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Circulant Triangular Fuzzy Number Matrices

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ABSTRACT

In this paper, we present some operations on circulant triangular fuzzy numbers matrices (TFNMs). The first row of the circulant matrices play important role in this study. We also study some properties of determinant and adjoint of circulant TFNMs. Finally, we investigate the distance of generalized TFNMs by a systematic process.

Keywords: *Fuzzy matrices, triangular number, circulant matrices*

1. Introduction

In most of cases of our life, the data obtained for decision making are only approximate. In 1965, Zadeh [21] introduce the concept of fuzzy set theory to meet those problems. The fuzzyness can be represented by different ways. One of the most useful representation is membership function. Also, depending the nature and shape of the membership function the fuzzy number can be classified in different forms, such as triangular fuzzy number (TFNs), trapezoidal fuzzy number, etc. Several researchers present various results on TFNs. In 1985, Chen [4] gives the concept of generalized TFNs. A brief review of fuzzy matrices and their variants is given below.

The fuzzy matrices introduced first time by Thomason [20], and discussed about the convergence of powers of fuzzy matrix. Several authors presented a number of results on the convergence of power sequence of fuzzy matrices [7, 8, 10]. Ragab et al. [14] presented some properties on determinant and adjoint of square fuzzy matrix. Ragab et al. [15] presented some properties of the min-max composition of fuzzy matrices. Kim [9] presented some important results on determinant of a square fuzzy matrices. Two new operators and some applications of fuzzy matrices are given in [16, 17, 18, 19]. Perhaps first time, Pal [11] introduced intuitionistic fuzzy determinant. Also, Pal et al. [12] introduced intuitionistic fuzzy matrices. Recently, Bhowmik and Pal [2, 3] introduced some results on intuitionistic fuzzy matrices, intuitionistic circulant fuzzy matrices and generalized intuitionistic fuzzy matrices. Pal and Shyamal [13] first time introduced triangular fuzzy matrices.

In Section 2 of this paper, we recall the definition of TFNs and some operations of TFNs. In Section 3, we recall the definition of triangular fuzzy number matrices (TFNMs) and some operations of TFNMs. In Section 4, we define the circulant TFNMs and some results of circulant TFNMs. Finally, we propose a systematic process to find the distance between general TFNMs.

2. Triangular Fuzzy Number

Sometimes it may happen that some data or numbers can not be specified precisely or accurately due to the error of the measuring technique or instruments, etc. Suppose, the height of a person is 160 cm. But, practically it is impossible to measure the height accurately; actually this height is about 160 cm; it is some more or less than 160 cm. Thus the height of that person can be written more precisely as $(160 - \alpha, 160, 160+\beta)$ where α and β are left and right spreads of 160. In general, this number can written as $(a - \alpha, a, a + \beta)$, where α and β are the left and right spreads of a respectively. These type of numbers are called triangular fuzzy numbers (TFNs) and alternately represented as (a, α, β). The mathematical definition of a TFN is given below.

Definition 1. A triangular fuzzy number represented as $\tilde{A}=\left(a^1,\tilde{a},a^u:\mu_{\tilde{A}}\right)$, where $a^1,\tilde{a},a^u$ are all real values, $\mu_{\tilde{A}}$ denotes the membership grade or height and $\mu_{\tilde{A}}\in[0,1]$. a^1,a^u are the left hand and right hand spreads of the mean value $\tilde{a}$ respectively and membership grade $\mu_{\tilde{A}}$ is defined as follows:

$$\mu_{\tilde{A}}(x)=\begin{cases}0 & for\ x\le \tilde{a}-a^l\\ 1-\dfrac{\tilde{a}-x}{a^1} & for\ \tilde{a}-a^l<x<\tilde{a}\\ 1 & for\ x=\tilde{a}\\ 1-\dfrac{x-\tilde{a}}{a^u} & for\ \tilde{a}<x<m+a^u\\ 0 & for\ x\ge \tilde{a}+a^u\end{cases}$$

A TFN $\tilde{A}$ is said to be normalized if $\mu_{\tilde{A}}=1$ and it can be represented as $\tilde{A}=\left(a^l,\tilde{a},a^u\right)$.

If $0\le a^l\le \tilde{a}\le a^u$, then $\tilde{A}$ is called standardized fuzzy number.

Through out this paper we used normalized TFN.

Note 1. A TFN is said to be symmetric if its both spreads are equal, i.e., if $a^l=a^u$ and it is sometimes denoted by $\tilde{A}=\left(\tilde{a},a^l\right)$.

Here we introduce the definition of arithmetic operations due to Dubois and

Prade [6] and Pal and Shyamal [13].

Let $\tilde{A}=\left(a^l,\tilde{a},a^u\right)$ and $\tilde{B}=\left(b^l,\tilde{b},b^u\right)$ be two TFNs.

1. **Addition:** $\tilde{A}+\tilde{B}=\left(\tilde{a}+b^l,\tilde{a}+\tilde{b},a^u+b^u\right)$.
2. **Scalar multiplication:** Let λ be a scalar, then
   $\lambda\tilde{A}=\left(\lambda a^l,\lambda\tilde{a},\lambda a^u\right)$, when $\lambda\geq 0$.
   $\lambda\tilde{A}=\left(-\lambda a^u,\lambda\tilde{a}_1,-\lambda a^2\right)$, when $\lambda\leq 0$.
3. **Subtraction:** $\tilde{A}-\tilde{B}=\left(a^l+b^u,\tilde{a}-\tilde{b},a^u+b^l\right)$.

   For two $\tilde{A}$ and $\tilde{B}$, their addition, subtraction and scalar multiplication, i.e., $\tilde{A}+\tilde{B}$, $\tilde{A}-\tilde{B}$ and $\lambda\tilde{A}$ are TFNs. But their product and inverse may not be TFNs.
4. **Multiplication:**
   (a) When $\tilde{A}\geq 0$ and $\tilde{B}\geq 0$ $\left(\tilde{A}\geq 0, \text{ if } \tilde{a}\geq 0\right)$
   $$\tilde{A}\tilde{B}=\left\langle a^l,\tilde{a},a^u\right\rangle\left\langle b^l,\tilde{b},b^u\right\rangle\simeq\left\langle \tilde{a}b^l+\tilde{b}a^l,\tilde{a}\tilde{b},\tilde{a}b^u+ba^u\right\rangle,$$
   (b) When $\tilde{A}\leq 0$ $\left(a\leq 0\right)$ and $\tilde{B}\geq 0$
   $$\tilde{A}\tilde{B}=\left\langle a^l,\tilde{a},a^u\right\rangle\left\langle b^l,\tilde{b},b^u\right\rangle\simeq\left\langle a^l\tilde{b}-\tilde{a}b^u,\tilde{a}\tilde{b},a^u\tilde{b}-\tilde{a}b^l\right\rangle,$$
   (c) When $\tilde{A}\leq 0$ and $\tilde{B}\leq 0$
   $$\tilde{A}\tilde{B}=\left\langle a^l,\tilde{a},a^u\right\rangle\left\langle b^l,\tilde{b},b^u\right\rangle\simeq\left\langle -a^u\tilde{b}-\tilde{a}b^u,\tilde{a}\tilde{b},-a^l\tilde{b}-\tilde{a}b^u\right\rangle$$

It can be shown that the shape of the membership function of $\tilde{A}\tilde{B}$ is not necessarily a triangular, but, if the spreads of $\tilde{A}$ and $\tilde{B}$ are small compared to their mean values $\tilde{a}$ and $\tilde{b}$ then the shape of membership function is closed to a triangle. When spreads are not small compared with mean values, the formula can be changed to

$$\left\langle a^l,\tilde{a},a^u\right\rangle\left\langle b^l,\tilde{b},b^u\right\rangle\simeq\left\langle \tilde{a}b^l+\tilde{b}a^l-a^lb^l,\tilde{a}\tilde{b},\tilde{a}b^u+\tilde{b}a^u-a^ub^u\right\rangle \quad \text{for} \quad \tilde{A}\geq 0 \quad \text{and}$$

$\tilde{B}\geq 0$.

Throughout the paper we used previous definition.

Now, we define inverse of a TFN based on the definition of multiplication.

5. **Inverse:** Inverse of a TFN $\tilde{A}=\left\langle a^l,\tilde{a},a^u\right\rangle,\tilde{a}>0$ is defined as,
   $$\tilde{A}^{-1}=\left\langle a^l,\tilde{a},a^u\right\rangle^{-1}\simeq\left\langle a^u\tilde{a}^{-2},\tilde{a}^{-1},\tilde{a}^l\tilde{a}^{-2}\right\rangle.$$
   This is also an approximate value of $\tilde{A}^{-1}$ and it is valid only a neighbourhood of $\frac{1}{\tilde{a}}$.

   The division of $\tilde{A}$ by $\tilde{B}$ is given by
   $$\frac{\tilde{A}}{\tilde{B}}=\tilde{A}.\tilde{B}^{-1}.$$
   Since inverse and product both are approximate, the division is also an

approximate value.
The formula definition of division is given below.

6. **Division:** $\frac{\tilde{A}}{\tilde{B}} = \tilde{A}.\tilde{B}^{-1} = \langle a^l, \tilde{a}, a^u \rangle . \langle b^u \tilde{b}^{-2}, \tilde{b}^{-1}, b^l \tilde{b}^{-2} \rangle$
$$\simeq \left\langle \frac{\tilde{a}}{\tilde{b}}, \frac{\tilde{a}b^u + a^l \tilde{b}}{b^2}, \frac{\tilde{a}b^l + a^u \tilde{b}}{b^2} \right\rangle$$
From the definition of multiplication of TFNs, the power of any TFN $\tilde{A}$ is defined in the following way.
7. **Exponentiation:** [13] Using the definition of multiplication it can be shown that $\tilde{A}^n$ is given by
$\tilde{A}^n = \langle a^l, \tilde{a}, a^u \rangle^n \simeq \langle \tilde{a}^n, -n\tilde{a}^{n-1}a^u, -n\tilde{a}^{n-1}a^l \rangle$, when n is negative.
$\simeq \langle \tilde{a}^n, n\tilde{a}^{n-1}a^l, n\tilde{a}^{n-1}a^u \rangle$, when n is positive.

3. Preliminaries and Definitions

Here we introduce some definitions due to Pal and Shyamal [13].

Definition 2. [13] (**Triangular fuzzy number matrix (TFNM)**). A TFNM of order $m \times n$ is defined as $\tilde{A} = (a_{ij})_{m \times n}$, where $a_{ij} = \langle a^l_{ij}, \tilde{a}_{ij}, a^u_{ij} \rangle$ is the ij[th] element of $\tilde{A}, \tilde{a}_{ij}$ is the mean value of $\tilde{A}$ and a^l_{ij}, a^u_{ij} are the left and right spreads of a_{ij} respectively.

Like classical matrices we define the following operations on TFNMs. Let $\tilde{A} = (a_{ij})$ and $\tilde{B} = (b_{ij})$ be two TFNMs of same order. The following operations are defined in [13]:

(i) $\tilde{A} + \tilde{B} = (a_{ij} + b_{ij})$

(ii) $\tilde{A} - \tilde{B} = (a_{ij} - b_{ij})$

(iii) $\tilde{A}.\tilde{B} = (c_{ij})_{m \times p}$ [where $\tilde{A} = (a_{ij})_{m \times n}, \tilde{B} = (b_{ij})_{n \times p}$

and $c_{ij} = \sum_{k=1}^{n} a_{ik}.b_{kj}$ for *i = 1,2,...,m* and *j = 1,2,...,p*.]

(iv) $\tilde{A}^{k+1} = \tilde{A}^k.\tilde{A}$

(v) $\tilde{A}' = (a_{ji})$ (the transpose of $\tilde{A}$)

(vi) $k.\tilde{A} = (ka_{ij})$, where *k* is a scalar.

Definition 3. (Null TFNM). A TFNM is said to be a null TFNM if all its entries are zero, i.e., all elements are $\langle 0,0,0 \rangle$. This matrix is denoted by $\tilde{O}$.

Definition 4. (Unit TFNM). A square TFNM is said to be a unit TFNM if all $a_{ii} = \langle 0,1,0 \rangle$ and $a_{ij} = \langle 0,0,0 \rangle, i \neq j$, for all *i, j*. It is denoted by $\tilde{I}$. A $n \times n$ unit TFNM is as follows:

$$\tilde{I}_n = \begin{bmatrix} \langle 0,1,0\rangle & \langle 0,0,0\rangle & . & . & \langle 0,0,0\rangle & \langle 0,0,0\rangle \\ \langle 0,0,0\rangle & \langle 0,1,0\rangle & . & . & \langle 0,0,0\rangle & \langle 0,0,0\rangle \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ \langle 0,0,0\rangle & \langle 0,0,0\rangle & . & . & \langle 0,1,0\rangle & \langle 0,0,0\rangle \\ \langle 0,0,0\rangle & \langle 0,0,0\rangle & . & . & \langle 0,0,0\rangle & \langle 0,1,0\rangle \end{bmatrix}$$

Definition 5. (Symmetric TFNM). A square TFNM $\tilde{A} = (a_{ij})$ is said to be symmetric if $\tilde{A} = \tilde{A}'$, i.e., if
$a_{ij} = a_{ji}$ for all i,j.

Now, we define triangular fuzzy number determinant (TFND) of a TFNM. Minor and cofactor of a TFNM defined as in classical matrices. But, TFND has some special properties due to the sub-distributive property of TFNs.

Definition 6. (Determinant of TFNM). The triangular fuzzy determinant of a TFNM $\tilde{A}$ of order $n \times n$ is denoted by $|\tilde{A}|$ or det $(\tilde{A})$ and is defined as,

$$\begin{aligned} |\tilde{A}| &= \sum_{\sigma \in S_n} \text{Sgn } \sigma \left\langle a^l_{1\sigma(1)}, \tilde{a}_{1\sigma(1)}, a^u_{1\sigma(1)} \right\rangle \ldots \left\langle a^l_{n\sigma(n)}, \tilde{a}_{n\sigma(n)}, a^u_{n\sigma(n)} \right\rangle \\ &= \sum_{\sigma \in S_n} \text{Sgn } \sigma \prod_{i=1}^{n} a_{i\sigma(i)} \end{aligned}$$

where $a_{i\sigma(i)} = \left\langle a^l_{i\sigma(i)}, \tilde{a}_{i\sigma(i)}, a^u_{i\sigma(i)} \right\rangle$ are TFNMs and S_n denotes the symmetric group of all permutations of the indices $\{1,2,...,n\}$ and Sgn $\sigma = 1$ or -1 according as the permutation $\sigma = \begin{pmatrix} 1 & 2 & \ldots & n \\ \sigma(1) & \sigma(2) & \ldots & \sigma(n) \end{pmatrix}$ is even or odd respectively.

The computation of $\det(\tilde{A})$ involves several products of TFNs. Since the product of two or more TFNs is an approximate TFN, the value of $\det(\tilde{A})$ is also an approximate TFN.

Definition 7. (Minor). Let $\tilde{A} = (a_{ij})$ be a square TFNM of order $n \times n$. The minor of an element a_{ij} in $\det(\tilde{A})$ is a determinant of order $(n-1)\times(n-1)$, which is obtained by deleting the ith row and the jth column from $\tilde{A}$ and is denoted by $\tilde{M}_{ij}$.

Definition 8. (Cofactor). Let $\tilde{A} = (a_{ij})$ be a square TFNM of order $n \times n$. The cofactor of an element a_{ij} in $\tilde{A}$ is denoted by $\tilde{A}_{ij}$ and is defined as, $\tilde{A}_{ij} = (-1)^{i+j} M_{ij}$.

Definition 9. (Adjoint). Let $\tilde{A} = (a_{ij})$ be a square TFNM and $\tilde{B} = (b_{ij})$ be a square TFNM whose elements are the co-factors of the corresponding elements in $|\tilde{A}|$ then the transpose of $\tilde{A}$ is called the adjoint or adjugate of $\tilde{A}$ and it is equal to $(\tilde{A}_{ji})$. The adjoint of $\tilde{A}$ is denoted by $\mathrm{adj}(\tilde{A})$.

Here $|\tilde{A}|$ contains n! terms out of which $\frac{n}{2}!$ are positive terms and the same number of terms are negative. All these n! terms contain n quantities at a time in product form, subject to the condition that from the n quantities in product one and only one is taken from each row and also single element is taken from each column.

Alternatively, a TFD of a TFNM $\tilde{A} = (a_{ij})$ may be expanded in the form $\sum_{j=1}^{n} a_{ij}.\tilde{A}_{ij}, i \in \{1, 2, ..., n\}$, where $\tilde{A}_{ij}$ is the cofactor of a_{ij}. Thus the TFD is the sum of the products of the elements of any row (or column) and the co-factors of the corresponding elements of the same row (or column). We refer this method as alternative method.

In classical mathematics, the value of a determinant is computed by any one of the aforesaid two processes and both yield same result. But, due to the failure of distributive laws of triangular fuzzy numbers, the value of a TFD, computed by the aforesaid two process will differ from each other. For this reason the value of a TFD should be determined according to the definition, i.e., using the following rule only

$$|\tilde{A}| = \sum_{\sigma \in S_n} \mathrm{Sgn}\ \sigma \left\langle a^{l}_{1\sigma(1)}, \tilde{a}_{1\sigma(1)}, a^{u}_{1\sigma(n)} \right\rangle ... \left\langle a^{l}_{n\sigma(n)}, \tilde{a}_{n\sigma(n)}, a^{u}_{n\sigma(1)} \right\rangle .$$

On the other hand, the value of a TFD computed by the alternative process yields incorrect result.

4. Circulant TFNM

Definition 10. A TFNM $\tilde{A}$ is said to be circulant TFNM if all the elements of $\tilde{A}$ can be determined completely by its first row. Suppose the first row of $\tilde{A}$ is

$$\left[\left\langle a^{l}_{1}, \tilde{a}_{1}, a^{u}_{1} \right\rangle, \left\langle a^{l}_{2}, \tilde{a}_{2}, a^{u}_{2} \right\rangle, \left\langle a^{l}_{3}, \tilde{a}_{3}, a^{u}_{3} \right\rangle ..., \left\langle a^{l}_{n}, \tilde{a}_{n}, a^{u}_{n} \right\rangle \right].$$

Then any element a_{ij} of $\tilde{A}$ can be determined (throughout the element of the first row) as

$$a_{ij} = a_{1(n-i+j+1)} \text{ with } a_{1(n+k)} = a_{1k} .$$

A circulant TFNM is the form of

$$\begin{bmatrix} \langle a_1^l, \tilde{a}_1, a_1^u \rangle & \langle a_2^l, \tilde{a}_2, a_2^u \rangle & . & . & \langle a_{(n-1)}^l, \tilde{a}_{(n-1)}, a_{(n-1)}^u \rangle & \langle a_{(n)}^l, \tilde{a}_{(n)}, a_{(n)}^u \rangle \\ \langle a_n^l, \tilde{a}_n, a_n^u \rangle & \langle a_1^l, \tilde{a}_1, a_1^u \rangle & . & . & \langle a_{(n-2)}^l, \tilde{a}_{(n-2)}, a_{(n-2)}^u \rangle & \langle a_{(n-1)}^l, \tilde{a}_{(n-1)}, a_{(n-1)}^u \rangle \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ \langle a_3^l, \tilde{a}_3, a_3^u \rangle & \langle a_4^l, \tilde{a}_4, a_4^u \rangle & . & . & \langle a_1^l, \tilde{a}_1, a_1^u \rangle & \langle a_2^l, \tilde{a}_2, a_2^u \rangle \\ \langle a_2^l, \tilde{a}_2, a_2^u \rangle & \langle a_3^l, \tilde{a}_3, a_3^u \rangle & . & . & \langle a_n^l, \tilde{a}_n, a_n^u \rangle & \langle a_1^l, \tilde{a}_1, a_1^u \rangle \end{bmatrix}$$

Remark 1. It is noted that the matrix TFNM $\tilde{A}$ is circulant if and only if $a_{ij} = a_{(k\oplus i)(k\oplus j)}$ for every $i, j, k \in \{1, 2, ..., n\}$, where $\oplus$ is sum modulo n. This supply that the elements of the diagonal are all equals.

Remark 2. For a circulant TFNM $\tilde{A}$ we notice that $a_{in} = a_{(i\oplus 1)1}$ and $a_{nj} = a_{1(j\oplus 1)}$ for every $i, j, \in \{1, 2, ..., n\}$.

Remark 3. For a circulant TFNM $\tilde{A}$ we notice that $a_{(i\oplus \overline{n-1})j} = a_{i(j\oplus 1)}$ for every $i, j \in \{1, 2, ..., n\}$.

Remark 4. For a circulant TFNM $\tilde{A}$ of order $n \times n$ with first row $\left[\langle a_1^l, \tilde{a}_1, a_1^u \rangle, \langle a_2^l, \tilde{a}_2, a_2^u \rangle, \langle a_3^l, \tilde{a}_3, a_3^u \rangle, ..., \langle a_n^l, \tilde{a}_n, a_n^u \rangle\right]$. Then the k^{th} column of $\tilde{A}$ is $\left[\langle a_k^l, \tilde{a}_k, a_k^u \rangle, \langle a_{(k-1)}^l, \tilde{a}_{(k-1)}, a_{(k-1)}^u \rangle, ..., \langle a_1^l, \tilde{a}_1, a_1^u \rangle, \langle a_n^l, \tilde{a}_n, a_n^u \rangle,\right.$

$$\left. \langle a_{(n-1)}^l, \tilde{a}_{(n-1)}, a_{(n-1)}^u \rangle, ..., \langle a_{(k+1)}^l, \tilde{a}_{(k+1)}, a_{(k+1)}^u \rangle \right]'.$$

Theorem 1. An $n \times n$ TFNM $\tilde{A}$ is circulant if and only if $\tilde{A}\tilde{C}_n = \tilde{C}_n\tilde{A}$, where $\tilde{C}_n$ is the permutation matrix of unit TFNM.

$$\tilde{C}_n = \begin{bmatrix} \langle 0,0,0 \rangle & \langle 0,0,0 \rangle & . & . & \langle 0,0,0 \rangle & \langle 0,1,0 \rangle \\ \langle 0,1,0 \rangle & \langle 0,0,0 \rangle & . & . & \langle 0,0,0 \rangle & \langle 0,0,0 \rangle \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ \langle 0,0,0 \rangle & \langle 0,0,0 \rangle & . & . & \langle 0,0,0 \rangle & \langle 0,0,0 \rangle \\ \langle 0,0,0 \rangle & \langle 0,0,0 \rangle & . & . & \langle 0,1,0 \rangle & \langle 0,0,0 \rangle \end{bmatrix}$$

Proof. Let $\tilde{A}$ be a TFNM and $\tilde{P} = \tilde{A}\tilde{C}_n$, then $p_{ij} = \sum_{k=1}^{n} \left(a_{ik} c_{kj} \right)$.

Since, only c_{1n} is $\langle 0,1,0\rangle$ and all other elements of the first row of $\tilde{C}_n$ is $\langle 0,0,0\rangle$. We get $p_{ij} = a_{1(j\oplus 1)}$.

Similarly, if $\tilde{T} = \tilde{C}_n\tilde{A}$, then $t_{ij} = a_{(i\oplus \overline{n-1})j}$.

So, by Remark 3 $p_{ij} = t_{ij}$ for all $i, j \in n$.

Hence $\tilde{A}\tilde{C}_n = \tilde{C}_n\tilde{A}$. So, $\tilde{A}$ is circulant TFNM.
Converse is straightforward.

Example 1. Let $\tilde{A}$ and $\tilde{C}$ be two circulant TFNMs of order 3×3, where

$$\tilde{A} = \begin{bmatrix} \langle 2,3,4\rangle & \langle 4,6,7\rangle & \langle 3,6,7\rangle \\ \langle 3,6,7\rangle & \langle 2,3,4\rangle & \langle 4,6,7\rangle \\ \langle 4,6,7\rangle & \langle 3,6,7\rangle & \langle 2,3,4\rangle \end{bmatrix} \text{ and } \tilde{C} = \begin{bmatrix} \langle 0,0,0\rangle & \langle 0,0,0\rangle & \langle 0,1,0\rangle \\ \langle 0,1,0\rangle & \langle 0,0,0\rangle & \langle 0,0,0\rangle \\ \langle 0,0,0\rangle & \langle 0,1,0\rangle & \langle 0,0,0\rangle \end{bmatrix}.$$

Then $$\tilde{A}\tilde{C} = \begin{bmatrix} \langle 4,6,7\rangle & \langle 3,6,7\rangle & \langle 2,3,4\rangle \\ \langle 2,3,4\rangle & \langle 4,6,7\rangle & \langle 3,6,7\rangle \\ \langle 3,6,7\rangle & \langle 2,3,4\rangle & \langle 4,6,7\rangle \end{bmatrix} \text{ and } \tilde{C}\tilde{A} = \begin{bmatrix} \langle 4,6,7\rangle & \langle 3,6,7\rangle & \langle 2,3,4\rangle \\ \langle 2,3,4\rangle & \langle 4,6,7\rangle & \langle 3,6,7\rangle \\ \langle 3,6,7\rangle & \langle 2,3,4\rangle & \langle 4,6,7\rangle \end{bmatrix}.$$

Thus $\tilde{A}\tilde{C} = \tilde{C}\tilde{A}$.

Theorem 2. For the circulant TFNMs $\tilde{A}$ and $\tilde{B}$.

(i) $\tilde{A} + \tilde{B}$ is a circulant TFNM.
(ii) $\tilde{A}'$ is a circulant TFNM.
(iii) $\tilde{A}\tilde{B}$ is also a circulant TFNM. In particular, $\tilde{A}^k$ is also a circulant TFNM.
(iv) $\tilde{A}\tilde{A}'$ is circulant TFNM.

Proof. (i) Proof is straightforward.

(ii) Since $\tilde{A}$ is circulant TFNM then $\tilde{A}$ commutes with $\tilde{C}_n$.

So, $\tilde{A}\tilde{C}_n = \tilde{C}_n\tilde{A}$. Transposing both sides of $\tilde{A}\tilde{C}_n = \tilde{C}_n\tilde{A}$, we get

$\tilde{C}_n'\tilde{A}' = \tilde{A}'\tilde{C}_n'$

or, $\tilde{C}_n\tilde{C}_n'\tilde{A}' = \tilde{C}_n\tilde{A}'\tilde{C}_n'$

or, $\tilde{A}' = \tilde{C}_n\tilde{A}'\tilde{C}_n'$ $\left[\text{since, } \tilde{C}_n'\tilde{C}_n = \tilde{I} = \tilde{C}_n\tilde{C}_n'\right]$

or, $\tilde{A}'\tilde{C}_n = \tilde{C}_n\tilde{A}'\tilde{C}_n'\tilde{C}_n = \tilde{C}_n\tilde{A}'$.

So, $\tilde{A}'$ is circulant TFNs.

(iii) Since, $\tilde{A}$ and $\tilde{B}$ are circulant TFNMs, each of $\tilde{A}$ and $\tilde{B}$ commutes with $\tilde{C}_n$. Hence, $\tilde{A}\tilde{B}$ commutes with $\tilde{C}_n$.

So, by Remark 3 and Theorem 1 we get $\tilde{A}\tilde{B}$ is circulant TFNM.
Proof is similar.

Theorem 3. If $\tilde{A}$ and $\tilde{B}$ are circulant TFNMs then $\tilde{A}\tilde{B} = \tilde{B}\tilde{A}$.

Proof. Let $\tilde{A}\tilde{B} = \tilde{C}$ and $\tilde{B}\tilde{A} = \tilde{D}$ then both the $\tilde{C}$ and $\tilde{D}$ are circulant by Theorem

2(iii) and their first rows are

$\left[\left\langle c_1^l,\tilde{c}_1,c_1^u\right\rangle,\left\langle c_2^l,\tilde{c}_2,c_2^u\right\rangle,\left\langle c_3^l,\tilde{c}_3,c_3^u\right\rangle,...,\left\langle c_n^l,\tilde{c}_n,c_n^u\right\rangle\right]$ and

$\left[\left\langle d_1^l,\tilde{d}_1,d_1^u\right\rangle,\left\langle d_2^l,\tilde{d}_2,d_2^u\right\rangle,\left\langle d_3^l,\tilde{d}_3,d_3^u\right\rangle,...,\left\langle d_n^l,\tilde{d}_n,d_n^u\right\rangle\right]$ respectively. Then the mean value of the kth element of the first row of C and D are respectively

$$\begin{aligned}\tilde{c}_k &= \left[\sum_{p=1}^{k}\left(\tilde{a}_p\tilde{b}_{(k-p+1)}\right)\right]+\left[\sum_{p=k+1}^{n}\left(\tilde{a}_p\tilde{b}_{(n-p+k+1)}\right)\right]\\ &= \left(\tilde{a}_1\tilde{b}_k\right)+\left(\tilde{a}_2\tilde{b}_{(k-1)}\right)+\left(\tilde{a}_3\tilde{b}_{(k-2)}\right)+...+\left(\tilde{a}_{(k-1)}\tilde{b}_2\right)+\left(\tilde{a}_k\tilde{b}_1\right)\\ &\quad+\left(\tilde{a}_{(k+1)}\tilde{b}_n\right)+...+\left(\tilde{a}_{(k+2)}\tilde{b}_{(n-1)}\right)+...+\left(\tilde{a}_{(n-1)}\tilde{b}_{(k+2)}\right)+\left(\tilde{a}_n\tilde{b}_{(k+1)}\right).\\ \tilde{d}_k &= \left[\sum_{p=1}^{k}\left(\tilde{b}_p\tilde{a}_{(k-p+1)}\right)\right]+\left[\sum_{p=k+1}^{n}\left(\tilde{b}_p\tilde{a}_{(n-p+k+1)}\right)\right]\\ &= \left(\tilde{a}_1\tilde{b}_k\right)+\left(\tilde{a}_2\tilde{b}_{(k-1)}\right)+\left(\tilde{a}_3\tilde{b}_{(k-2)}\right)+...+\left(\tilde{a}_{(k-1)}\tilde{b}_2\right)+\left(\tilde{a}_k\tilde{b}_1\right)\\ &\quad+\left(\tilde{a}_{(k+1)}\tilde{b}_n\right)+...+\left(\tilde{a}_{(k+2)}\tilde{b}_{(n-1)}\right)+...+\left(\tilde{a}_{(n-1)}\tilde{b}_{(k+2)}\right)+\left(\tilde{a}_n\tilde{b}_{(k+1)}\right).\end{aligned}$$

It can be easy to see that $\tilde{c}_k=\tilde{d}_k$.

The left hand spread of the k[th] element of the first row of C and D are

$$\begin{aligned}c_k^l &= \sum_{p=1}^{k}\left(a_p^l\tilde{b}_{(k-p+1)}+\tilde{a}_p b_{(k-p+1)}^l\right)+\sum_{p=k+1}^{n}\left(a_p^l\tilde{b}_{(n-p+k+1)}+\tilde{a}_p b_{(n-p+k+1)}^l\right)\\ &= \left(a_1^l\tilde{b}_k+\tilde{a}_1 b_k^l\right)+\left(a_2^l\tilde{b}_{(k-1)}+\tilde{a}_2 b_{(k-1)}^l\right)+...+\left(a_{(n-1)}^l\tilde{b}_{(k+2)}\right)\\ &\quad+\left(\tilde{a}_{(n-1)}b_{(k+2)}^l\right)+\left(a_n^l\tilde{b}_{(k+1)}+\tilde{a}_n b_{k+2}^l\right).\\ d_k^l &= \sum_{p=1}^{k}\left(b_p^l\tilde{a}_{(k-p+1)}+\tilde{b}_p a_{(k-p+1)}^l\right)+\sum_{p=k+1}^{n}\left(b_p^l\tilde{a}_{(n-p+k+1)}+\tilde{b}_p a_{(n-p+k+1)}^l\right)\\ &= \left(b_1^l\tilde{a}_k+\tilde{a}_1 b_k^l\right)+\left(b_2^l\tilde{a}_{(k-1)}+\tilde{b}_2 a_{(k-1)}^l\right)+...+\left(b_{(n-1)}^l\tilde{a}_{(k+2)}\right)\\ &\quad+\left(\tilde{b}_{(n-1)}a_{(k+2)}^l\right)+\left(b_n^l\tilde{a}_{(k+1)}+\tilde{b}_n a_{k+2}^l\right).\end{aligned}$$

It can be easy to see that $c_k^l=d_k^l$.

Similarly we can see, the right hand spread of the kth element of the first row of C and D are equal i.e., $c_k^u=d_k^u$.

Since $\tilde{C}$ and $\tilde{D}$ are circulant, we have $c_{ij}=d_{ij}$ and hence the theorem is proved.

Theorem 4. A circulant TFNM $\tilde{A}$ is symmetric iff $a_{1i}=a_{1(n-i+2)}$ for every $i\in\{1,2,...,n\}$

Proof. Let $\tilde{A}$ be symmetric, then

$$a_{1i}=a_{(1\oplus k)(i\oplus k)}=a_{i1}=a_{(i\oplus k)(1\oplus k)} \text{ for every } i,k\in\{1,2,...,n\}.$$

Taking $k=n-i$, then

$$a_{(1\oplus(n-1))(i\oplus(n-i))}=a_{(i\oplus(n-i))(1\oplus(n-i))}=a_{n(n-i+1)}=a_{1(n-i+2)} \text{ by remark 2.}$$

Conversely, suppose $a_{1i} = a_{1(n-i+2)}$ for every $i \in \{1,2,...,n\}$, then $a_{i1} = a_{(i+k)(1+k)}$ for every $i,k \in \{1,2,...,n\}$.

Taking $k = n-i$, we get

$a_{il} = a_{n(n-i+1)} = (a+1)_{(n-i+2)} = a_{1i}$. But since $\tilde{A}$ is circulant and $a_{1i} = a_{i1}$. We have $a_{ij} = a_{ji}$ for every $i,k \in \{1,2,...,n\}$ and $\tilde{A}$ is symmetric.

Theorem 5. If a TFNMs $\tilde{A}$ is circulant, then $\tilde{E}\tilde{A}$ is symmetric where $\tilde{E}$ is a permutation matrix of unit TFNM and the form of $\tilde{E}$ is

$$\tilde{E} = \begin{bmatrix} \langle 0,0,0\rangle & \langle 0,0,0\rangle & . & . & \langle 0,0,0\rangle & \langle 0,1,0\rangle \\ \langle 0,0,0\rangle & \langle 0,0,0\rangle & . & . & \langle 0,1,0\rangle & \langle 0,0,0\rangle \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ \langle 0,0,0\rangle & \langle 0,1,0\rangle & . & . & \langle 0,0,0\rangle & \langle 0,0,0\rangle \\ \langle 0,1,0\rangle & \langle 0,0,0\rangle & . & . & \langle 0,0,0\rangle & \langle 0,0,0\rangle \end{bmatrix}.$$

Proof. Let $\tilde{R} = \tilde{E}\tilde{A}$, then $r_{ij} = \sum_{k=1}^{n} e_{ik}a_{kj}$ for all $i,j,k = 1,2,...,n$.

Since, $\tilde{E}$ is a permutation matrix of unit TFNM and only the elements $e_{1n}, e_{1(n-1)}, e_{1(n-2)}, ..., e_{n1}$ are $\langle 0,1,0\rangle$ and all others elements are $\langle 0,0,0\rangle$, we get

$$r_{ij} = \sum_{k=1}^{n} e_{ik}a_{kj} = a_{(n-i+1)j}$$

Now, since $\tilde{A}$ is circulant, we know

$$r_{ij} = a_{(n-i+1)j} = a_{((n-i+1)\oplus k)(k\oplus j)} \text{ for all } i,j,k \in \{1,2,...,n\}.$$

when, $k = i$, then

$$r_{ij} = a_{(n-i+1)j} = a_{(n\oplus 1)(i\oplus j)} = a_{1(i\oplus j)}$$

$$\text{and } r_{ji} = a_{(n-j+1)i} = a_{((n-j+1)\oplus k)(k\oplus i)} \text{ for } i,j,k \in \{1,2,...,n\}$$

Taking $k = j$, then

$$r_{ji} = a_{(n-j+1)j} = a_{(n\oplus 1)(i\oplus j)} = a_{1(i\oplus j)}.$$

Hence $r_{ij} = r_{ji}$ and thus $\tilde{R}$ is symmetric.

Theorem 6. Let $\tilde{A}$ be a circulant TFNM of order $n \times n$. Then

(i) adj $\tilde{A}$ is also circulant TFNM.

(ii) If $\tilde{A}$ is a square TFNM then $|\tilde{A}| = |\tilde{A}'|$.

(iii) adj $\tilde{A}$ = $(\text{adj}\tilde{A})'$.

Proof. (i) We have to prove co-factor of the elements $a_{i(j\oplus 1)}$ and $a_{(i\oplus \overline{n-1})j}$ for all

$i, j \in n$ are same.

Since $\tilde{A}$ is circulant then by Remark 3, $a_{i(j\oplus 1)} = a_{(i\oplus \overline{n-1})j}$ and so the minor of $a_{i(j\oplus 1)}$ and $a_{(i\oplus \overline{n-1})j}$ are will be same.

Now, co-factor of $a_{i(j\oplus 1)} = (-1)^{i+(j\oplus 1)} \sum_{\sigma\in S_n} \text{Sgn}\ \sigma \prod_{\substack{k=1 \\ k\neq i \\ k\neq j\oplus 1}}^{n} a_{k\sigma(k)}$ and

$$a_{(i\oplus \overline{n-1})j} = (-1)^{(i\oplus \overline{n-1})+j} \sum_{\sigma\in S_n} \text{Sgn}\ \sigma \prod_{\substack{k=1 \\ k\neq j \\ k\neq (i\oplus \overline{n-1})}}^{n} a_{k\sigma(k)}$$

It is obvious that for fixed n, the sign of $(-1)^{i+(j\oplus 1)}$ and $(-1)^{(i\oplus \overline{n-1})+j}$ is same for all $i, j \in \{1, 2 ... n\}$.

So, the co-factor of $a_{i(j\oplus 1)}$ and $a_{(i\oplus \overline{n-1})j}$ are same.

Hence adj $\tilde{A}$ is also circulant TFNM.

(ii) Let $\tilde{A} = (a_{ij})_{n\times n}$ be a square TFNM and $\tilde{A} = \tilde{B} = (b_{ij})_{n\times n}$. Then,

$$|\tilde{B}| = \sum_{\sigma\in Sn} \text{Sgn}\ \sigma\ b_{1\sigma(1)}, b_{2\sigma(2)}, ... b_{n\sigma(n)} = \sum_{\sigma\in Sn} \text{Sgn}\ \sigma\ a_{\sigma(1)1}, a_{\sigma(2)s}, ... a_{\sigma(n)n}.$$

Let ϕ be a permutation of $\{1, 2, ..., n\}$ such that $\phi\sigma = I$, the identity permutation. Then $\phi = \sigma^{-1}$. Since σ runs over the whole set of permutations, ϕ also runs over the same set of permutation. Let $\sigma(i) = j$ then $i = \sigma^{-1}(j)$ and $a_{\sigma(i)i} = a_{j\phi(j)}$ for all i,j. Therefore,

$$|\tilde{B}| = \sum_{\sigma\in S_n} \text{Sgn}\ \sigma \left\langle a^{l}_{1\sigma(1)}, \tilde{a}_{1\sigma(1)}, a^{u}_{1\sigma(1)} \right\rangle \left\langle a^{l}_{2\sigma(2)}, \tilde{a}_{2\sigma(2)}, a^{u}_{2\sigma(2)} \right\rangle ... \left\langle a^{l}_{n\sigma(n)}, \tilde{a}_{n\sigma(n)}, a^{u}_{n\sigma(n)} \right\rangle$$

$$= \sum_{\phi\in S_n} \text{Sgn}\ \phi \left\langle a^{l}_{1\sigma(1)}, \tilde{a}_{1\sigma(1)}, a^{u}_{1\sigma(1)} \right\rangle \left\langle a^{l}_{2\sigma(2)}, \tilde{a}_{2\sigma(2)}, a^{u}_{2\sigma(2)} \right\rangle ... \left\langle a^{l}_{n\sigma(n)}, \tilde{a}_{n\sigma(n)}, a^{u}_{n\sigma(n)} \right\rangle$$

$= |\tilde{A}|$. Hence, $|\tilde{A}| = |\tilde{A}'|$.

(iv) Similar to (i) and (ii).

5. Distance Between Normal Generalized TFNMs

Definition 11. (Generalized triangular number fuzzy matrix (GTFNM)). A TFNM of order $m\times n$ is defined as $\tilde{A} = (a_{ij})_{m\times n}$, where $a_{ij} = \langle a^{l}_{ij}, \tilde{a}_{ij}, a^{u}_{ij} \rangle$ is the ijth element of $\tilde{A}, \tilde{a}_{ij}$ is the mean value of $\tilde{A}$ and a^{l}_{ij}, a^{u}_{ij} are the left and right spreads of a_{ij} respectively. It is said to be generalized if $a^{l}_{ij} \le \tilde{a}_{ij} \le a^{u}_{ij}$.

In this section we proposed a method to make a score value by standardizing each element $a_{ij} = \langle a^{l}_{ij}, \tilde{a}_{ij}, a^{u}_{ij} \rangle$ of a TFNM A as follows:

Step 1: Each generalized TFN is standardized as follows:

$$\tilde{a}_{ij} = \left\langle \frac{a_{ij}^{l}}{a_{ij}^{u}}, \frac{\tilde{a}_{ij}}{a_{ij}^{u}}, \frac{a_{ij}^{u}}{a_{ij}^{u}} \right\rangle$$
$$= \left\langle a_{ij}^{l*}, \tilde{a}_{ij}^{*} a_{ij}^{u*} \right\rangle$$

Step 2: Calculate the defuzified value $x_{\tilde{a}_{ij}^*}$ using the following rule:

$$x_{\tilde{a}^*_{ij}} = \frac{a_{ij}^{l*} + \tilde{a}_{ij}^{*} + a_{ij}^{u*}}{3}$$

Step 3: Calculate the spread $std_{\tilde{a}_{ij}^*}$ of the $\tilde{a}_{ij}^*$ as follows:

$$std_{\tilde{a}_{ij}^*} = \sqrt{\frac{\left(a_{ij}^{l*} - x_{\tilde{a}_{ij}^*}\right)^2 + \left(\tilde{a}_{ij}^{*} - x_{\tilde{a}_{ij}^*}\right)^2 + \left(a_{ij}^{u*} - x_{\tilde{a}_{ij}^*}\right)^2}{3}}$$

Step 4: Calculate, score $\left(\tilde{a}_{ij}^*\right)$, the score value of the standardized generalized TFN as follows:

$$\text{score}\left(\tilde{a}_{ij}^*\right) = x_{\tilde{a}_{ij}^*} . \left(1 - \alpha . std_{\tilde{a}_{ij}^*}\right)$$

Note 2. It is noted that score value of any generalized TFN must be a real number and it’s value belongs to the interval [0,1].

Also, it is noted that, α is a parameter for adjusting the degree of importance of the spread of a generalized TFN and $\alpha = \{0.5, 1.5\}$. If the exparte consider the spread is more important than α is taken as 1.5 otherwise α equal to 0.5.

Now we define two basic distances between TFNMs. The distance δ is a mapping from the set of TFNMs (M) to the set of real number (R).

$$\delta : M \times M \to R$$

Score-distance (SD): The SD between two TFNMs $\tilde{A}$ and $\tilde{B}$ of order $m \times n$ is

$$SD\left(\tilde{A}, \tilde{B}\right) = \sum_{i=1}^{m} \sum_{j=1}^{n} \left| \text{score}\left(\tilde{a}_{ij}^*\right) - \text{score}\left(\tilde{b}_{ij}^*\right) \right|.$$

It is obvious $0 \le SD\left(\tilde{A}, \tilde{B}\right) \le m.n$. The score distance $SD : M \times M \to R$ satisfy the following conditions:

(i) $SD\left(\tilde{A}, \tilde{B}\right) \ge 0$ for all $A, B \in M$.

(ii) $SD\left(\tilde{A}, \tilde{B}\right) = 0$ iff $\tilde{A} = \tilde{B}$ for all $A, B \in M$.

(iii) $SD\left(\tilde{A}, \tilde{B}\right) = SD\left(\tilde{B}, \tilde{A}\right)$ for all $\tilde{A}, \tilde{B} \in M$.

(iv) $SD\left(\tilde{A}, \tilde{B}\right) \le SD\left(\tilde{A}, \tilde{C}\right) + SD\left(\tilde{C}, \tilde{B}\right)$ for all $\tilde{A}, \tilde{B}, \tilde{C} \in M$ (triangular property).

Thus the SD is metric on M.

Normalized Score-distance (SD*): The normalized SD is defined as:

$$SD^*\left(\tilde{A}, \tilde{B}\right) = \frac{SD\left(\tilde{A}, \tilde{B}\right)}{m.n}, \text{ where } 0 \le SD^*\left(\tilde{A}, \tilde{B}\right) \le 1.$$

Euclidian score-distance (E): The Euclidian SD is defined as :

$$E\left(\tilde{A},\tilde{B}\right)=\sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\text{score}\left(\tilde{a}_{ij}^{*}\right)-\text{score}\left(\tilde{b}_{ij}^{*}\right)\right]^{2}},\ \text{where}\ 0\le SD^{*}\left(\tilde{A},\tilde{B}\right)\le\sqrt{mn}\ .$$

The Euclidian distance is also a metric on M.

Normalized Euclidian score-distance (SD): The normalized Euclidian SD is defined as :

$$E^{*}\left(\tilde{A},\tilde{B}\right)=\frac{E\left(\tilde{A},\tilde{B}\right)}{\sqrt{m.n}},\ \text{where}\ 0\le E^{*}\left(\tilde{A},\tilde{B}\right)\le 1\ .$$

Example 2. Let $\tilde{A}=\begin{bmatrix}\langle 1,3,4\rangle & \langle 1,4,5\rangle & \langle 2,3,5\rangle\\ \langle 2,3,6\rangle & \langle 1,3,5\rangle & \langle 2,3,8\rangle\\ \langle 1,4,5\rangle & \langle 1,3,4\rangle & \langle 2,4,5\rangle\end{bmatrix}$ and

$$\tilde{B}=\begin{bmatrix}\langle 3,5,6\rangle & \langle 1,3,7\rangle & \langle 2,2,5\rangle\\ \langle 2,3,8\rangle & \langle 1,4,5\rangle & \langle 2,4,5\rangle\\ \langle 3,4,7\rangle & \langle 1,3,5\rangle & \langle 2,3,6\rangle\end{bmatrix}$$

Then score $\left(\tilde{A}^{*}\right)$ = $\begin{bmatrix}0.35 & 0.24 & 0.43\\ 0.35 & 0.30 & 0.27\\ 0.32 & 0.35 & 0.47\end{bmatrix}$

score $\left(\tilde{B}^{*}\right)$ = $\begin{bmatrix}0.40 & 0.24 & 0.39\\ 0.27 & 0.32 & 0.48\\ 0.42 & 0.31 & 0.35\end{bmatrix}$

Then $SD\left(\tilde{A},\tilde{B}\right)=0.66$ by taking $\alpha = 1.5$.

6. Conclusion

In this article some important properties of circulant triangular fuzzy number matrices (TFNM) are defined. The concept of adjoint of circulant TFNM is discussed and some properties of determinant of circulant TFNM are presented in this article. Finally we define generalized TFNM and also investigate the distance measure of generalized circulant TFNMs. It is true fact that circulant TFNM is very much useful in our daily life. In a subsequent paper we will try to develop the measure of sign-distance of circulant TFNMs and comparative results of our proposed distance, sign-distance and Cheng distance [5].

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