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A Note on the Diophantine Equation $3^x + 63^y = z^2$

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ABSTRACT

Diophantine equations are gradually drawing attention in the study of hydrogen spectrum, eco- nomics, Biology, quantum Hall effect, chemistry, cryptography etc. Different types of schemes are employed to find solution of Diophantine equations. Some special types of Diophantine equations could be addressed with the help of Catalan's conjecture and Congruence theory. The Diophantine equation $3^x + 63^y = z^2$ is addressed in this paper to find the solution(s) in non-negative integers. It is found that the equation has only two solutions of (x, y, z) as $(1, 0, 2)$ and $(0, 1, 8)$ in non-negative integers.

Keywords: Integer solution, Congruence, Modular arithmetic.

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1. Introduction

Linear or non-linear types of Diophantine equations find intense application in different fields of real life as well as in research, e.g., economics, cryptography, quantum Hall effect, Hygrogen spectrum, computer science, applied algebra etc. [1–8]. It is believed that the Indian mathematician Bramhagupta first described the linear Diophantine equation. Great mathematicians like Fermat, Gauss, Euler also studied Diophantine problems in grate details [5].

The Diophantine equations of the form $p^x + q^y = r^z$ have been studied for several years. The conditional solution $(p, x, y, z) = (2, 0, 3, 3)$ has been found for the Diophantine equation $3^x + p^y = z^2$ where, $p \equiv 2 (mod\, 3)$ [9]. The Diophantine equation $n^x + 10^y = z^2$ has two solutions $(n, x, y, z) = (2, 3, 0, 3)$ and $(x, y, z) = (3, 2, 15)$ for some specific conditions [10]. The solution $(x, y, z) = (1, 0, 12)$ is found for the Diophantine equation $143^x + 45^y = z^2$ [11]. There is no solution for the Diophantine equation $13^{2m} + (6r + 1)^n = z^2$ [12]. Different conditional solutions of the Diophantine equation $n^x + (5p)^y = z^2$ are also found [13]. There exist the solutions of the Diophantine equation $2^{2nx} - p^y = z^2$ for odd as well as even prime value of *p*. The trivial solution of that Diophantine equation also exists for $p \equiv 3 (mod\, 4)$ [14]. Non-trivial non-negative integer solutions are found for the Diophantine equation $p^x - 2^y = z^2$ with $p = k^2 + 4$ is a prime number and $k \geq 1$ [15].

Among this type of Diophantine equations, the form $3^x + q^y = z^2$ has been drawing attention of the researcher for last few years. Two positive integer solutions for (x, y, z) of two Diophantine equations $3^x + 91^y = z^2$ and $3^x + 19^y = z^2$ are established [16]. Four non-negative integer solutions are found for the Diophantine equation $3^x + 13^y = z^2$ [17]. The solution of the Diophantine equation $3^x + 7^y = z^2$ is found as $(1, 0, 2)$ and $(2, 1, 14)$ [18]. Four solutions $(1, 0, 2)$, $(3, 1, 12)$, $(7, 1, 48)$ and $(7, 2, 126)$ describe the set of solutions of the Diophantine equation $3^x + 117^y = z^2$ [19]. The solutions of the Diophantine equation $3^x + 35^y = z^2$ are found to be $(1, 0, 2)$ and (0, 1, 6) [20]. It is found that unique solution $(1, 0, 2)$ exists for the Diophantine equations $3^x + 5^y = z^2$ and $3^x + 17^y = z^2$ [21, 22].The Diophantine equation $3^x + b^y = z^2$ has been studied for the condition $b \equiv 5 (mod\ 30)$ or $b \equiv 5 (mod\ 20)$ [23].

Finding the solution (if any) of one of such equations $3^x + 63^y = z^2$, where x, y and z are non- negative integers, is the aim of this paper. The paper is organized as follows: In Sec. II, Catalan conjecture has been considered. Then two lemmas have been introduced. In Sec. III, main theorem with its proof is presented. Finally, conclusion is given in section IV.

2. Preliminaries

Here, Catalan's conjecture [24] is used to prove the Lemmas 2.2 and 2.3.

Proposition 2.1. $(3, 2, 2, 3)$ is a unique solution of (p, q, x, y) for the Diophantine equation $p^x - q^y = 1$, where p, q, x and y are integers with $\min(p, q, x, y) > 1$ [24,25].

Lemma 1. The Diophantine equation $1 + 63^y = z^2$, where y and z are non-negative integers, has a unique solution of (y, z) as $(1,8)$.

Proof: Let y and z be non-negative integers such that $1 + 63^y = z^2$. For $y = 0$, $63^0 + 1 = 2 = z^2$, that is impossible. It follows that $y \geq 1$. Thus, $z^2 = 63^x + 1 \geq 63^1 + 1 = 64$. Hence, $z \geq 8$. Again, the equation $63^y + 1 = z^2$ can be written as $z^2 - 63^y = 1$. By proposition 2.1, $z = 8$ for $y = 1$. Therefore, a unique solution of the Diophantine equation $1 + 63^y = z^2$ for (y, z) is (1, 8).

Lemma 2. The Diophantine equation $3^x + 1 = z^2$, where x and z are non-negative integers, has a unique solution of (x, z) as $(1,2)$.

Proof: Let x and z be non-negative integers such that $3^x + 1 = z^2$. For $x = 0$, $3^0 + 1 = 2 = z^2$, that is impossible. It shows that $x \geq 1$. Thus, $z^2 = 3^x + 1 \geq 3^1 + 1 = 4$. Hence, $z \geq 2$. Again, the equation $3^x + 1 = z^2$ can be expressed as $z^2 - 3^x = 1$. By proposition 2.1, $z = 2$ for $x = 1$. Therefore, a unique solution of the Diophantine equation $3^x + 1 = z^2$ for (x, z) is $(1,2)$.

3. Result and discussion

It will be shown in the following section that the Diophantine equation $3^x + 63^y = z^2$ has two unique non-negative integer solutions.

Theorem 3.1. The non-linear Diophantine equation $3^x + 63^y = z^2$ has only two non-negative integer solutions of (x, y, z) as $(1, 0, 2)$ and $(0, 1, 8)$, where x, y and z are non-negative integers.

Proof: Three cases are considered here.

Case – I: For $x = 0$, it can be concluded from Lemma 2.2 that the solution of the Diophantine equation $3^x + 63^y = z^2$ for (x, y, z) is $(0, 1, 8)$.

Case – II: For $y = 0$, it can be concluded from Lemma 2.3 that the solution of the Diophantine equation $3^x + 63^y = z^2$ for (x, y, z) is $(1, 0, 2)$.

Case – III: If $x \geq 1$ and $y \geq 1$, consequently 3^x and 63^y both are odd. Therefore, z^2 is even. As a result, z is even. Now it can be shown for the odd values of x that $3^x \equiv 3 (mod\ 4)$, and $3^x \equiv 1 (mod\ 4)$ for the even values of x. In a similar way, $63^y \equiv 3 (mod\ 4)$ for the odd values of y, and $63^y \equiv 1 (mod\ 4)$ for the even values of y. Therefore, $3^x + 63^y \equiv 2 (mod\ 4)$ for even and odd values of x and y; and $z^2 \neq 2$. Thus, it can be concluded that the Diophantine equation has no solution for $x, y \geq 1$.

4. Conclusion

Different methods are applied to various types of Diophantine equation as there is no unique method to find the solution of different types of Diophantine equation. In this paper, Catalan's conjecture and congruence theory are applied to find the solutions of the Diophantine equation $3^x + 63^y = z^2$. Th equation has two solutions for non-negative integers (x, y, z) as $(1, 0, 2)$ and $(0, 1, 8)$. This study may be useful for deriving a correlation among different Diophantine equation of the type $3^x + p^y = z^2$.

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REFERENCES

1. J. Klaska, Real-world applications of Number theory, *South Bohemia Mathematical Letters*, 25 (1) (2017), 39 – 47.
2. T. Andreescu, D. Andrica, and I. Cucurezeanu, *An Introduction to Diophantine Equations: A Problem-Based Approach,* 1st Edition, GIL Publishing House, (2002).
3. L. J. Mordell, *Diophantine Equations*, Academic Press, London, New York (1969).
4. W. Sierpinski, *Elementary Theory of Numbers*, 2nd Edition, North-Holland, Amsterdam (1988).
5. D.M. Burton, *Elementary Number Theory*, 6th edition, McGraw-Hill Education (2007).
6. R. D. Mauldin, A generalization of Fermat's last theorem: the Beal conjecture and prize problem, *Notices of the AMS*, 44 (11) (1997), 1436 – 1439.
7. T. H. Gronwall, A Diophantine Equation Connected with the hydrogen Spectrum, *Phys. Rev*., 36 (1930), 167.
8. M. Kohmoto, B. I. Halperin, and Y. S. Wu, Diophantine equation for the three-dimensional quantum Hall effect, *Phys. Rev. B*, 45(1992), 13488.
9. W. Tangjai and C. Chubthaisong, On the Diophantine equation $3^x + p^y = z^2$ where $p \equiv 2(mod\,3)$, *WSEAS TRANSACTIONS on MATHEMATICS*, 20 (2021), 283-287.
10. S. Tadee, On Diophantine equation $n^x + 10^y = z^2$, *WSEAS TRANSACTIONS on MATHEMATICS*, 22 (2023), 150 -153.
11. S. Aggarwal, C. Swarup, D. Gupta, and S. Kumar, Solution of the Diophantine equation $143^x + 45^y = z^2$, *J. Adv. Res. Appl. Math. Stat*. **7**, 1 (2022).
12. S. Aggarwal, A. T. Shahida, E. Pandey, and A. Vyas, On the problem of solution of non-linear (exponential) Diophantine equation $\beta^x + (\beta + 18)^y = z^2$, *Math. Stat*. 11 (2023), 834.
13. S. Tadee, *On the Diophantine equation* $n^x + (5p)^y = z^2$, Integers, *A*68 (24) (2024).
14. W. Orosram and A. Unchai, *On the Diophantine Equation* $2^{2nx} - p^y = z^2$ *,where p is a prime*, International Journal of Mathematics and Computer Science, 17(1) (2022) 447–451.
15. M. Buosi, A. Lemos, A. L. P. Porto and D. F. G. Santiago, *On the exponential Diophantine equation* $p^x - 2^y = z^2$ *with* $p = k^2 + 4$ *, a prime number*, Palestine Journal of Mathematics, 11(4), (2022) 130–135.
16. J. F. T. Rabago, *On two Diophantine equations* $3^x + 19^y = z^2$ *and* $3^x + 91^y = z^2$, International journal of mathematics and scientific computing, 3(1) (2013) 28.
17. S. Asthana and M. M. Singh, *On the Diophantine equation* $3^x + 13^y = z^2$, International journal of pure and applied mathematics, 114 (2) (2017) 301-304.
18. C. G. Rao, *On the Diophantine equation* $3^x + 7^y = z^2$, EPRA International Journal of Research and Development, 3(6) (2018) 93–95.
19. S. Asthana and M. M. Singh,*On the Dio[phantine equation* $3^x + 117^y = z^2$, Ganita, 70 (2) (2020) 43 .

20. D. Biswas, *Does the Solution to the Non-linear Diophantine Equation* $3^x + 35^y = z^2$ *Exist?* J. Sci. Res., 14 (3) (2022) 861-865.
21. B. Sroysang, *On the Diophantine equation* $3^x + 5^y = z^2$, International journal of pure and applied mathematics, 81(4) (2012) 605-608.
22. B. Sroysang, *On the Diophantine equation* $3^x + 17^y = z^2$, International journal of pure and applied mathematics, 89(1) (2013) 111-114.
23. K. Srimud and S. Tadee , *On the Diophantine equation* $3^x + b^y = z^2$, International Journal of Mathematics and Computer Science, 18(1) (2023) 137–142 .
24. E. Catalan, J. für die Reine und Angew. Math., 27 (1844), 192 .
25. P. Mihailescu, *Primary cyclotomic units and a proof of Catalan's conjecture* J. für die Reine und Angew. Math., 572 (2004) 167-195.