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On Regular Fuzzy Graphs

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ABSTRACT

In this paper, regular fuzzy graphs, total degree and totally regular fuzzy graphs are introduced. Regular fuzzy graphs and totally regular fuzzy graphs are compared through various examples. A necessary and sufficient condition under which they are equivalent is provided. A characterization of regular fuzzy graphs on a cycle is provided. Some properties of regular fuzzy graphs are studied and they are examined for totally regular fuzzy graphs.

Keywords: *degree of a vertex, regular fuzzy graph, total degree, totally regular fuzzy graph.*

1. Introduction

Fuzzy graph theory was introduced by Azriel Rosenfeld in 1975. Though it is very young, it has been growing fast and has numerous applications in various fields. In this paper, we introduce regular fuzzy graphs, total degree and totally regular fuzzy graphs. We make a comparative study between regular and totally regular fuzzy graphs though various examples. We provide a necessary and sufficient condition under which they become equivalent. Then we provide a characterization of regular fuzzy graphs in which the underlying crisp graph is a cycle. Also we study some properties of regular fuzzy graphs and examine whether they hold for totally regular fuzzy graphs.

First we go through some definitions and results which can be found in [1]– [7].

Definition 1.1: A fuzzy graph G is a pair of functions $G:(\sigma,\mu)$ where σ is a fuzzy subset of a non empty set V and μ is a symmetric fuzzy relation on σ . The underlying crisp graph of $G:(\sigma,\mu)$ is denoted by $G^*(V,E)$ where $E \subseteq V\times V$. A fuzzy graph G is complete if $\mu(uv) = \sigma(u) \wedge \sigma(v)$ for all $u,v\in V$ where uv denotes the edge between u and v.

Definition 1.2: Let $G:(\sigma,\mu)$ be a fuzzy graph. The degree of a vertex u is $d_G(u)= \sum_{u\neq v}\mu(uv)$. Since $\mu(uv)> 0$ for $uv\in E$ and $\mu(uv) = 0$ for $uv \notin E$, this is equivalent to $d_G(u)=\sum_{uv\in E}\mu(uv)$. The minimum degree of G is $\delta(G)=\wedge\{d(v)/ v \in V\}$. The maximum degree of G is $\Delta(G) = \vee\{ d(v)/ v \in V\}$.

Definition 1.3: The strength of connectedness between two vertices u and v is $\mu^{\infty}(u,v)=\sup\{ \mu^k(u,v)/k=1,2,\ldots\}$ where $\mu^k(u,v)= \sup\{\mu(uu_1)\wedge \mu(u_1u_2)\wedge\ldots \wedge \mu(u_{k-1}v) / u_1,\ldots,u_{k-1}\in V\}$.

Definition 1.4: An edge uv is a fuzzy bridge of G:(σ,μ) if deletion of uv reduces the strength of connectedness between pair of vertices.

Definition 1.5: A vertex u is a fuzzy cutvertex of G:(σ,μ) if deletion of u reduces the strength of connectedness between some other pair of vertices.

Definition 1.6: Let G:(σ,μ) be a fuzzy graph such that G*:(V,E) is a cycle. Then G is a fuzzy cycle if and only if there does not exist a unique edge xy such that $\mu(xy)=\wedge\{\mu(uv)/(uv) >0\}$.

Definition 1.7: The order of a fuzzy graph G is $O(G) = \sum_{u\in V}\sigma(u)$.

The size of a fuzzy graph G is $S(G) = \sum_{uv\in E}\mu(uv)$.

Lemma 1.8: Let G:(σ,μ) be a fuzzy graph on a cycle G*. Then G is a fuzzy cycle if and only if G is not a fuzzy tree.

Lemma 1.9: Let G:(σ,μ) be a fuzzy graph on a cycle G* and let $t = \wedge\{\mu(uv) / \mu(uv)>0\}$. Then all edges uv such that $\mu(uv) > t$ are fuzzy bridges of G.

Lemma 1.10: Let G:(σ,μ) be a fuzzy graph such that G*:(V,E) is a cycle. Then a vertex is a fuzzy cutvertex of G if and only if it is a common vertex of two fuzzy bridges.

2. Regular and totally regular fuzzy graphs

Definition 2.1: Let G:(σ,μ) be a fuzzy graph on G*:(V,E). If $d_G(v)=k$ for all $v\in V$, (i.e) if each vertex has same degree k, then G is said to be a regular fuzzy graph of degree k or a k-regular fuzzy graph. This is analogous to the definition of regular graphs in crisp graph theory.

Example 2.2: Any connected fuzzy graph with two vertices is regular.

Remark 2.3: G is a k-regular fuzzy graph iff $\delta=\Delta=k$.

Remark 2.4: In crisp graph theory, any complete graph is regular. But this result does not carry over to the fuzzy case. A complete fuzzy graph need not be regular.

For example, consider G*:(V,E) where V={u,v,w} and E={uv,vw,wu}. Define G:(σ,μ) by σ(u)=0.5, σ(v) = 0.7 σ(w) = 0.6 and μ(uv)=0.5, μ(vw)=0.6, μ(wu)=0.5. Then G is a complete fuzzy graph. But d(u)= μ(uv) + μ(uw) = 0.5 + 0.5 = 1 and d(v)=d(w)=1.1. So G is not regular.

Definition 2.5: Let G:(σ,μ) be a fuzzy graph on G*. The total degree of a vertex u∈V is defined by $td_G(u)= \sum_{u\neq v}\mu(uv)+\sigma(u) = \sum_{uv\in E}\mu(uv)+\sigma(u) = d_G(u) + \sigma(u)$.
If each vertex of G has the same total degree k, then G is said to be a totally regular fuzzy graph of total degree k or a k-totally regular fuzzy graph.

Example 2.6: Consider G*:(V,E) where V= $\{v_1,v_2,v_3,v_4\}$ and E= $\{v_1v_2,v_2v_3,v_3v_4,v_4v_1\}$. Define G:(σ,μ) by $\sigma(v_1)=0.5$, $\sigma(v_2) = 0.4$, $\sigma(v_3) = 0.7$, $\sigma(v_4) = 0.5$ and $\mu(v_1v_2)=0.2$, $\mu(v_2v_3)=0.4$, $\mu(v_3v_4)=0.2$, $\mu(v_4v_1)=0.4$. Then $d(v_i) = 0.6$ for all i = 1,2,3,4. So G is a regular fuzzy graph. But $td(v_1) = 1.1 \neq 1 = td(v_2)$. So G is not totally regular.

Example 2.7: Consider G*:(V,E) where V=$\{v_1,v_2,v_3\}$ and E=$\{v_1v_2,v_1v_3\}$. Define G:(σ,μ) by $\sigma(v_1) = 0.4$, $\sigma(v_2) = 0.8$, $\sigma(v_3) = 0.7$ and $\mu(v_1v_2) = 0.3$, $\mu(v_1v_3) = 0.4$. Then $td(v_i) =1.1$ for all i = 1,2,3. So G is a totally regular fuzzy graph. But $d(v_1) = .7 \neq 0.3 = d(v_2)$. So G is not regular.

Example 2.8: Consider G*:(V,E) where V =$\{v_1,v_2,v_3\}$ and E=$\{v_1v_2,v_2v_3, v_3v_1\}$. Define G:(σ,μ) by $\sigma(v_1) = \sigma(v_2) = \sigma(v_3) = 0.4$ and $\mu(v_1v_2) = \mu(v_2v_3) = \mu(v_3v_1) = 0.3$. Then $d(v_i) = 0.6$ for all i = 1,2,3. So G is a regular fuzzy graph. Also $td(v_i) = 1$ for all i = 1,2,3. Hence G is also a totally regular fuzzy graph.

Example 2.9: Consider G*:(V,E) where V=$\{v_1,v_2,v_3\}$ and E=$\{v_1v_2,v_1v_3\}$. Define G:(σ,μ) by $\sigma(v_1)=0.3$, $\sigma(v_2) = \sigma(v_3) = 0.4$ and $\mu(v_1v_2)=0.1$, $\mu(v_1v_3)=0.2$. Then $d(v_1) = 0.3 \neq 0.1 = d(v_2)$. So G is not a regular fuzzy graph. Also $td(v_1) = 0.6 \neq 0.5 = td(v_2)$. So G is not totally regular.

Remark 2.10: From the above examples, it is clear that in general there does not exist any relationship between regular fuzzy graphs and totally regular fuzzy graphs. However, a necessary and sufficient condition under which these two types of fuzzy graphs are equivalent is provided in the following theorem.

Theorem 2.11: Let G:(σ,μ) be a fuzzy graph on G*:(V, E).Then σ is a constant function if and only if the following are equivalent:
(1). G is a regular fuzzy graph.
(2). G is a totally regular fuzzy graph.

Proof: Suppose that σ is a constant function. Let σ(u) = c, a constant, for all u ∈ V.

Assume that G is a k_1 – regular fuzzy graph.

Then $d(u) = k_1$, for all u ∈ V.

So $td(u) = d(u) + \sigma(u)$, for all u ∈ V.

$\Rightarrow$ $td(u) = k_1 + c$, for all $u \in V$.

Hence G is a totally regular fuzzy graph. Thus (1) $\Rightarrow$ (2) is proved.

Now, suppose that G is a k_2 – totally regular fuzzy graph.

Then $td(u) = k_2$, for all $u \in V$.

$\Rightarrow$ $d(u) + \sigma(u) = k_2$, for all $u \in V$.

$\Rightarrow$ $d(u) + c = k_2$, for all $u \in V$.

$\Rightarrow$ $d(u) = k_2 - c$, for all $u \in V$.

So G is a regular fuzzy graph. Thus (2) $\Rightarrow$ (1) is proved. Hence (1) and (2) are equivalent.

Conversely, assume that (1) and (2) are equivalent. (i.e) G is regular if and only if G is totally regular.

Suppose σ is not a constant function. Then $\sigma(u) \neq \sigma(w)$ for at least one pair of vertices $u,w \in V$.

Let G be a k-regular fuzzy graph. Then $d(u) = d(w) = k$

So $td(u) = d(u) + \sigma(u) = k + \sigma(u)$ and $td(w) = d(w) + \sigma(w) = k + \sigma(w)$

Since $\sigma(u) \neq \sigma(w)$, we have $td(u) \neq td(w)$.

So G is not totally regular which is a contradiction to our assumption.

Now let G be a totally regular fuzzy graph.

Then $td(u) = td(w) \Rightarrow d(u) + \sigma(u) = d(w) + \sigma(w)$

$\Rightarrow d(u) - d(w) = \sigma(w) - \sigma(u) \neq 0$

$\Rightarrow d(u) \neq d(w)$.

So G is not regular which is a contradiction to our assumption. Hence σ is a constant function.

Theorem 2.12: If a fuzzy graph G is both regular and totally regular, then σ is a constant function.

Proof: Let G be a k_1 – regular and k_2 – totally regular fuzzy graph.

So $d(u)=k_1$, for all $u \in V$ and $td(u) = k_2$, for all $u \in V$.

Now $td(u) = k_2$, for all $u \in V$.

$\Rightarrow$ $d(u) + \sigma(u) = k_2$, for all $u \in V$.

$\Rightarrow$ $k_1 + \sigma(u) = k_2$, for all $u \in V$.

$\Rightarrow$ $\sigma(u) = k_2 - k_1$, for all $u \in V$.

Hence σ is a constant function.

Remark 2.13: Converse of theorem 2.12 need not be true.

For example, Consider $G^*:(V,E)$ where $V=\{v_1,v_2,v_3\}$ and $E=\{v_1v_2,v_1v_3\}$. Define $G:(\sigma,\mu)$ by $\sigma(v_i)=0.5$, $i=1,2,3$ and $\mu(v_1v_2)=0.1$, $\mu(v_1v_3)=0.2$. Then σ is a constant function. But $d(v_2)=0.1\neq 0.2=d(v_3)$. Also $td(v_2)=0.6\neq 0.7=td(v_3)$. So G is neither regular nor totally regular.

3. A characterization of regular fuzzy graphs on a cycle

Theorem 3.1 and Theorem 3.3 provide a characterization of a regular fuzzy graph G:(σ,μ) such that G*:(V,E) is a cycle.

Theorem 3.1: Let G:(σ,μ) be a fuzzy graph where G*:(V,E) is an odd cycle. Then G is regular iff μ is a constant function.

Proof: If μ is a constant function, say μ(uv)=c, for all uv∈E, then d(v)=2c, for every v∈V. So G is regular.

Conversely, suppose that G is a k-regular fuzzy graph. Let $e_1,e_2,\ldots,e_{2n+1}$ be the edges of G* in that order.

Let $\mu(e_1) = k_1$. Since G is k-regular,

$\mu(e_2) = k-k_1$

$\mu(e_3) = k - (k-k_1) = k_1$

$\mu(e_4) = k-k_1$

and so on.

Therefore $\mu(e_i) = \begin{cases} k_1, & \text{if } i \text{ is odd} \\ k - k_1, & \text{if } i \text{ is even} \end{cases}$.

Hence $\mu(e_1)= \mu(e_{2n+1})=k_1$.

So if e_1 and e_{2n+1} incident at a vertex u, then d(u)=k.

So $d(e_1)+d(e_{2n+1}) = k$

$\Rightarrow \quad k_1 + k_1 = k$

$\Rightarrow \quad 2k_1 = k$

$\Rightarrow \quad k_1 = k / 2$

Hence $k-k_1= k/2$. So $\mu(e_i) = k/2$**,** for all i. Hence μ is a constant function.

Remark 3.2: The above theorem does not hold for totally regular fuzzy graphs. For example, Consider the cycle G*:(V,E) where $V=\{v_1,v_2,v_3\}$ and $E =\{v_1v_2,v_2v_3, v_3v_1\}$. Define G:(σ,μ) by $\sigma(v_1) = 0.5$, $\sigma(v_2) = 0.6$, $\sigma(v_3) = 0.4$ and $\mu(v_1v_2) = 0.1$, $\mu(v_2v_3) = 0.2$, $\mu(v_3v_1) = 0.3$. Then $td(v_i) = 0.9$ for all i = 1,2,3. Hence G is a totally regular fuzzy graph. But μ is not a constant function.

Theorem 3.3: Let G:(σ,μ) be a fuzzy graph where G* is an even cycle. Then G is regular iff either μ is a constant function or alternate edges have same membership values.

Proof: If either μ is constant function or alternate edges have same membership values, then G is a regular fuzzy graph.

Conversely, Suppose G is a k-regular fuzzy graph. Let $e_1,e_2,\ldots,e_{2n}$ be the edges of the even cycle G*in that order. Proceeding as in theorem 3.1,

$\mu(e_i) = \begin{cases} k_1, & \text{if } i \text{ is odd} \\ k - k_1, & \text{if } i \text{ is even} \end{cases}$

If $k_1=k-k_1$, then μ is a constant function.

If $k_1\neq k-k_1$, then alternate edges have same membership values.

Remark 3.4: The above theorem does not hold for totally regular fuzzy graphs.

For example, Consider the cycle $G^*:(V,E)$ where $V=\{v_1,v_2,v_3,v_4\}$ and $E= \{v_1v_2,v_2v_3,v_3v_4,v_4v_1\}$. Define $G:(\sigma,\mu)$ by $\sigma(v_1)=0.9$, $\sigma(v_2) = 0.7$, $\sigma(v_3) = 0.5$, $\sigma(v_4) = 0.7$ and $\mu(v_1v_2)=0.2$, $\mu(v_2v_3)=0.5$, $\mu(v_3v_4)=0.4$, $\mu(v_4v_1)=0.3$. Then $td(v_i) = 1.4$ for all $i = 1,2,3,4$. So G is totally regular. But in G, neither μ is a constant nor alternate edges have the same μ-values.

4. Properties of regular fuzzy graphs

Theorem 4.1: The size of a k-regular fuzzy graph $G:(\sigma,\mu)$ on $G^*:(V, E)$ is $\frac{pk}{2}$ where $p= | V |$.

Proof: The size of G is $S(G) = \sum_{uv\in E}\mu(uv)$.

Since G is k-regular, $d_G(v) = k$, for all $v\in V$.

We have $\sum_{v\in V} d_G(v) = 2\sum_{uv\in E}\mu(uv) = 2\,S(G)$.

So $2\,S(G) = \sum_{v\in V} d_G(v) = \sum_{v\in V} k = pk$.

Hence $S(G) = \frac{pk}{2}$.

Theorem 4.2: If $G:(\sigma,\mu)$ is a r- totally regular fuzzy graph, then $2S(G) + O(G) = pr$ where $p= | V |$.

Proof: Since G is a r-totally regular fuzzy graph,

$$r = td(v) = d(v) + \sigma(v) \text{ for all } v\in V.$$

$$\text{So } \sum_{v\in V} r = \sum_{v\in V} d_G(v) + \sum_{u\in V}\sigma(u)$$

$$\Rightarrow pr = 2S(G) + O(G)$$

Corollary 4.3: If G is a k-regular and a r- totally regular fuzzy graph, then $O(G) = p(r-k)$.

Proof: From Theorem 4.1, $S(G) = \frac{pk}{2}$ (or) $2S(G) = pk$.

From Theorem 4.2, $2S(G) + O(G) = pr$

So $O(G) = pr - 2S(G) = pr - pk = p(r-k)$.

Theorem 4.4: Let $G:(\sigma,\mu)$ be a regular fuzzy graph where G^* is a cycle. Then G is a fuzzy cycle. It can not be a fuzzy tree.

Proof: Assume that $G:(\sigma,\mu)$ is a regular fuzzy graph on a cycle G^*. Then by Theorem3.1 and Theorem3.3, either μ is a constant function or alternate edges have same membership values.

So there does not exist a unique edge xy such that $\mu(xy) = \wedge\{\mu(uv) / \mu(uv) > 0\}$.

Therefore G is a fuzzy cycle.

Hence by Lemma 1.8, G can not be a fuzzy tree.

Remark 4.5: The above theorem does not hold for totally regular fuzzy graphs.

For example, Consider G:(σ,μ) in Remark 3.4. G is totally regular. But G has a unique edge v_1v_2 such that $\mu(v_1v_2) = \wedge\{\mu(xy) / xy \in E\}$. So G is a fuzzy tree.

Theorem 4.6: A regular fuzzy graph on an odd cycle does not have a fuzzy bridge. Hence it does not have a fuzzy cutvertex.

Proof: Assume that G:(σ,μ) is a regular fuzzy graph on an odd cycle G*. Then μ is a constant function. So removal of any edge does not reduce the strength of connectedness between any pair of vertices. Hence G has no fuzzy bridge.

Hence by Lemma 1.10, G does not have a fuzzy cutvertex.

Remark 4.7: The above theorem does not hold for totally regular fuzzy graphs.

For example, Consider the fuzzy graph G:(σ,μ) in Remark 3.2. G is totally regular. Also $\wedge\{\mu(xy) / xy \in E\} = 0.1$ and $\mu(v_2v_3) = 0.2 > 0.1$, $\mu(v_3v_1) = 0.3 > 0.1$. So by Lemma 1.9, v_2v_3 and v_3v_1 are the fuzzy bridges of G. Hence by Lemma 1.10, v_3 is a fuzzy cutvertex.

Theorem 4.8: Let G:(σ,μ) be a regular fuzzy graph on an even cycle G*:(V, E). Then either G does not have a fuzzy bridge or it has q/2 fuzzy bridges where q = | E |. Also G does not have a fuzzy cutvertex.

Proof: Assume that G:(σ,μ) is a regular fuzzy graph on an even cycle G*. Then by Theorem 3.3, either μ is a constant function or alternate edges have same membership values.

Case 1: μ is a constant function. Then the removal of any edge does not reduce the strength of connectedness between any pair of vertices. So G does not have a fuzzy bridge and hence does not have a fuzzy cutvertex.

Case 2: Alternate edges have same membership values. Then by Lemma 1.9, edges with greater membership values are the fuzzy bridges of G. There are q/2 such edges where q = | E |. Hence G has q/2 fuzzy bridges. But then no vertex is a common vertex of two fuzzy bridges. So G does not have a fuzzy cutvertex.

Remark 4.9: The above theorem does not hold for totally regular fuzzy graphs.

For example, consider the following examples:

(1). Consider G:(σ,μ) in Remark 3.4. G is totally regular. v_2v_3 , v_3v_4 and v_4v_1 are the fuzzy bridges of G. Hence v_3 and v_4 are the fuzzy cutvertices of G. Here G has more than q/2(=2) fuzzy bridges.

(2).Consider G*:(V,E) where $V=\{v_1,v_2,v_3,v_4,v_5,v_6\}$ and $E=\{v_1v_2,v_2v_3,v_3v_4,v_4v_5,v_5v_6,v_6v_1\}$. Consider G:(σ,μ) on G* such that $\sigma(v_1)=0.9$, $\sigma(v_2)=0.7$, $\sigma(v_3)=0.8$, $\sigma(v_4)=\sigma(v_5)=\sigma(v_6)=1$ and $\mu(v_1v_2) = 0.3$, $\mu(v_2v_3) = 0.4$, $\mu(v_3v_4) = \mu(v_4v_5) = \mu(v_5v_6) = \mu(v_6v_1) = 0.2$. Then $td(v_i) = 1.4$ for all i = 1,2,3,4. So G is totally regular. v_1v_2 and v_2v_3 are the fuzzy bridges of G. Hence v_2 is a fuzzy cutvertex of G. Here G has fewer than q/2(=3) fuzzy bridges.

Theorem 4.10: A connected k-regular fuzzy graph where k > 0 with p≥3 can not have an end vertex (a vertex of degree 1 in G*).

Proof: Since k > 0, $d_G(v) > 0$ for every $v \in V$. So each vertex is adjacent to atleast one vertex. If possible, let u be an end vertex and let $uv \in E$. Then $d(u) = k = \mu(uv)$. Since G is connected and $p \geq 3$, v is adjacent to some other vertex $w \neq u$.

Then $d(v) \geq \mu(uv) + \mu(vw)$

$> \mu(uv)$

So $d(v) > k$, which is a contradiction. Hence G can not have an end vertex.

Remark 4.11: The above theorem does not hold for totally regular fuzzy graphs. For example, Consider $G:(\sigma,\mu)$ in Example 2.7. G is a totally regular fuzzy graph, which has two end vertices v_2 and v_3.

Remark 4.12: It is evident from theorem 2.11 that Theorems 3.1, 3.3, 4.4, 4.6, 4.8 and Theorem 4.10 hold for totally regular fuzzy graphs when σ is a constant function.

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