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A Characterization of Neutrosophic Hom-Group

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ABSTRACT

Hom-groups are non-associative kinds of special and essential algebraic forms of some group structures. In actual fact, they represent the generalizations of some kinds of groups. These groups have a kind of characteristic features as well as special properties in which their associativity, as well as the unitality features, seem to be twisted, and this is just by a form of compatible bijective map. In this work, efforts are intensified to introduce neutrosophy into concepts of Hom – group. Furthermore, the basic properties involving the neutrosophic hom-groups and their subgroups with relevant examples are treated. This has been extended to some of the characterizations involving the neutrosophic Hom–groups. The hom-group can sometimes play special roles in the fields of physics, chemistry, and engineering as well as some other aspects of general physical sciences as required.

Keywords: hom-groups, hom-subgroups, neutrosophic hom–subgroups, associativity, algebraic structure, homomorphisms

AMS Mathematics Subject Classification (2010): 20A05, 20E34, 20A99

1. Introduction

Hartwig and associates were known to be responsible for the introduction of the concepts of the hom-Lie algebras in general. They carried out general studies of the deformations of Witt and Virasoro algebras. Later, Jiang et.al.. studied Hom-Lie algebras, the Hom-Lie groups, as well as the presentations of some useful results in that direction. Laurent-Gengouxa et al.. introduced the ideas behind the hom-group. This was actually done by using their work as an essential foundation and fundamentally based. The twisting map given and represented by α was taken by Hassanzadeh to be an invertible map and then, the invertibility of α was later used in the complete study as well as the establishments of several fundamental and useful properties of the indigenous hom-groups along with many interesting forms useful and essential examples. Also, some other researchers have been able to successfully give certain examples of neutrosophic hom-groups, together with some other fundamental properties. The neutrosophic morphisms of neutrosophic hom-groups are also very essential and indispensable in the course of the study of those properties presented. In the long run, it could be discovered that solid relationships exist between the given neutrosophic hom-groups as well as the Cartesian products of certain hom-groups.

Definition 1. (see [3]) Suppose that V is a group. Then V is a Hom-group if and only that V has or is in possession of some set V. This is together with a certain distinguished membership of the special identity denoted by 1 in V. We also have a bijective set mapping. This is also given by $\alpha: V \to V$, and then, the given operation, which is binary and here in this particular context represented by the expression of the mapping given here as follows: $\mu : V \times V \to V$. In practice, these essential forms of structures satisfy some axioms. Such axioms are given as follows:
(1) Majorly, the given product mapping $\alpha: V \to V$ is supposed to satisfy the condition known as the Hom-associativity property. The characteristic feature is such that : $\mu(\alpha(v), \mu(t, l)) = \mu(\mu(v, t), \alpha(l))$.

Explanations:
Since μ is the binary operation of the group, set μ = " $\odot$ " for a general case.We have the first axiom given by the following :
For every v_1, v_2 and $v_3, \in V$,

$\alpha(v_1) \odot (v_2 \odot v_3) = (v_1 \odot v_2) \odot \alpha(v_3)$We have the following examples :

(i.) Let $\odot = +$, the ordinary addition, we have, the first axiom given by: For every $v_1, v_2, v_3, \in V, \alpha(v_1) + (v_2 + v_3) = (v_1 + v_2) + \alpha(v_3)$
(ii.) Let $\odot = x$, the ordinary multiplication, we have, that For every v_1, v_2, and $v_3, \in V$, $\alpha(v_1)(v_2v_3) = (v_1v_2)\alpha(v_3)$. Here, μ indicates the binary operation under which the group V assumes its closure as a set. Hence, for the sake of simplicity, the multiplication sign μ may be omitted where necessary.

This particular mapping α is a multiplicative one and $\alpha(v_1v_2) = \alpha(v_1)\alpha(v_2)$.

Element 1 is called unit and it satisfies the Hom-unitarity conditions$v_1 1 = 1v_1 = \alpha(v_1)$.
(4) For each element v in V, there exists an element $v^{-1} \in V$ such that
$v_1v_1^{-1} = v_1^{-1}v_1 = 1$. Without loss of generality, denote the Hom-group by the pair (V, α).

2. Main results

Theorem 1. Suppose that we have the given group $V(I)$ as a neutrosophic *Hom – Group* such that p is a fixed element of the neutrosophic Hom – Group given by $V(I)$. It would then imply that, the given map of $V(I) \to V(I)$ given by α : $x \to px$ is a bijection
Proof: We first show the homomorphism property : $\forall v \in V(I)$ with the identity element $\{1\}$,
$v.1 = 1.v = \alpha(v)$. Comparing this with the given mapping above , $\alpha(v) = pv = 1.v = g.1$
$\Rightarrow$ our fixed element in this special case $p = 1$ We have that if $a = (a_1 , a_2I)$, then , $\alpha(a).= \alpha(a_1 , a_2I).1 = 1. (a_1 , a_2I)$
$= (a_1 , a_2I)$. $\Rightarrow$ $\alpha(a_1 , a_2I) = (a_1 , a_2I)$, meaning that $\alpha(a_1)+ \alpha(a_2)I = a_1 + a_2I \;\; a_2I)$

$\Rightarrow$ $\alpha(a_1) = a_1$ and $\alpha(a_2) = a_2 \quad a_2I$).
Secondly, on the Hom-associativity property i .e . for a, b, c $\in$ G(I) where $a = a_1 + a_2I$, $b = b_1 + b_2I$ and $c = c_1 + c_2I$, $\alpha(a)(bc)) = (ab)\alpha(c)$.
Hence, LHS = $\alpha(a_1 + a_2I)(b_1 + b_2I)(c_1 + c_2I)) = g(a_1 + a_2I)(b_1 + b_2I)(c_1 + c_2I)$
$= 1.(a_1 + a_2I)(b_1 + b_2I)(c_1 + c_2I)$
$= [a_1b_1c_1 + (a_1b_1c_2 + a_1b_2c_1 + a_1b_2c_2 + a_2b_1c_1 + a_2b_1c_2 + a_2b_2c_1 + a_2b_2c_2)]$
Also,
RHS = $.(a_1 , a_2I)(b_1 , b_2I)(p(c_1 , c_2I)) = .(a_1 , a_2I)(b_1 , b_2I)(1.(c_1 , c_2I))$
$= .(a_1 , a_2I)(b_1 , b_2I)(c_1 , c_2I)$ = LHS
To show that the map is bijective, we need to prove two distinct conditions. The first one is to show that the mapping is a monomorphism. i.e. it is one-to-one, and then, secondly, to show that it is an onto mapping

Monomorphism: Elementarily, if a mapping is one-to-one, it implies that: if given a mapping f, then, we are going to have that $f(v) = f(u)$ means that $u = v$.
Given the map α : G(I) $\rightarrow$ G(I) given by α : $x \rightarrow px$, for $x = (x_1 + x_2I) \in$ G(I).
Let $\alpha(x) = \alpha(y)$, where $x_1 + x_2I = x$ and $y_1 + y_2I = y$. This means that
$\alpha(x_1 , x_2I) = \alpha(y_1 , y_2I) \Rightarrow p(x_1 , x_2I) = p(y_1 , y_2I) \Rightarrow (x_1 , x_2I) = (y_1 , y_2I)$
i.e. $x_1 = y_1$ and $x_2 = y_2 \Rightarrow x = y$

Onto : For a fixed p and every pg , there always exists a $g \in$ Dom(α) such that $\alpha(g)$ = pg. i.e. for $g = (g_1 + g_2I) \in$ G(I), and fixed p , $\alpha(g_1 + g_2I) = p(g_1 + g_2I)$.
Hence, α is a bijection.

Definition 2. Suppose that $W_1(I)$, $W_2(I)$, . . . , $W_n(I)$ is neutrosophic Hom – groups and consider $W(I) = W_1(I) \times W_2(I) \times \ldots \times W_n(I) = \{ a_1 , a_2 , \ldots a_n \mid a_i \in W_i(I)\}$. Define the operation * on the neutosophic Hom - Group given by W(I) as : $(a_1 , a_2 , \ldots a_n)*(b_1 , b_2 , \ldots b_n) = (a_1*b_1 , a_2*b_2 , \ldots a_n*b_n)$. Then, W(I) is called the direct product of the $W_i(I)$'s

Theorem 2. The pair (W(I) , *) is a neutrosophic hom-group.
Proof: We are first going to investigate the axioms of a typical neutrosophic Hom – group. By induction, setting n = 2 , we consider $W(I) = W_1(I) \times W_2(I)$.such that for every x , y $\in$ W(I), we have that $x = (a_1 + b_1I , a_2 + b_2I)$,and $y = (c_1 + d_1I , c_2 + d_2I)$
So, $x \oplus y = [(a_1 + c_1) + (b_1 + d_1)I , (a_2 + c_2) + (b_2 + d_2)I]$. Here, the identity element is given by : $0 = [(0 , 0I) , (0 , 0I)]$. Hence, we have as follows :
$\alpha(x) = 0 \oplus x = x \oplus 0$. And for the twisted associative property, it means that for every x, y, z, $\in$ G(I), where $x = (a_1 + b_1I, a_2 + b_2I)$, $y = (c_1 + d_1I, c_2 + d_2I)$ and $z = (e_1 + f_1I, e_2 + f_2I)$. We have that $\alpha(a_1 + b_1I , a_2 + b_2I)[(c_1 + d_1I , c_2 + d_2I)(e_1 + f_1I , e_2 + f_2I)]$
$= [(a_1 + b_1I , a_2 + b_2I)(c_1 + d_1I , c_2 + d_2I)]\alpha(e_1 + f_1I , e_2 + f_2I)$
$\Rightarrow \alpha(a_1 + b_1I , a_2 + b_2I)[(c_1e_1 + (c_1f_1 + d_1e_1 + d_1f_1)I , (c_2e_2 + (c_2f_2 + d_2e_2 + d_2f_2)I]$
$[(a_1c_1 + (a_1d_1 + b_1c_1 + b_1d_1)I , (a_2c_2 + (a_2d_2 + b_2c_2 + b_2d_2)I]\alpha(e_1 + f_1I , e_2 + f_2I)$.
And for n in general , $N(I) = N_1(I) \times N_2(I) \times \ldots \times N_n(I) = \prod_{i=1}^{n} N_i$, x, y, z, N(I),
$x = (x_{11} + x_{12}I , x_{21} + x_{22}I , \ldots , x_{n1} + x_{n2}I)$, $y = (y_{11} + y_{12}I , y_{21}+ y_{22}I , \ldots , y_{n1} + y_{n2}I)$,
and $z = (z_{11} + z_{12}I , z_{21} + z_{22}I , \ldots , z_{n1} + z_{n2}I)$, we have :
(i.) $\alpha(x) = \alpha(x_{11} + x_{12}I , x_{21} + x_{22}I , \ldots , x_{n1} + x_{n2}I) = 0 \oplus (x_{11} + x_{12}I , x_{21} + x_{22}I, \ldots, x_{n1} + x_{n2}I)$

= $(x_{11} + x_{12}I, x_{21} + x_{22}I, \ldots, x_{n1} + x_{n2}I)\oplus 0$

(ii.) $\alpha(x_{11} + x_{12}I, x_{21} + x_{22}I, \ldots, x_{n1} + x_{n2}I)[(y_{i1} + y_{i2}I)(z_{i1} + z_{i2}I), (y_{(i+1)1} + y_{(i+1)2}I)(z_{(i+1)1} + z_{(i+1)2}I) \ldots, (y_{n1} + y_{n2}I)(z_{n1} + z_{n2}I)]$

= $[(x_{i1} + x_{i2}I)(y_{i1} + y_{i2}I), (x_{(i+1)1} + x_{(i+1)2}I)(y_{(i+1)1} + y_{(i+1)2}I) \ldots, (x_{n1} + x_{n2}I)(y_{n1} + y_{n2}I)]\alpha(z_{11} + z_{12}I, z_{21} + z_{22}I, \ldots, z_{n1} + z_{n2}I)$

or

$$\left(\begin{array}{l} \alpha(x_{11} + x_{12}I, x_{21} + x_{22}I, \ldots, x_{n1} + x_{n2}I)[\sum_{i=1}^{n}(y_{i1} + y_{i2}I)(z_{i1} + z_{i2}I)] \\ = [\sum_{i=1}^{n}(x_{i1} + x_{i2}I)(y_{i1} + y_{i2}I)]\alpha(z_{11} + z_{12}I, z_{21} + z_{22}I, \ldots, z_{n1} + z_{n2}I) \end{array}\right)$$

LHS = $\alpha(x_{11} + x_{12}I, x_{21} + x_{22}I, \ldots, x_{n1} + x_{n2}I)[(y_{i1} + y_{i2}I)(z_{i1} + z_{i2}I), (y_{(i+1)1} + y_{(i+1)2}I)(z_{(i+1)1} + z_{(i+1)2}I) \ldots, (y_{n1} + y_{n2}I)(z_{n1} + z_{n2}I)]$

= $0\oplus(x_{11} + x_{12}I, x_{21} + x_{22}I, \ldots, x_{n1} + x_{n2}I)[(y_{i1} + y_{i2}I)(z_{i1} + z_{i2}I), (y_{(i+1)1} + y_{(i+1)2}I)(z_{(i+1)1} + z_{(i+1)2}I) \ldots, (y_{n1} + y_{n2}I)(z_{n1} + z_{n2}I)]$ ……….(1)

RHS = $(x_{i1} + x_{i2}I)(y_{i1} + y_{i2}I), (x_{(i+1)1} + x_{(i+1)2}I)(y_{(i+1)1} + y_{(i+1)2}I) \ldots, (x_{n1} + x_{n2}I)(y_{n1} + y_{n2}I)]((z_{11} + z_{12}I, z_{21} + z_{22}I, \ldots, z_{n1} + z_{n2}I)\oplus 0)$

= $[(x_{i1} + x_{i2}I)(y_{i1} + y_{i2}I), (x_{(i+1)1} + x_{(i+1)2}I)(y_{(i+1)1} + y_{(i+1)2}I) \ldots, (x_{n1} + x_{n2}I)(y_{n1} + y_{n2}I)]\alpha(z_{11} + z_{12}I, z_{21} + z_{22}I, \ldots, z_{n1} + z_{n2}I)$ …………..(2)

Using (i) for (1) and (2) LHS $\Rightarrow$ RHS

Theorem 3. Suppose that Let ***S*** is a set. Also, let M(I) be a neutrosophic Hom – group, and that $M^S(I)$ represents the set for every map from S to M(I). Define a binary opersation "*" on G^S by $(\beta*\delta)(x) = \beta(x)\delta(x)$. Then, $(M^S, *)$ is a neutrosophic Hom – group.

Proof: The mapping is well defined since $(\beta*\delta)(x) = \beta(x)*\delta(x)$ a homomorphism property.

Next, we have that

(i) For each $\delta \in (M^S, *)$, $e*\delta = \delta*e = \alpha(\delta)$

and $\alpha(\beta)[\delta*\gamma] = [\beta*\delta]\alpha(\gamma)$.

LHS = $\alpha(\beta)[\delta*\gamma] = e*\beta*[\delta*\gamma] = e*\beta*\delta*\gamma$ …………(1)

RHS = $[\beta*\delta]\alpha(\gamma) = [\beta*\delta]*\gamma*e = \beta*\delta*\gamma*e$ …………(2)

Using (i.) for both (1) and (2) proves the result.

Definition 3. Given that V(I) is a neutrosophic hom – group and U(I) a neutrosophic hom – subgroup of the neutrosophic hom – group V(I). i.e. $U(I) \leq V(I)$.

Let $g = (a + bI) \in V(I)$, then, gU(I) is the set of left cosets of U(I) in V(I). This set is denoted by V(I)/U(I). Hence, $V(I)/U(I) = \{ x \mid x = gU(I), g = (a + bI) \in V(I)$, for some a, b in the classical group V} (the right cosets can as well be defined analogously).

Now, define a binary operation "*" on V(I)/U(I). i.e. Let (V(I)/U(I) , *) be a groupoid. Then , $\forall$ x, y $\in$ V(I)/U(I), x = gU(I) and y = hU(I) for
g = (a + bI), h = (c + dI) $\in$ V(I). We have that x*y = gU(I)*hU(I) = gh*U(I)
= (a + bI)(c + dI)*H(I) = (ac + [ad + bc + bd]I)*H(I) $\equiv$ (u + vI)*H(I) = k*H(I), where k = (u + vI) = (ac + [ad + bc + bd]I) $\in$ V(I), and
u = ac , v = (ad + bc + bd) $\in$ V. Hence, the operation "*" is well-defined.
Next, we show that if G(I) is a neutrosophic Hom – group and U(I) $\leq$ V(I) where U(I) is a neutrosophic Hom – subgroup of the neutrosophic hom – group V(I) as defined, then V(I)/U(I) is a neutrosophic Hom – group.

(i.) Let x $\in$ V(I)/U(I). Then, x = gU(I).
We have that for every x $\in$ V(I)/U(I), e* gU(I) = gU(I)*e = α(x) = α(gU(I)) = α((a + bI)U(I))
= α((aU(I)) + α(bU(I)I) = α((aU(I)) + α(bU(I))I

(ii.) Let x = g_1U(I) , y = g_2U(I) , z = g_3U(I) $\in$ V(I)/U(I).
We show that α(x)[yz] = [xy]α(z)

Now, LHS = α(x)[yz] = α(g_1U(I))[g_2U(I)g_3U(I)]= e*g_1U(I)[g_2U(I)g_3U(I)],by (i)
= e*($g_1g_2g_3$U(I)) ………………. ……………….(1)
Also, from the RHS of (ii.) [xy]α(z) = [g_1U(I)g_2U(I)]α(g_3U(I))
= [g_1U(I)g_2U(I)]g_3U(I)*e
= ($g_1g_2g_3$U(I))*e ……………….…………….(2)

Comparing (1) and (2) with (i.) ⇒ LHS = e*($g_1g_2g_3$U(I)) = RHS = ($g_1g_2g_3$U(I))*e
The axioms are satisfied. Hence, V(I)/U(I) is a neutrosophic hom – group. This group is called the neutrosophic Hom-quotient group.
Finally, we now show that every element x = gU(I), g = (a + bI) $\in$ V(I), is neutrosophic i. e. x = gU(I) $\equiv$ m + nI for m, n $\in$ V(I)/U(I). This could simply be done by exposing the contents of gU(I)
Recall that since U(I) $\leq$ V(I) , U(I) = { h | h = g = (p + qI) $\in$ U(I).
Here, p, q $\in$ U.
Hence, gU(I) = gh = (a + bI)(p + qI) = ap + (aq + bp + bq)I $\in$ V(I)/U(I).
Clearly, = ap + (aq + bp + bq)I $\equiv$ m + nI for m, n $\in$ V(I)/U(I).
Therefore, the factor neutrosophic group given by the algebraic structure, as represented and expressed by V(I)/U(I), can be seen and recognised as a neutrosophic hom–quotient group.

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