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Galerkin Method for the Numerical Solution of One-Dimensional Differential Equations using Gegenbauer Wavelets

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ABSTRACT

Differential equations are important because for many physical systems, one can, subject to suitable idealizations, formulate a differential equation that describes how the system changes in time. Understanding the solutions of the differential equation is then of paramount interest. Wavelet analysis is a new branch of mathematics widely applied in signal analysis, image processing, numerical analysis, etc. This paper presents the Galerkin method for the numerical solution of one-dimensional differential equations using weight functions are Gegenbauer wavelets (GWGM). The performance of the proposed method is better than that of the existing ones in terms of convergence. Some of the test problems are taken to demonstrate the validity and efficiency of the proposed method.

Keywords: Gegenbauer wavelets; Function approximation; Galerkin method; One-dimensional differential equations.

AMS Mathematics Subject Classification (2010): 65T60, 97N40, 30E25

1. Introduction

Differential equations play a crucial role in mathematics and the sciences as they are capable of representing a broad range of actual-life scenarios. The numerical approach enables the resolution of complex problems through relatively simple operations. A significant advantage of numerical methods, in contrast to analytical methods, is their ease of implementation on modern computers, which allows for quicker solutions compared to those obtained through analytical techniques. Galerkin's method is part of a broader category of numerical techniques [1]. In the literature, these equations are solved by many researchers have attempted to obtain higher accuracy rapidly by using numerous methods. Some of the methods are available in the literature concerning their numerical solution [2 – 4].

The applications of wavelet theory in numerical methods for solving differential equations are nearly 20 years old. In the early 90s, people were very optimistic because it seemed that many good properties of wavelets would automatically lead to competent numerical method for differential equations. The reason for this optimism was the fact that many differential equations have solutions containing local phenomena and interactions between several scales. Such solutions can be well represented in wavelet bases because of their good properties such as compact support (locality in space) and vanishing moment (locality in scale) [5].

The Galerkin's method is well-regarded in applied mathematics for its efficiency and practicality. Utilizing wavelets with the Galerkin method offers major advantages over traditional finite difference and finite element methods, resulting in wide applications across various fields of science and engineering. The wavelet approach is a strong substitute for the finite element method to a certain extent. Additionally, the wavelet technique offers a useful substitute for solving differential equations numerically [6–7].

This study presents the development of the Galerkin method utilizing Gegenbauer wavelets (GWGM) for addressing one-dimensional differential equations numerically.

Galerkin's approach and the characteristics of Gegenbauer wavelets allow us to identify the unknown coefficients, which in turn leads to solve the differential equations numerically.

The following is an outline of the paper's structure: In section 2, An overview of Gegenbauer wavelets and function approximation is given. Section 3 focuses on the Galerkin method utilizing Gegenbauer wavelets. Section 4 includes a numerical illustration. Lastly, Section 5 presents a discussion regarding the conclusions derived from the research conducted.

2. Gegenbauer wavelets and Function approximation

Gegenbauer wavelets:

Gegenbauer wavelets $\psi_{n,m}(t\ )\psi = k\ \ (n$ involve four arguments: $n = 1, 2, ..., 2^{k-1}$, m is the degree of Gegenbauer polynomials and x is the normalized time, k is any positive integer. They are defined on the interval $[0\ ,1)$, Gegenbauer wavelets are defined as [8–9]

$$\psi_{n,m}(t) = \begin{cases} \dfrac{2^{\frac{k}{2}}}{\sqrt{R_m^\lambda}} G_m^\lambda(2^k t - \hat{n}), & \dfrac{\hat{n}-1}{2^k} \le t < \dfrac{\hat{n}+1}{2^k}, \\ 0, & \text{otherwise}, \end{cases} \tag{2.1}$$

where $\lambda > -\frac{1}{2}$, $\hat{n} = 2n - 1, m = 0,1,2,...., M-1$ & R_m^λ is the normalization factor given by

$$R_m^\lambda = \begin{cases} 2^{1-2\lambda}\pi \dfrac{\Gamma(m+2\lambda)}{m!\,(m+\lambda)\left(\Gamma\lambda\right)^2}, & \lambda \ne 0, m \ne 0 \\ \dfrac{2\pi}{m^2}, & \lambda = 0, m \ne 0 \\ \pi, & \lambda = 0, m = 0 \end{cases} \tag{2.2}$$

and the Gegenbauer polynomials G_{m+1}^λ defined as

$$\begin{cases} G_0^{\lambda}(t) = 1, \\ G_1^{\lambda}(t) = 2\lambda t, \\ G_{m+1}^{\lambda}(t) = \dfrac{1}{(m+1)}\left[2t(m+\lambda)G_m^{\lambda}(t) - (m+2\lambda-1)G_{m-1}^{\lambda}(t)\right], \\ m = 0,1,2,\ldots., M-1 \end{cases}$$

The first few Gegenbauer wavelet bases for $k = 1$, $M = 3$ & $\lambda = 2$ are as follows:

$$\psi_{1,0}(t) = \frac{4}{\sqrt{3\pi}} \quad , \quad \psi_{1,1}(t) = 4\sqrt{\frac{2}{\pi}}(-1 + 2t) \quad ,$$

$$\psi_{1,2}(t) = \frac{8}{\sqrt{15\pi}}\left(5 - 24t + 24t^2\right),$$

$$\psi_{1,3}(t) = 4\sqrt{\frac{2}{3\pi}}\left(-5 + 42t - 96t^2 + 64t^3\right) \text{ and so on.}$$

Function approximation:

Suppose $y(t) \in L^2[0\ ,\ 1)$ is expanded in terms of Euler wavelets as:

$$y(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\,\psi_{n,m}(t) \tag{2.5}$$

Truncating the above infinite series, we get

$$y(t) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{n,m}\,\psi_{n,m}(t) \tag{2.6}$$

3. Method of solution

The one-dimensional equation is of the of the form,

$$\frac{\partial^2 y}{\partial t^2} + \alpha\frac{\partial y}{\partial t} + \beta\, y = f(t) \tag{3.1}$$

With boundary conditions $\quad u(0) = a, \quad u(1) = b \qquad (3.2)$

Here $f(t)$ be a continuous function t and α & β are constants.

The residual of the Eq. (3.1) is

$$R(t) = \frac{\partial^2 y}{\partial t^2} + \alpha\frac{\partial y}{\partial t} + \beta\, y - f(t) \tag{3.3}$$

The residual of the equation $R(t)$ is found here. The boundary conditions will be satisfied and if $R(t) = 0$ for the exact solution $y(t)$.

It is possible to expand the trial series solution $y(t)$ to Eq. (3.1), defined over $[0\ ,1)$, as modified Euler wavelets, meeting the specified boundary requirements. This involves the following unknown coefficients

$$y(t) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \psi_{n,m}(t) \tag{3.4}$$

where $c_{n,m}$'s are unidentified coefficients that need to be found.

By selecting Euler wavelet polynomials of higher degree, the accuracy of the solution is improved.

Now, differentiate Eq. (3.4) twice w.r.t. t and in Eq. (3.4) put these values i.e. $y\,,\ \frac{\partial y}{\partial t}\,,\ \frac{\partial^2 y}{\partial t^2}$.

To determine the values of $c_{n,m}$'s, by selecting the weight functions as assumed base elements and integrating the residual to zero together with the boundary values [10].

i.e. $$\int_0^1 \psi_{1,m}(t)\ R(t)\ dt = 0\,,\ m = 0\,,1\,,2\,,........$$

A system of linear algebraic equations is derived from the equation above and solving this system, the unknown coefficients are obtained. The numerical solution of Eq. (3.1) was then produced by substituting these unknowns in the trail solution i.e. Eq. (3.4).

To determine the correctness of WRMEW for one-dimensional equations, utilize the error measure i.e. maximum absolute error and will be computed as

$$E_{\max} = \max \left| \ y(t)_n - y(t)_e \right|,$$

where $y(t)_n$ and $y(t)_e$ are respectively the numerical and exact solutions.

4. Numerical Illustration

Problem 4.1. Consider the differential equation [11],

$$\frac{\partial^2 y}{\partial t^2} + y = -t, \quad 0 \le t \le 1 \tag{4.1}$$

$$\text{BCs: } y(0) = 0\,,\quad y(1) = 0 \tag{4.2}$$

Now, Eq. (4.1) should be implemented according to the method described in section 3:
Using Eq. (4.1), the residual is given as:

$$R(t) = \frac{\partial^2 y}{\partial t^2} + y + t \tag{4.3}$$

Then, the weight function $w(t) = t(1-t)$ should be selected for Euler wavelet bases in order to meet the specified boundary conditions Eq. (4.2),

$$\Psi_{1,0}(t) = \psi_{1,0}(t) \times t(1-t) = \frac{4}{\sqrt{3\pi}} t(1-t)$$

$$\Psi_{1,1}(t) = \psi_{1,1}(t) \times t(1-t) = 4\sqrt{\frac{2}{\pi}}(-1 + 2t)\, t(1-t)$$

$$\Psi_{1,2}(t) = \psi_{1,2}(t) \times t(1-t) = \frac{8}{\sqrt{15\pi}}\left(5 - 24t + 24t^2\right) t(1-t)$$

Considering that Eq. (4.1)'s trail solution for $k = 1$ & $m = 2$ and is provided by

$$y(t) = c_{1,0}\, \Psi_{1,0}(t) + c_{1,1}\, \Psi_{1,1}(t) + c_{1,2}\, \Psi_{1,2}(t) \tag{4.4}$$

Then the Eq. (4.4) becomes

$$y(t) = c_{1,0}\left\{\frac{4}{\sqrt{3\pi}}t(1-t)\right\} + c_{1,1}\left\{4\sqrt{\frac{2}{\pi}}(-1+2t)t(1-t)\right\} + c_{1,2}\left\{\frac{8}{\sqrt{15\pi}}\left(5-24t+24t^2\right)t(1-t)\right\}$$

$$\Rightarrow y(t) = c_{1,0}\frac{4}{\sqrt{3\pi}}\left(t-t^2\right) + c_{1,1}4\sqrt{\frac{2}{\pi}}\left(-t+3t^2-2t^3\right) + c_{1,2}\frac{8}{\sqrt{15\pi}}\left(5t-29t^2+48t^3-24t^4\right) \tag{4.5}$$

Differentiating Eq. (4.5) twice with respect to variable t and substituting the values $y, \frac{\partial^2 y}{\partial t^2}$ into Eq. (4.3), residual of Eq. (4.1) is found. Using the weighted residual approach to go to the subsequent considerations if the weight functions in the trial solution are equal to the basis functions:

$$\int_0^1 \psi_{1,j}(t)\,R(t)\,dt = 0, \quad j = 0, 1, 2 \tag{4.6}$$

In Eq. (4.6), put $j = 0, 1, 2$

i.e.
$$\left.\begin{aligned} \int_0^1 \psi_{1,0}(t)\,R(t)\,dt &= 0 \\ \int_0^1 \psi_{1,1}(t)\,R(t)\,dt &= 0 \\ \int_0^1 \psi_{1,2}(t)\,R(t)\,dt &= 0 \end{aligned}\right\} \tag{4.7}$$

From Eq. (4.7), a system of algebraic equations is formed that includes unknown coefficients such as $c_{1,0}$, $c_{1,1}$ and $c_{1,2}$. By solving this system, then find the values for $c_{1,0} = 0.2134$, $c_{1,1} = 0.0189$ and $c_{1,2} = -0.0008$. Obtained the numerical solution on substituting the values $c_{1,0}$, $c_{1,1}$ and $c_{1,2}$ in Eq. (4.5). Table 1 shows a comparison between the numerical solution and the absolute errors, whereas Figure 1 presents the numerical solution alongside the exact solution of Eq. (4.1) $y(t) = \frac{\sin(t)}{\sin(1)} - t$.

Table 1: Comparison of exact, method [11] and GWGM the absolute errors of problem 4.1

t	Numerical solution			Exact solution	Absolute error		
	Ref [11]	**Ref [12]**	**GWGM**		**Ref [11]**	**Ref [12]**	**GWGM**
0.1	0.0186708	0.0185968	0.018644	0.0186420	2.88e-05	4.50e-05	2.00e-06
0.2	0.0361655	0.0360428	0.036125	0.0360977	6.78e-05	5.50e-05	2.70e-05
0.3	0.0512714	0.0511785	0.051210	0.0511948	7.66e-05	1.60e-05	1.50e-05
0.4	0.0628316	0.0627884	0.062787	0.0627829	4.87e-05	5.50e-06	4.10e-06
0.5	0.0697452	0.0697454	0.069746	0.0697470	1.84e-06	1.60e-06	1.00e-06
0.6	0.0709672	0.0710047	0.070999	0.0710184	5.12e-05	1.40e-05	1.90e-05
0.7	0.0655087	0.0655570	0.065563	0.0655851	7.64e-05	2.80e-05	2.20e-05
0.8	0.0524367	0.0524753	0.052504	0.0525025	6.58e-05	2.70e-05	2.00e-06
0.9	0.0308742	0.0308913	0.030912	0.0309019	2.77e-05	1.10e-05	1.00e-05

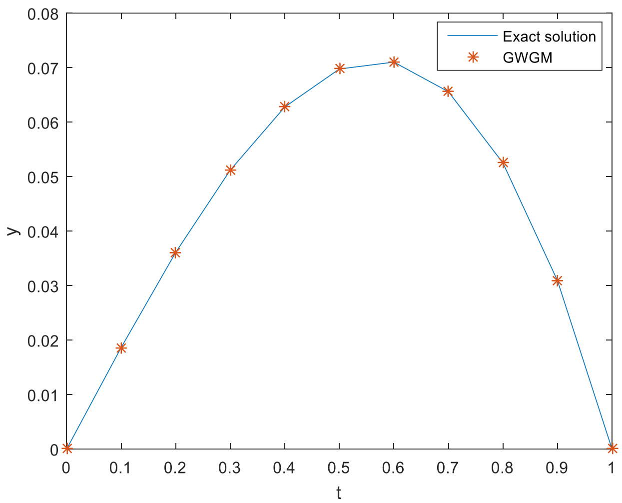

Figure 1: Comparison of GWGM with the exact solution of problem 4.1.

Problem 4.2. Next, a different differential equation [14],

$$\frac{\partial^2 y}{\partial t^2} - \pi^2 y = -2\pi^2 \sin(\pi t), \quad 0 \leq t \leq 1 \qquad (4.8)$$

$$\text{BCs}: \quad y(0) = 0, \quad y(1) = 0 \qquad (4.9)$$

As detailed in section 3 and the preceding test problem, the values of $c_{1,0} = 2.9542$, $c_{1,1} = 0.0$ and $c_{1,2} = -0.1264$ are determined. The numerical solution was then derived by substituting the values of $c_{1,0}$, $c_{1,1}$ and $c_{1,2}$ in Eq. (4.5). Figure 2 compares the numerical solution to the exact solution of Eq. (4.8) $y(t) = \sin(\pi t)$, whereas Table 2 compares the numerical solution to the absolute errors.

Table 2: Comparison of method [12] and GWGM with exact solution and the absolute errors for problem 4.2

t	Numerical solution		Exact solution	Absolute error	
	Ref [13]	**GWGM**		**Ref [13]**	**GWGM**
0.1	0.3079992	0.3087720	0.309016	1.02e-03	2.44e-04
0.2	0.5880739	0.5885235	0.588772	7.00e-04	2.49e-04
0.3	0.8094184	0.8092579	0.809016	4.00e-04	2.40e-04
0.4	0.9515192	0.9506633	0.951056	4.60e-04	3.93e-04
0.5	1.0001543	0.9998625	1.000000	1.50e-04	1.40e-04
0.6	0.9513935	0.9507633	0.951056	3.40e-04	2.93e-04
0.7	0.8092985	0.8091698	0.809016	2.80e-04	1.50e-04
0.8	0.5878225	0.5877638	0.587785	3.80e-05	2.10e-05
0.9	0.3084107	0.3087720	0.309016	6.10e-04	2.44e-04

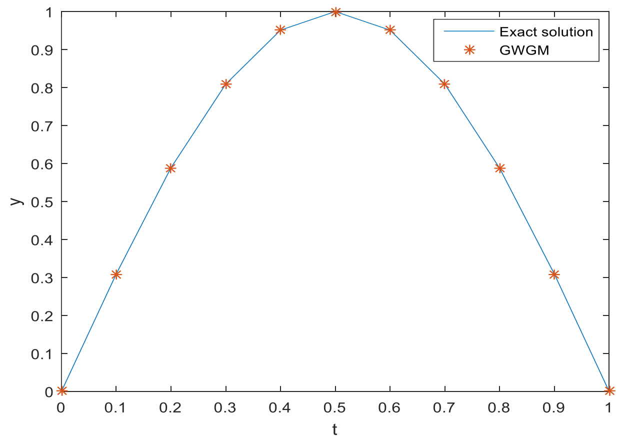

Figure 2: Comparison of GWGM with the exact solution for problem 4.2.

Problem 4.3. Another differential equation [15],

$$\frac{\partial^2 y}{\partial t^2} - 4y = 4\cosh(1), \quad 0 \le t \le 1 \tag{4.10}$$

$$\text{BCs:} \quad y(0) = 0, \quad y(1) = 0 \tag{4.11}$$

Section 3 and the earlier problems are followed in order to determine the values of the unknown coefficients i.e. $c_{1,0} = -1.6897$, $c_{1,1} = 0.0$ and $c_{1,2} = -0.0252$. To find the numerical solution, enter the values of $c_{1,0}$, $c_{1,1}$ and $c_{1,2}$ in Eq. (4.5). Figure 3 shows the numerical solution to the exact solution of Eq. (4.10) $y(t) = \cosh(2t-1) - \cosh(1)$ as well as a comparison of the numerical solution to the absolute errors in Table 3.

Table 3: Comparison of GWGM and absolute error with the exact solution for problem 4.3.

t	Numerical solution		Exact solution	Absolute error	
	Ref [12]	**GWGM**		**Ref [12]**	**GWGM**
0.1	-0.2056232	-0.2056484	-0.2056457	2.20e-05	2.70e-06
0.2	-0.3576501	-0.3576031	-0.3576124	3.80e-05	9.30e-06
0.3	-0.4620069	-0.4620072	-0.4620083	1.40e-06	1.10e-06
0.4	-0.5229269	-0.5230220	-0.5230139	8.70e-05	8.10e-06
0.5	-0.5429500	-0.5430525	-0.5430806	1.30e-04	2.81e-05
0.6	-0.5229233	-0.5230220	-0.5230139	9.10e-05	8.10e-06
0.7	-0.4620007	-0.4620146	-0.4620083	7.60e-06	6.30e-06
0.8	-0.3576430	-0.3576031	-0.3576124	3.10e-05	9.30e-06
0.9	-0.2056179	-0.2056484	-0.2056457	2.80e-05	2.70e-06

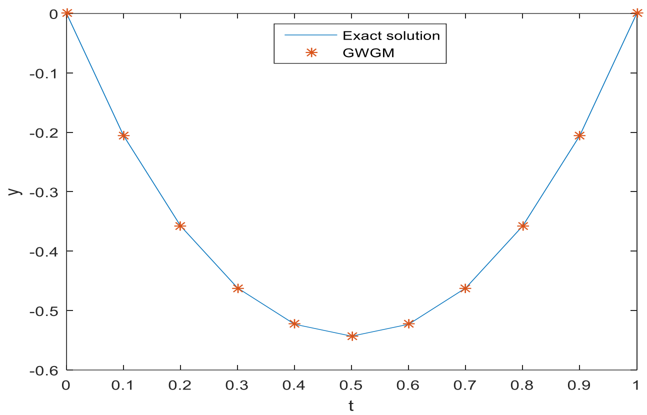

Figure 3: Comparison of GWGM and the exact solution for problem 4.3.

Problem 4.4. Finally, the non-linear differential equation [16],

$$\frac{\partial^2 y}{\partial t^2} - y^2 = 2\pi^2 \cos(2\pi t) - \sin^4(2\pi t), \quad 0 \le t \le 1 \tag{4.12}$$

$$\text{BCs:} \quad y(0) = 0, \quad y(1) = 0 \tag{4.13}$$

The exact solution of Eq. (4.11) $y(t) = \sin^2(\pi t)$ is shown in table 3 and figure 3 together with the numerical solution, which was derived as described in section 3.

Table 4: Comparison of GWGM and absolute error with the exact solution for problem 4.4.

t	Numerical solution		Exact solution	Absolute error	
	Ref [13]	**GWGM**		**Ref [13]**	**GWGM**
0.1	0.096728	0.096566	0.0954920	1.24E-03	1.07E-03
0.2	0.359769	0.351027	0.3454920	1.43E-02	5.53E-03
0.3	0.659432	0.657536	0.6545082	4.92E-03	3.03E-03
0.4	0.909876	0.906723	0.9045082	5.37E-03	2.21E-03
0.5	0.998784	0.999895	1	1.22E-03	1.05E-03
0.6	0.910518	0.910439	0.9045082	6.01E-03	5.93E-03
0.7	0.657385	0.654694	0.6545082	2.88E-03	1.86E-03
0.8	0.348918	0.347936	0.3454920	3.43E-03	2.44E-03
0.9	0.099824	0.097774	0.0954920	4.33E-03	2.28E-03

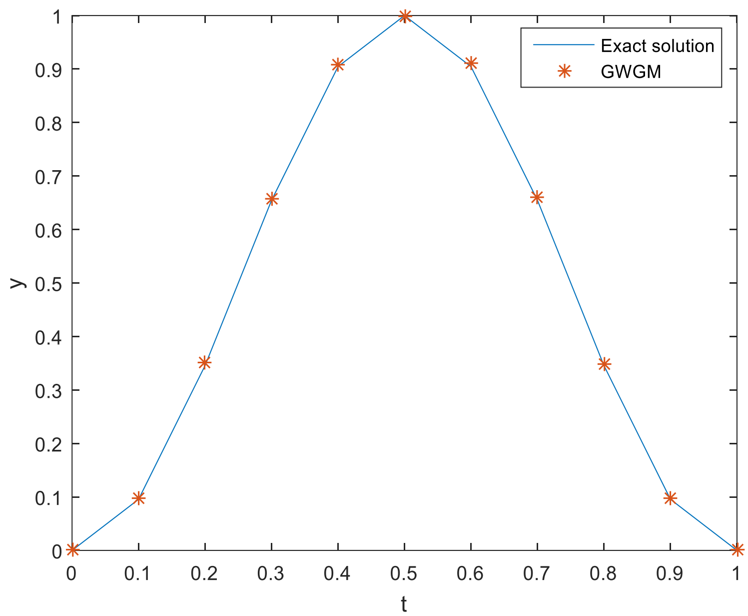

Figure 4: Comparison of GWGM and the exact solution for problem 4.4.

5. Conclusion

The Galerkin method using Gegenbauer wavelets (GWGM) for the numerical solution of one-dimensional differential equations was presented in this study. The results derived from this method, along with the associated data, tables, and figures, indicate that the numerical solutions achieved through this approach do better than those generated by the previously recognized methods (Ref [11], Ref [12] & Ref [13]) and show a closer connection to the exact solution. Moreover, the absolute error associated with this technique is significantly lower in comparison to the existing methods (Ref [11], Ref [12] & Ref [13]). Consequently, the Galerkin method that incorporates Gegenbauer wavelets has been confirmed to be highly efficient in solving one-dimensional differential equations.

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