_1(G)$. Thus,

$(n-1)(n-2)/2$ paths of length $(n-1)$ are obtained. So, totally there are $(n-1)(n-2)/2+(n-1) = n(n-1)/2$ paths of length $(n-1)$. This proves the theorem.

Following are some important remarks in the path partition of K_n.

Remark 3.1 These $n(n-1)/2$ paths has the following properties:
(1) Each path contains exactly one edge from K_n.
(2) If n is odd, each path starts from a point vertex and ends with a line vertex.
(3) If n is even, each path starts with a point vertex and ends with a point vertex.

In the next theorem, we study the path partition or cycle partition of $BG_1(G)$ when $G = (n/2)K_2$.

Theorem 3.4 If $G = (n/2)K_2$, edges of $BG_1(G)$ can be partitioned into $(n/2)C_{n-1}$, if n/2 is even or $(n/2)P_n$, if n/2 is odd, such that each path or cycle contains exactly one edge from G.
Proof: Let $v_1, v_2, ..., v_n \in V(G)$, $e_{12} = v_1v_2$, $e_{34} = v_3v_4$, ..., $e_{(n-1)n} = v_{n-1}v_n \in E(G)$.
<u>Case 1:</u> <u>n/2 is odd.</u>
Consider the following partitions:
(1) $v_1\ v_2\ e_{34}\ v_{n-1}\ e_{56}\ v_{n-3} \ldots e_{(n/2)((n/2)+1)}\ v_{((n/2)+2)}\ e_{((n/2)+4)((n/2)+5)}\ v_{(n/2)+1} e_{((n/2)+6)((n/2)+7)}\ v_{(n/2)-1} \ldots e_{n-1n}\ v_6\ e_{12}\ v_4$.
(2) $v_3\ v_4\ e_{56}\ v_1 \ldots e_{((n/2)+2)((n/2)+3)}\ v_{(n/2)+4}\ e_{((n/2)+6)((n/2)+7)}\ v_{(n/2)+3}\ e_{((n/2)+8)((n/2)+9}\ v_{(n/2)+1} \ldots e_{12}\ v_8\ e_{34}\ v_6$..
(n/2) $v_{n-1}\ v_n\ e_{12}\ v_{n-3}\ e_{34}\ v_{n-5}\ e_{56} \ldots e_{(n/2)-2)(n/2)-1}\ v_{(n/2)}\ e_{((n/2)+2)((n/2)+3)}\ v_{(n/2)-1}\ e_{((n/2)+4)((n/2)+5)}\ v_{(n/2)-3} \ldots e_{(n-3)(n-2)}\ v_4\ e_{(n-1)n}\ v_2$.
Thus, edges of $BG_1(G)$ can be partitioned into n/2 paths of length n–1, that is $(n/2)P_n$.

<u>Case 2:</u> <u>n/2 is even.</u>
Consider the following partitions:
(1) $v_1\ v_2\ e_{34}\ v_n\ e_{56}\ v_{n-2}\ e_{78}\ v_{n-4} \ldots e_{((n/2)+1)((n/2)+2)}\ v_{((n/2)-1)}\ e_{((n/2)+3)((n/2)+4)}\ v_{((n/2)-3)} \ldots v_3\ \ e_{(n-1)n}\ v_1$.
(2) $v_3\ v_4\ e_{56}\ v_2\ e_{78}\ v_n\ e_{9(10)}\ v_{n-2}$.. $e_{((n/2)+3)((n/2)+4)}\ v_{(n/2)+1}\ e_{((n/2)+5)((n/2)+6)}\ v_{(n/2)-1} \ldots e_{12}\ v_3$. .
(n/2) $v_{n-1}\ v_n\ e_{12}\ v_{n-2}\ e_{34}\ v_{n-4}\ e_{56} \ldots e_{((n/2)-1)(n/2)}\ v_{(n/2)-3}\ e_{((n/2)+1)((n/2)+2)}\ v_{(n/2)-1} \ldots e_{(n-3)(n-2)} v_{n-1}$.
Thus, edges of $BG_1(G)$ can be partitioned into n/2 cycle of length n–1.

The following theorem gives the partition of $BG_1(K_n)$ into cycles when n is a multiple of four.

Theorem 3.5 When n is a multiple of four, edges of $BG_1(K_n)$ can be partitioned into $n(n-1)/2$ cycles of length $(n-1)$. That is $BG_1(K_n)$ is C_{n-1} packable.
Proof: n is a multiple of 4. Therefore, n is even and $(n-2)/2$ is odd. Edges of K_n can be partitioned into $(n-2)/2$ cycles of length n and $(n/2)K_2$′s. Consider the cycle C_n formed by $v_1, v_2, v_3, ..., v_n$, let $e_{12} = v_1v_2$, $e_{23} = v_2v_3$,... , $e_{n1} = v_nv_1 \in E(G)$. Now, the edges of $BG_1(C_n)$ except the edges of the cycle C_n can be partitioned into
(1) $v_1\ e_{(n-1)n}\ v_{n-2}\ e_{n1}\ v_{n-3}\ e_{12}\ v_{n-4}\ e_{23}\ v_{n-5} \ldots e_{((n/2)-3)((n/2)-2)}\ v_{n/2}$.
(2) $v_2\ e_{n1}\ v_{n-1}\ e_{12}\ v_{n-2}\ e_{23}\ v_{n-3}\ e_{34} \ldots e_{((n/2)-2)((n/2)-1)}\ v_{((n/2)+1)}$..

(n) $v_n\, e_{(n-2)(n-1)}\, v_{n-3}\, e_{(n-1)n}\, v_{n-4}\, e_{n1}\, v_{n-5} \ldots e_{((n/2)-4)((n/2)-3)}\, v_{((n/2)-1)}$.
n-paths of length n–2. Now, add v_1 at the last in (1), v_2 at the last in (2), ..., v_n in (n). The edges $v_{n/2}\, v_1$, $v_{(n/2)+1}v_2$, ..., $v_{((n/2)-1)}\, v_n$ are in some other cycles (which form K_n). Thus, there are n cycles of length n−1 corresponding to the cycle $v_1v_2v_3...v_nv_1$ in G. Similarly, for the other cycles in K_n also, such partitions can be formed. Thus, there are n(n–2)/2 cycles of length (n–1).

Now, consider the $(n/2)K_2$′s in K_n. As in the previous theorem, in $BG_1(G)$, there are n/2 cycles of length (n–1) corresponding to this $(n/2)K_2$′s. Thus, totally there are (n(n–2)/2)+(n/2) = n(n–1)/2 cycles of length (n–1).

Next, we study the path partition of $BG_1(K_{n,n})$.

Theorem 3.6 Edges of $BG_1(K_{n,n})$ can be partitioned into n^2 paths of length 2n–1, such that each path contains exactly one edge from G = $K_{n,n}$. That is $BG_1(K_{n,n})$ is P_{2n-1} packable.

Proof: **Case 1:** n is even.
Edges of $K_{n,n}$ can be partitioned into n/2 cycles of length 2n. But it is already proved that $BG_1(C_n)$ can be partitioned into n paths of length n−1. Hence, $BG_1(K_{n,n})$ can be partitioned into (n/2)2n paths of length 2n−1.

Case 2: n is odd.
In this case, edges of $K_{n,n}$ can be partitioned into (n–1)/2 cycles of length 2n and nK_2. It is proved that the edges of $BG_1(C_n)$ can be partitioned into n paths of length (n−1) and edges $BG_1(nK_2)$ can be partitioned into n paths of length 2n−1. Therefore, edges of $BG_1(K_{n,n})$ can be partitioned into ((n−1)/2)2n paths of length 2n−1 and n paths of length 2n−1, that is, n^2 paths of length 2n–1.

Now we pose a open problem in the partition of edges of $BG_1(G)$.

Open Problem: Let G be a finite, simple, undirected (p, q) graph. Edges of $BG_1(G)$ can be partitioned into q paths of length p–1, each containing exactly one edge from G. That is $BG_1(G)$ is P_{p-1} packable.

Other properties of $BG_1(G)$ such as domination parameters of $BG_1(G)$ and graph equations connecting $BG_1(G)$, Total graphs and Line graphs are also studied and are submitted for publication.

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