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Optimising IoT Networks Using $\mathfrak{L}$-Graph Products: A Study on Strong and Modular Products

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ABSTRACT

The graph product $G \odot H$ defines a combination of two graphs into a new graph, with a set of vertices and edges derived from the original graphs, providing a structured representation of relationships. This paper discusses graph products using lattice graphs ($\mathfrak{L}$-graphs), with a particular focus on strong and modular products. We studied $\mathfrak{L}$-graph products, focusing on their main features and how they can be compared with each other. We used theoretical analysis and mathematical proofs to show their structures and real-world importance. Our findings indicate that strong products support efficient communication across network levels, whereas modular products optimize shared pathways, reducing congestion and energy consumption in IoT networks. These findings improve our understanding of complicated interactions in IoT systems, leading to improved network design and performance.

Keywords: Residuated Lattices, $\mathfrak{L}$-Graph, Isomorphism, strong product, modular product

AMS Mathematics Subject Classification (2010): 05CCX, 05C76, 05C90

1. Introduction

Leonhard Euler invented graph theory in 1736 in order to address the Königsberg Bridge problem. Since the development of graph theory, many scholars have used this concept to solve real problems in a variety of fields. An L-graph, commonly known as a line graph, is a specific graph in graph theory. The line graph is formed by connecting each edge of the original graph to a vertex. Zadeh introduced the concept of a fuzzy subset and used the set type to reflect imprecise natural occurrences. Several researchers (including Kuzmin and Bezdek) have since examined numerous applications for this notion.

Later, Bhattacharya added some remarks about fuzzy graphs. Mordeson and Peng [1] introduced some operations on fuzzy graphs. Meenakshi and Shivangi [2,3] used cubic fuzzy graphs to explain the modular product and presented their work on graph products utilizing correlation and regression coefficients. Radha and Arumugam [4] studied the strong product of two fuzzy graphs, analyzing structural properties and modeling possibilities. Ramachandran and Thomas [5] explained isomorphisms on $\mathfrak{L}$-fuzzy graphs and defined their counterpart in the framework of equivalence relations. Raisi Sarbizhan and Zahedi [6,7] studied the kronecker product of RL graph and the maximal product L-graph automata, their applications and automata theory. Vijaya [8] examined the regularity of the modular product of two fuzzy graphs. Talebi et al. [9] studied the concept of isomorphism in vague graphs, establishing necessary and sufficient conditions for two vague graphs. Rashmanlou et al. [10,11] explored an extensive study on vague graphs, analysing their properties and the product operations on interval-valued fuzzy graphs, focusing on degree-based properties. Ali et al. [12] investigated vertex connectivity in fuzzy graphs and applied their findings to human trafficking networks. Meenakshi et al. [13] utilized a neutrosophic approach for network optimization and presented decision-making scenarios under uncertainty. Meenakshi and Dhanushiya [14,15] studied the graph operation concept using intuitionistic fuzzy graph and its vertex order colouring.

This study seeks to establish a graph based on a residuated lattice known as the $\mathfrak{L}$-graph. An $\mathfrak{L}$ graph typically refers to a graph associated with a lattice, where the graph vertices are elements, and the edges represent relationships between these elements. The current work aims to explore the innovative graph products of two $\mathfrak{L}$-graphs, illustrating one of its applications. These graph products combine the vertices and edges of the original graphs in various ways, preserving their structural and modular properties. The graph product isomorphism has been investigated using the residuated lattice framework, which takes into account an appropriate natural relationship between lattice operations. We investigated the isomorphism of strong and modular product $\mathfrak{L}$ - graphs for IoT applications. The applications of IoT in the $\mathfrak{L}$-graph products are evidence of the significant influence technology will have on how industries develop in the future. As technology advances, we can expect even more innovative applications in these fields. The concept of a $\mathfrak{L}$-graph establishes links between edges and their connectivity patterns, which aids in creating the final graph products. Some examples and theorems are also presented to study the graph products of two $\mathfrak{L}$-graphs.

The manuscript is structured as follows: In section 2 we have discussed the fundamental concepts of the $\mathfrak{L}$-graph. Section 3 presents the strong product of $\mathfrak{L}$-graph related to some theorems and illustrations. In section 4, we have studied the modular product of $\mathfrak{L}$-graph. Section 5 presents the applications of IoT-based industry and irrigation systems. In Section 6, we wrap up the manuscript by identifying possible future research topics, outlining the consequences of the main findings, and summarizing them.

2. Preliminaries

This section discusses the fundamental concepts of the $\mathfrak{L}$-graph.

Definition 2.1. [7] Let $\mathfrak{L} = (\mathfrak{L}, \vee, \wedge, \odot, \rightarrow, 0, 1)$ be a residuated lattice if it satisfies the following conditions:

i) $\mathfrak{L} = (\mathfrak{L}, \vee, \wedge, 0, 1)$ is a bounded lattice with lowest element 0 and largest element 1 .

ii) $\mathfrak{L} = (\mathfrak{L}, \odot, 0, 1)$ is a commutative monoid (i.e., $\odot$ is commutative, associative, and $\mathfrak{c} \odot 1 = \mathfrak{c}$ holds),

iii) $\mathfrak{c} \odot \mathfrak{d} \leq \mathfrak{e}$ if and only if $\mathfrak{c} \leq \mathfrak{d} \rightarrow \mathfrak{e}$ holds (adjointness condition).

Proposition 2.2. [7] Let $\mathfrak{L} = (\mathfrak{L}, \vee, \wedge, \odot, \to, 0, 1)$ be a residuated lattice then it holds following conditions:
$(R_1) 1 \star \mathfrak{c} = \mathfrak{c}$ where $\star \epsilon \{\wedge, \odot, \to\}$,
$(R_2) \mathfrak{c} \odot 0 = 0, 1' = 0, 0' = 1$,
$(R_3) \mathfrak{c} \odot \mathfrak{d} \leq 1, \mathfrak{c} \wedge y \leq \mathfrak{c}, \mathfrak{d}$, and $\mathfrak{d} \leq (\mathfrak{c} \to \mathfrak{d})$,
$(R_4) \mathfrak{c} \leq \mathfrak{d}$ implies $\mathfrak{c} \star \mathfrak{e} \leq \mathfrak{d} \star \mathfrak{e}$, where $\star \in \{\wedge, \vee, \odot\}$
$(R_5) \mathfrak{e} \odot (\mathfrak{c} \wedge \mathfrak{d}) \leq (\mathfrak{e} \odot \mathfrak{c}) \wedge (\mathfrak{e} \odot \mathfrak{d})$,
$(R_6) \mathfrak{c} \odot (\mathfrak{d} \vee \mathfrak{e}) = (\mathfrak{c} \odot \mathfrak{d}) \vee (\mathfrak{c} \odot \mathfrak{e})$,
$(R_7) (\mathfrak{c} \vee \mathfrak{d}) \to \mathfrak{e} = (\mathfrak{c} \to \mathfrak{d}) \wedge (\mathfrak{c} \to \mathfrak{e})$.

Definition 2.3. [7] If $\mathfrak{G} = (\vartheta^\star, \tau^\star)$ is called an $\mathfrak{L}$ - graph on $\mathfrak{G}^\star = (\mathbb{V}, \mathbb{E})$ if $\vartheta^\star: \mathbb{V} \to \mathfrak{L}$ and $\tau^\star: \mathbb{E} \to \mathfrak{L}$ are function with $\tau^\star(k\mathfrak{l}) \leq \vartheta^\star(k) \odot \vartheta^\star(\mathfrak{l})$ for every $(k\mathfrak{l}) \epsilon \mathbb{E}$.

Definition 2.4. [7] Let $\mathfrak{G}_1 = (\vartheta_1^\star, \tau_1^\star)$ and $\mathfrak{G}_2 = (\vartheta_2^\star, \tau_2^\star)$ be two $\mathfrak{L}$-graphs on $\mathfrak{G}_1^\star = (\mathbb{V}_1, \mathbb{E}_1)$ and $\mathfrak{G}_2^\star = (\mathbb{V}_2, \mathbb{E}_2)$ respectively, and $c \epsilon \mathfrak{L} \setminus \{1\}$
Then $\mathfrak{G}_1$ and $\mathfrak{G}_2$ are isomorphic with threshold c, which is denoted by $\mathfrak{G}_1 \odot \mathfrak{G}_2$ only if there exist a bijection $\mathfrak{b}: \mathbb{V}_1 \to \mathbb{V}_2$ such that the following conditions hold for all $u, v \in \mathbb{V}_1$,
(i) $uv \epsilon \mathbb{E}_1$ if and only if $\mathfrak{b}(u)\mathfrak{b}(v) \epsilon \mathbb{E}_2$,
(ii) $\vartheta_1^\star(u) > c$ if and only if $\vartheta_2^\star(\mathfrak{b}(u)) > c$,
(iii) $\tau_1^\star(uv) > c$ if and only if $\tau_2^\star(\mathfrak{b}(u)\mathfrak{b}(v))$
(iv) A function $\mathfrak{b}$ is considered isomorphic if and only if $\mathfrak{b}$ is isomorphic with threshold c for every $c \in \mathfrak{L} \setminus \{1\}$.

3. The strong product of two $\mathfrak{L}$-graphs

Throughout this study, let $\mathfrak{L}$ denote a residuated lattice, and let S represent an $\mathfrak{L}$-graph, denoted as $S = (\vartheta_1^\star, \tau_1^\star)$, on $S^\star = (\mathbb{V}_1, \mathbb{E}_1)$, where $\vartheta_1^\star$ and $\tau_1^\star$ are operations defined on $S^\star$. Similarly, let P denote another $\mathfrak{L}$-graph, expressed as $P = (\vartheta_2^\star, \tau_2^\star)$, on $P^\star = (\mathbb{V}_2, \mathbb{E}_2)$.

Definition 3.1. Let $\mathfrak{G}_1$ and $\mathfrak{G}_2$ denote two $\mathfrak{L}$ - graphs. Then the strong products of two $\mathfrak{L}$-graphs $\mathfrak{G}_1$ and $\mathfrak{G}_2$ is defined by $\mathfrak{G}_1 \odot \mathfrak{G}_2 = \big((\vartheta^\star, \tau^\star)\big)$ and $(\mathfrak{G}_1 \odot \mathfrak{G}_2)^\star = (\mathbb{V}, \mathbb{E})$, where
(i) $\mathbb{V} = \mathbb{V}_1 \odot \mathbb{V}_2$.
(ii) $\mathbb{E} = \mathbb{E}_1 \odot \mathbb{E}_2$.
(iii) $\vartheta^\star(s_i, s_j) = \vartheta_1^\star(s_i) \wedge \vartheta_2^\star(s_j)$ for all $(s_i, s_j) \epsilon \mathbb{V}$.
(iv) $\mathbb{E} = (s_i, s_j)(s'_m, s'_n) \mid s_i = s'_m, s_j s'_n \in \mathbb{E}_2$ or $s_j = s'_n, s_i s'_m \epsilon \mathbb{E}_1$ or $s_i s'_m \epsilon \mathbb{E}_1$ and $s_j s'_n \epsilon \mathbb{E}_2$.

(v) $$\tau^\star(s_i, s_j, s'_m, s'_n) = \begin{cases} \vartheta_1^\star(s_i) \odot \tau_2^\star(s_j, s'_n), & \text{if } (s_i) = (s'_m) \text{ and } (s_j, s'_n) \in \mathbb{E}_2, \\ \vartheta_1^\star(s_j) \odot \tau_1^\star(s_i, s'_m), & \text{if } (s_j) = (s'_n) \text{ and } (s_i, s'_m) \in \mathbb{E}_1 \\ \tau_1^\star(s_i, s'_m) \odot \tau_2^\star(s_j, s'_n), & \text{if } (s_i, s'_m) \in \mathbb{E}_1 \text{ and } S(s_j, s'_n) \in \mathbb{E}_2 \end{cases}$$

Theorem 3.2. Let S and P be two $\mathfrak{L}$ - graphs. Then
(i) $S \odot P$ is an $\mathfrak{L}$ - graphs,
(ii) If (s, s') is a vertex of $S \odot P$, then $d_{S\odot P}(s, s') = d_{S(s)} + d_{P(s')}$
(iii) $|\mathbb{E}| = |\mathbb{E}_1| \times |\mathbb{V}_2| + |\mathbb{V}_1| \times |\mathbb{E}_2|$
Proof:
(i) According to the definition of $\mathfrak{L}$-graph, this is proved by showing that
$\tau^\star\left((s_i, s_j)(s'_m, s'_n)\right) \leq \vartheta^\star(s_i, s_j) \odot \vartheta^\star(s'_m, s'_n)$.
We know that $\tau^\star\left((s_i, s_j)(s'_m, s'_n)\right)$ has three cases.
Case 1. If $s_i = s'_m, (s_j, s'_n) \in \mathbb{E}_2$ then

$$\tau^\star\left((s_i, s_j)(s'_m, s'_n)\right) = \vartheta_1^\star(s_i) \odot \tau_2^\star(s_j, s'_n) \text{ by definition of } \tau^\star$$
$$= \left(\vartheta_1^\star(s_i) \wedge \vartheta_1^\star(s'_m)\right) \odot \tau_2^\star(s_j s'_n)$$
$$\leq \left(\vartheta_1^\star(s_i) \wedge \vartheta_1^\star(s'_m)\right) \odot \left(\vartheta_2^\star(s_j) \wedge \left(\vartheta_2^\star(s'_n)\right) \text{ by definition of } \mathfrak{L} - \text{ graph },\right.$$
$$\leq \left(\vartheta_1^\star(s_i) \wedge \left(\vartheta_1^\star(s_j) \odot \left(\vartheta_2^\star(s'_m) \wedge \left(\vartheta_2^\star(s'_n)\right.\right.\right.\right. \text{ by } (R_3) \text{ proposition (2.2)}$$
$$= \left(\vartheta_1^\star(s_i) \wedge \left(\vartheta_1^\star(s_j) \odot \left(\vartheta_2^\star(s'_m) \wedge \left(\vartheta_2^\star(s'_n)\right.\right.\right.\right. \text{ by definition of } \vartheta^\star$$
$$\tau^\star\left((s_i, s_j)(s'_m, s'_n)\right) = \vartheta^\star(s_i, s_j) \odot \vartheta^\star(s'_m, s'_n)$$

case 2. If $s_j = s'_n, (s_i, s'_m) \in \mathbb{E}_2$

$$\tau^\star\left((s_i, s_j)(s'_m, s'_n)\right) = \vartheta_1^\star(s_j) \odot \tau_2^\star(s_i, s'_m) \text{ by definition of } \tau^\star$$

case 3. If $s_i s'_m \epsilon \mathbb{E}_1, s_j s'_n \epsilon \mathbb{E}_2$

$$\tau^\star\left((s_i, s_j)(s'_m, s'_n)\right) = \tau_1^\star((s_i, s_j) \odot \tau_2^\star(s'_m, s'_n))$$
$$\tau^\star\left((s_i, s_j)(s'_m, s'_n)\right) = \vartheta^\star(s_i, s_j) \odot \vartheta^\star(s'_m, s'_n)$$

The above results are obtained using the following cases, therefore, we can say that $S \odot P$ is an $\mathfrak{L}$-graphs.
(ii) Each vertex (s, s') in the $\mathfrak{L}$-graphs $S \odot P$ is connected to the vertex (s_i, s'_j), where $s_i = s$ and s' is adjacent to s'_j, or $s' = s'_j$ and s_i is adjacent to s, or both vertices (s, s') and (s_i, s'_j) are adjacent in the $\mathfrak{L}$-graph. In this context, s represents the vertices adjacent to s_i in the $\mathfrak{L}$-graph S, and s' represents the vertices adjacent to s'_j in $\mathfrak{L}$-graph P. Therefore, the distance function $d_{SP}(s, s') = d_S(s) + d_P(s')$.
(iii) consider $\mathcal{M} = \{(s, s'_m)(s, s'_n) \mid s \epsilon \mathbb{V}_1, (s'_m s'_n) \epsilon \mathbb{E}_2\}$, and $\mathcal{R} = \{(s_i, s')(s_j, s') \mid s' \epsilon \mathbb{V}_2, (s_i, s_j) \epsilon \mathbb{E}_1\}$. As $= \mathcal{M} \cup \mathcal{R}, |\mathbb{E}| = |\mathcal{M}| + |\mathcal{R}| = |\mathbb{E}_1| \times |\mathbb{V}_2| + |\mathbb{V}_1| \times |\mathbb{E}_2|$.

Theorem 3.3. Let $\mathfrak{G}$ and $\mathfrak{G}'$ be isomorphic $\mathfrak{L}$-graphs, and let $\mathcal{H}$ and $\mathcal{H}'$ be isomorphic $\mathfrak{L}$-graphs. Then, the strong product of $\mathfrak{G} \odot \mathcal{H}$ is isomorphic to $\mathfrak{G}' \odot \mathcal{H}'$.
Proof: Given isomorphic $\mathfrak{L}$-graphs $\mathfrak{G}$ and $\mathfrak{G}'$, there exists an isomorphism $\mathfrak{h}: \mathbb{V}(\mathfrak{G}) \to \mathbb{V}(\mathfrak{G}')$, such that for any two vertices (u, v) in $\mathbb{V}(\mathfrak{G})$, the edge (u, v) is in $\mathbb{E}(\mathfrak{G})$ if and only if the edge $\mathfrak{h}(u), \mathfrak{h}(v)$ is in $\mathbb{E}(\mathfrak{G}')$. Similarly, for isomorphic $\mathfrak{L}$-graphs $\mathcal{H}$ and $\mathcal{H}'$, there exists an isomorphism $\mathfrak{h}: \mathbb{V}(\mathcal{H}) \to \mathbb{V}(\mathcal{H}')$, such that for any two vertices (x, y) in $\mathbb{V}(\mathcal{H})$, the edge (x, y) is in $\mathbb{E}(\mathcal{H})$ if and only if the edge $(\mathfrak{h}(\mathfrak{c}), \mathfrak{h}(\mathfrak{d}))$ is in $\mathbb{E}(\mathcal{H}')$. Now, consider the strong product of $\mathfrak{G} \odot \mathcal{H}$ and $\mathfrak{G}' \odot \mathcal{H}'$. The vertex set of $\mathfrak{G} \odot \mathcal{H}$ is $\mathbb{V}(\mathfrak{G} \odot \mathcal{H}) = \mathbb{V}(\mathfrak{G}) \odot \mathbb{V}(\mathcal{H})$, and the vertex set of $\mathfrak{G}' \odot \mathcal{H}'$ is $\mathbb{V}(\mathfrak{G}' \odot \mathcal{H}') =$

$\mathbb{V}(\mathfrak{G}') \odot \mathbb{V}(\mathcal{H}')$. Define the function $\mathfrak{h}: \mathbb{V}(\mathfrak{G} \odot \mathcal{H}) \to \mathbb{V}(\mathfrak{G}' \odot \mathcal{H}')$ as follows:
$\mathfrak{h}(u, x) = (\mathfrak{h}(u), \mathfrak{h}(\mathfrak{c}))$ This function $\mathfrak{h}$ is an isomorphism because it preserves adjacency:
For any two vertices $((u_1, \mathfrak{c}_1))$ and $((u_2, \mathfrak{c}_2))$ in $\mathbb{V}(\mathfrak{G} \odot \mathcal{H})$, $((u_1, \mathfrak{c}_1))$ is adjacent to $((u_2, \mathfrak{c}_2))$ if and only if either $(u_1) = (u_2)$ or $(\mathfrak{c}_1) = (\mathfrak{c}_2)$.
Similarly, $\mathfrak{h}(u_1, \mathfrak{c}_1)$ is adjacent to $\mathfrak{h}(u_2, \mathfrak{c}_2)$ if and only if $\mathfrak{h}(u_1) = \mathfrak{h}(u_2)$ or $\mathfrak{h}(\mathfrak{c}_1) = \mathfrak{h}(\mathfrak{c}_2)$, which preserves adjacency. It is a bijection, meaning $\mathfrak{h}$ is a one-to-one and onto function. Therefore, the strong product $(\mathfrak{G} \odot \mathcal{H})$ is isomorphic to $(\mathfrak{G}' \odot \mathcal{H}')$ by the isomorphism $\mathfrak{h}$.

Example 3.4. Suppose $\mathfrak{L} = (\mathfrak{L}, \vee, \wedge, \odot, \to, 0, 1)$ and two $\mathfrak{L}$ - graphs S and P, where

$$\mathfrak{g} \odot \mathfrak{b} = \begin{cases} \mathfrak{g} + \mathfrak{b} - 1, & \text{if } \mathfrak{g} + \mathfrak{b} \geq 1 \\ 0, & \text{if } \mathfrak{g} + \mathfrak{b} < 1 \end{cases}$$

$$\mathfrak{g} \to \mathfrak{b} = \begin{cases} 1, & \text{if } \mathfrak{b} - \mathfrak{g} \geq 0 \\ 1 - \mathfrak{g} + \mathfrak{b}, & \text{if } \mathfrak{b} - \mathfrak{g} < 0 \end{cases}$$

$\mathbb{V}_1 = \{s_1, s_2, s_3, s_4\}, \mathbb{E}_1 = \{s_1s_2, s_1s_3, s_2s_4, s_3s_4\}, \tau_1(s_i, s_j) = \vartheta_1^\star(s_i) \odot \vartheta_1^\star(s_j)$, for every $(s_is_j) \epsilon \mathbb{E}_1, \vartheta_1^\star(s_1) = 0.8, \vartheta_1^\star(s_2) = 0.6, \vartheta_1^\star(s_3) = 0.7, \vartheta_1^\star(s_4) = 0.5$,
$\tau_1^\star(s_1s_2) = 0.4, \tau_1^\star(s_1s_3) = 0.5, \tau_1^\star(s_2s_4) = 0.1, \tau_1^\star(s_3s_4) = 0.2$,
$\mathbb{V}_2 = \{s_1', s_2', s_3', s_4'\}, \mathbb{E}_2 = \{s_1's_2', s_2's_4', s_1's_3', s_3's_4'\}$,
$\tau_2^\star(s_m's_n') = (\vartheta_2^\star(s_m') \wedge \vartheta_2^\star(s_n')) \odot (\vartheta_2^\star(s_m') \wedge \vartheta_2^\star(s_n'))$,for every $(s_m's_n') \epsilon \mathbb{E}_2, \vartheta_2^\star(s_1') = 0.5, \vartheta_2^\star(s_2') = 0.9, \vartheta_2^\star(s_3') = 0.7, \vartheta_2^\star(s_4') = 0.6, \tau_2^\star(s_1's_2') = 0.4, \tau_2^\star(s_1's_3') = 0.2, \tau_2^\star(s_3's_4') = 0.3, \tau_2^\star(s_2's_4') = 0.5$. Then $S \odot P$ is the strong product of S and P (figure 2) where, $\mathbb{V} = \{(s_i, s_j') \mid 1 \leq i, j \leq 4\}$, $\mathbb{E} =$
$\{\{(s_i, s_1')(s_i, s_2'), (s_is_1')(s_is_3'), (s_is_2')(s_is_4'), (s_is_4')(s_is_3'),$
$(s_1, s_j')(s_3, s_j'), (s_1, s_j')(s_2, s_j'), (s_2, s_j')(s_4, s_j'), (s_4, s_j')(s_3, s_j') \mid 1 \leq i, j \leq 4\},$
$\{(s_2, s_2')(s_4, s_1'), (s_2, s_4')(s_4, s_3'), (s_2, s_3')(s_4s_4'), (s_4, s_1')(s_3s_2'), (s_4, s_2')(s_3s_4'),$
$(s_4, s_4')(s_3s_3'), (s_1, s_1')(s_2s_2'), (s_1, s_1')(s_3, s_3'), (s_1, s_2')(s_2, s_1'),$
$(s_1, s_2')(s_2, s_4'), (s_1, s_4')(s_2, s_2'), (s_1, s_4')(s_3, s_3'), (s_1, s_4')(s_2, s_3'),$
$(s_1, s_3')(s_3, s_1'), (s_1, s_3')(s_2, s_4'), (s_2, s_1')(s_4, s_2'), (s_2, s_2')(s_4, s_4')\}\}$
$\vartheta^\star(s_1, s_1') = 0.5, \vartheta^\star(s_1, s_2') = 0.8, \vartheta^\star(s_1, s_4') = 0.6, \vartheta^\star(s_1, s_3') = 0.7,$
$\vartheta^\star(s_2, s_1') = 0.5, \vartheta^\star(s_2, s_2') = 0.6, \vartheta^\star(s_2, s_4') = 0.6, \vartheta^\star(s_2, s_3') = 0.6,$
$\vartheta^\star(s_4, s_1') = 0.5, \vartheta^\star(s_4, s_2') = 0.5, \vartheta^\star(s_4, s_4') = 0.5, \vartheta^\star(s_4, s_3') = 0.5,$
$\vartheta^\star(s_3, s_1') = 0.5, \vartheta^\star(s_3, s_2') = 0.7, \vartheta^\star(s_3, s_4') = 0.6, \vartheta^\star(s_3, s_3') = 0.7.$

Table 1: Membership values $\tau^\star$

$\tau^\star((s_1s_1')(s_1s_2')) = 0.2$	$\tau^\star((s_1s_3')(s_2s_3')) = 0.1$	$\tau^\star((s_1s_1')(s_1s_3')) = 0$
$\tau^\star((s_1s_2')(s_1s_4')) = 0.3$	$\tau^\star((s_2s_1')(s_4s_1')) = 0.3$	$\tau^\star((s_1s_4')(s_1s_3')) = 0.1$
$\tau^\star((s_1s_1')(s_3s_1')) = 0$	$\tau^\star((s_2s_1')(s_2s_3')) = 0$	$\tau^\star((s_1s_1')(s_2s_1')) = 0$

$\tau^{\star}\big((s_1s_1')(s_2s_2')\big) = 0$	$\tau^{\star}\big((s_2s_2')(s_4s_4')\big) = 0$	$\tau^{\star}\big((s_1s_1')(s_3s_3')\big) = 0$
$\tau^{\star}\big((s_1s_2')(s_2s_1')\big) = 0$	$\tau^{\star}\big((s_2s_2')(s_4s_1')\big) = 0$	$\tau^{\star}\big((s_1s_2')(s_2s_2')\big) = 0.3$
$\tau^{\star}\big((s_1s_2')(s_3s_2')\big) = 0.4$	$\tau^{\star}\big((s_2s_4')(s_4s_4')\big) = 0$	$\tau^{\star}\big((s_1s_2')(s_2s_4')\big) = 0$
$\tau^{\star}\big((s_1s_4')(s_2s_2')\big) = 0$	$\tau^{\star}\big((s_2s_3')(s_4s_4')\big) = 0$	$\tau^{\star}\big((s_1s_4')(s_3s_4')\big) = 0.1$
$\tau^{\star}\big((s_1s_4')(s_3s_3')\big) = 0$	$\tau^{\star}\big((s_4s_1')(s_3s_1')\big) = 0$	$\tau^{\star}\big((s_1s_4')(s_2s_4')\big) = 0$
$\tau^{\star}\big((s_1s_4')(s_2s_3')\big) = 0$	$\tau^{\star}\big((s_4s_1')(s_4s_3')\big) = 0$	$\tau^{\star}\big((s_1s_3')(s_3s_1')\big) = 0$
$\tau^{\star}\big((s_1s_3')(s_2s_4')\big) = 0$	$\tau^{\star}\big((s_4's_2')(s_3s_2')\big) = 0$	$\tau^{\star}\big((s_1s_3')(s_3s_3')\big) = 0.2$
$\tau^{\star}\big((s_2s_1')(s_2s_2')\big) = 0$	$\tau^{\star}\big((s_4s_2')(s_3s_4')\big) = 0$	$\tau^{\star}\big((s_2s_4')(s_4s_3')\big) = 0$
$\tau^{\star}\big((s_2s_1')(s_4s_2')\big) = 0$	$\tau^{\star}\big((s_4s_4')(s_3s_4')\big) = 0$	$\tau^{\star}\big((s_2s_3')(s_4s_3')\big) = 0$
$\tau^{\star}\big((s_2s_2')(s_4s_2')\big) = 0$	$\tau^{\star}\big((s_4s_4')(s_4s_3')\big) = 0$	$\tau^{\star}\big((s_4s_1')(s_4s_2')\big) = 0$
$\tau^{\star}\big((s_2s_2')(s_2s_4')\big) = 0.1$	$\tau^{\star}\big((s_3s_1')(s_3s_3')\big) = 0$	$\tau^{\star}\big((s_4s_1')(s_3s_2')\big) = 0$
$\tau^{\star}\big((s_2s_4')(s_2s_3')\big) = 0$	$\tau^{\star}\big((s_3s_4')(s_3s_3')\big) = 0$	$\tau^{\star}\big((s_4's_2')(s_3s_1')\big) = 0$
$\tau^{\star}\big((s_4s_2')(s_4s_4')\big) = 0$	$\tau^{\star}\big((s_3s_1')(s_3s_2')\big) = 0.1$	
$\tau^{\star}\big((s_4s_4')(s_3s_3')\big) = 0$	$\tau^{\star}\big((s_3s_2')(s_3s_4')\big) = 0.2$	

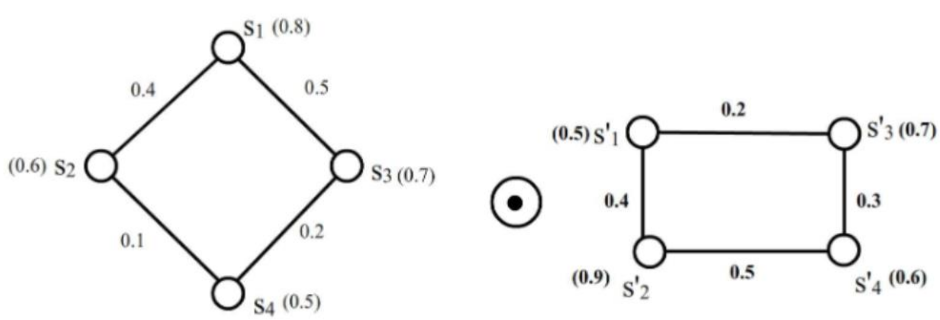

Figure 1: The $\mathfrak{L}$ - Graphs S and P

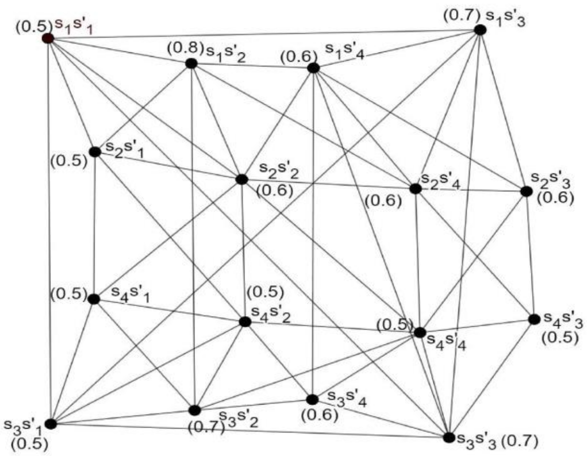

Figure 2: $S \odot P$ strong product of two $\mathfrak{L}$ - Graphs S and P

Example 3.5. Suppose $\mathfrak{L} = (\mathfrak{L}, \vee, \wedge, \odot, \rightarrow, 0, 1)$; then $P \odot S$ be the isomorphic strong product of P and S (figure 3) where, $\mathbb{V} = \{(s'_j, s_i) \mid 1 \leq i, j \leq 4\}$,
$\vartheta^\star(s'_1, s_1) = 0.5, \vartheta^\star(s'_2, s_1) = 0.8, \vartheta^\star(s'_4, s_1) = 0.6, \vartheta^\star(s'_3, s_1) = 0.7,$
$\vartheta^\star(s'_1, s_2) = 0.5, \vartheta^\star(s'_2, s_2) = 0.6, \vartheta^\star(s'_4, s_2) = 0.6, \vartheta^\star(s'_3, s_2) = 0.6,$
$\vartheta^\star(s'_1, s_4) = 0.5, \vartheta^\star(s'_2, s_4) = 0.5, \vartheta^\star(s'_4, s_4) = 0.5, \vartheta^\star(s'_3, s_4) = 0.5,$
$\vartheta^\star(s'_1, s_3) = 0.5, \vartheta^\star(s'_2, s_3) = 0.7, \vartheta^\star(s'_4, s_3) = 0.6, \vartheta^\star(s'_3, s_3) = 0.7.$

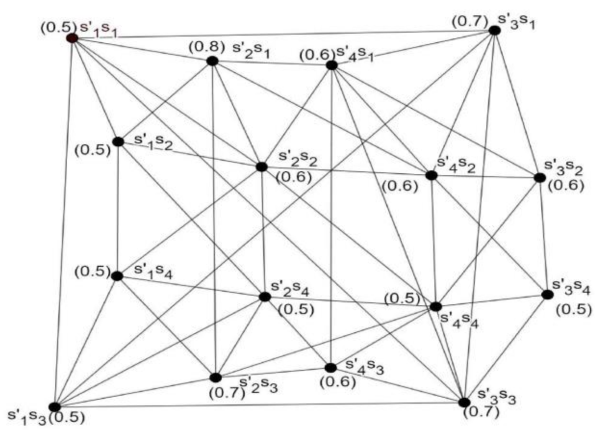

Figure 3: $P \odot S$ isomorphic Strong product of two $\mathfrak{L}$ - Graphs P and S

4. The modular product of 𝔏-graphs

In this section, we introduce the modular product using $\mathfrak{L}$-graph and its isomorphic graph. Let $\mathcal{M}$ represent an $\mathfrak{L}$-graph $\mathcal{M} = (\vartheta_1^\star, \tau_1^\star)$ on $\mathcal{M}^\star = (\mathbb{V}_1, \mathbb{E}_1)$, and let $\mathcal{P}$ denote an $\mathfrak{L}$-graph $\mathcal{P} = (\vartheta_2^\star, \tau_2^\star)$ on $\mathcal{P}^\star = (\mathbb{V}_2, \mathbb{E}_2)$.

Definition 4.1. Let $\mathcal{M}$ and $\mathcal{P}$ be two $\mathfrak{L}$-graphs. Then the modular product of two $\mathfrak{L}$-graphs $\mathcal{M}$ and $\mathcal{P}$ is defined by $\mathcal{M} \boxtimes \mathcal{P} = (\vartheta^\star, \tau^\star)$ and $(\mathcal{M} \boxtimes \mathcal{P})^\star = (\mathbb{V}, \mathbb{E})$, where

(i) $\mathbb{V} = \mathbb{V}_1 \boxtimes \mathbb{V}_2$
(ii) $\mathbb{E} = \mathbb{E}_1 \boxtimes \mathbb{E}_2$
(iii) $\mathbb{V}_1 \boxtimes \mathbb{V}_2 = (t_i, t'_m)\ t_i \in \mathbb{V}_1, t'_m \epsilon \mathbb{V}_2$
(iv) $\vartheta^\star(t_i, t'_m) = \vartheta_1^\star(t_i) \wedge \vartheta_2^\star(t'_m)$ for all $(t_i, t'_m) \epsilon \mathbb{V}$
(v) $\mathbb{E}_1 \boxtimes \mathbb{E}_2 = (t_i, t'_m)(t_j, t'_n), (t_i, t_j) \in \mathbb{E}_1, (t'_m, t'_n) \epsilon \mathbb{E}_2$ or $(t_i, t_j) \notin \mathbb{E}_1, (t'_m, t'_n) \notin \mathbb{E}_2$
(vi)$\tau^\star(t_i, t'_m)(t_j, t'_n)$

$$= \begin{cases} \tau_1^\star(t_i, t_j) \\ \left(\vartheta_1^\star(t_i) \wedge \vartheta_1^\star(t_j)\right) \boxtimes (\vartheta_2^\star, t_n^\star), \text{if } (t'_m) \wedge t'_m\big) \in \vartheta_2^\star(t'_n)\big), \text{ if } (t_i, t'_m) \notin \mathbb{E}_1 \text{ and } (t_j, t'_n) \notin \mathbb{E}_2 \end{cases}$$

Theorem 4.2. Let $\mathcal{M}$ and $\mathcal{P}$ be two $\mathfrak{L}$ - graphs. Then
(i) $\mathcal{M} \boxtimes \mathcal{P}$ is an $\mathfrak{L}$ - graph,
(ii) If (t, t') is a vertex of $\mathcal{MP}$, then $d_{\mathcal{MP}}(t, t') = d_{\mathcal{M}(t)} + d_{p(t')}$
(iii) $|\mathbb{E}| = |\mathbb{E}_1| \times |\mathbb{V}_2| + |\mathbb{V}_1| \times |\mathbb{E}_2|$
Proof:
(i) According to the definition of $\mathfrak{L}$-graph, this is proved by showing $\tau^\star\left((t_i, t'_m)(t_j, t'_n)\right) \leq \vartheta^\star(t_i, t'_m) \boxtimes \vartheta^\star(t_j, t'_n)$. We know that $\tau^\star\left((t_i, t'_m)(t_j, t'_n)\right)$ has two cases.

Case 1. $(t_i, t_j) \in \mathbb{E}_1and(t'_m, t'_n) \in \mathbb{E}_2$ then

$$\tau^\star\left((t_i, t'_m)(t_j, t'_n)\right) = \tau_1^\star(t_i, t_j)\ \tau_2^\star(t'_m, t'_n \text{ by definition of } \tau^\star$$
$$= \left(\vartheta_1^\star(t_i) \wedge \vartheta_1^\star(t_j)\right) \boxtimes \left(\vartheta_2^\star(t'_m) \wedge \vartheta_2^\star(t'_n)\right) \text{ by definition of } \mathfrak{L} - \text{ graph}$$
$$\leq \left(\vartheta_1^\star(t_i) \boxtimes \vartheta_2^\star(t'_m)\right) \wedge \left(\vartheta_1^\star(t_j) \boxtimes \vartheta_2^\star(t'_n) \text{ by (} R_5 \text{) proposition (2.2)}\right.$$
$$\leq \left(\vartheta_1^\star(t_i) \boxtimes \left(\vartheta_1^\star(t_j) \boxtimes \vartheta_2^\star(t'_n)\right)\right) \wedge \left(\vartheta_2^\star(t'_m) \boxtimes \left(\vartheta_1^\star(t_j) \boxtimes \vartheta_2^\star(t'_n)\right)\right) \text{ by (} R_5 \text{) proposition (2.2)}$$
$$\leq \left(\vartheta_1^\star(t_i) \boxtimes \left(\vartheta_1^\star(t_j) \wedge \vartheta_2^\star(t'_n)\right)\right) \wedge \left(\vartheta_2^\star(t'_m) \boxtimes \left(\vartheta_1^\star(t_j) \wedge \vartheta_2^\star(t'_n)\right)\right) \text{ by (} R_3 \text{) proposition (2.2)}$$
$$\leq \left(\vartheta_1^\star(t_i) \wedge \vartheta_2^\star(t'_m)\right) \boxtimes \left(\vartheta_1^\star(t_j) \wedge \vartheta_2^\star(t'_n)\right) \text{ by } (R_5) \text{ proposition (2.2)}$$
$$\tau^\star\left((t_i, t'_m)(t_j, t'_n)\right) \leq \vartheta^\star(t_i, t'_m) \boxtimes \vartheta^\star(t_j, t'_n)$$

case 2. $(t_i, t_j) \notin \mathbb{E}_1$ and $(t'_m, t'_n) \notin \mathbb{E}_2$ Assume the first case. Hence,

$$\tau^\star\left((t_i, t'_m)(t_j, t'_n)\right) = \left(\vartheta_1^\star(t_i) \wedge \vartheta_1^\star(t_j)\right) \boxtimes \left(\vartheta_2^\star(t'_m) \wedge \vartheta_2^\star(t'_n)\right) \text{ by definition of } \tau^\star$$
$$\leq \left(\vartheta_1^\star(t_i) \boxtimes \vartheta_2^\star(t'_m)\right) \wedge \left(\vartheta_1^\star(t_j) \boxtimes \vartheta_2^\star(t'_n)\right) \text{ by (R5) proposition (2.2)}$$
$$\leq \left(\vartheta_1^\star(t_i) \boxtimes \left(\vartheta_1^\star(t_j) \boxtimes \vartheta_2^\star(t'_n)\right)\right) \wedge \left(\vartheta_2^\star(t'_m) \boxtimes \left(\vartheta_1^\star(t_j) \boxtimes \vartheta_2^\star(t'_n)\right)\right)$$
$$\leq \left(\left(\vartheta_1^\star(t_i) \wedge \vartheta_2^\star(t'_m)\right) \boxtimes \left(\vartheta_1^\star(t_j) \wedge \vartheta_2^\star(t'_n)\right)\right) \text{ by (R5) proposition (2.2)}$$
$$\tau^\star\left((t_i, t'_m)(t_j, t'_n)\right) \leq \vartheta^\star(t_i, t'_m) \boxtimes \vartheta^\star(t_j, t'_n) \text{ by definition of } \vartheta^\star$$

Above results are obtained using the second case, therefore we can say that $\mathcal{M} \boxtimes \mathcal{P}$ is an $\mathfrak{L}$-graph.

(ii) Each vertex (t, t') of $\mathfrak{L}$-graph $\mathcal{M} \boxtimes \mathcal{P}$ is connected to the vertex (t_i, t') that t_i adjacent vertices of t' in the $\mathfrak{L}$-graph $\mathcal{M}$ and connected to the vertex (t', t_j) that t_j are the adjacent vertices of t' in the $\mathfrak{L}$-graph $\mathcal{P}$. So, $d_{\mathcal{MP}}(t, t') = d_{\mathcal{M}}(t) + d_{\mathcal{P}(t')}$. (iii) consider $\mathcal{Q} = \{(t, t'_m)(t, t'_n) \mid t \in \mathbb{V}_1, t'_m t'_n \in \mathbb{E}_2\}$, and $\mathcal{R} = \{(t_i, t')(t_j, t') \mid t' \epsilon \mathbb{V}_2, (t_i, t_j) \in \mathbb{E}_1\}$. As $= \mathcal{Q} \cup \mathcal{R}, |\mathbb{E}| = |\mathcal{Q}| + |\mathcal{R}| = |\mathbb{E}_1| \times |\mathbb{V}_2| + |\mathbb{V}_1| \times |\mathbb{E}_2|$.

Example 4.3. Suppose $\mathfrak{L} = (\mathcal{L}, \vee, \wedge, \sqcup, \rightarrow, 0, 1)$ and two $\mathfrak{L}$ - graphs $\mathcal{M}$ and $\mathcal{P}$ where

$$\mathfrak{g} \boxtimes \mathfrak{b} = \begin{cases} \mathfrak{g} + \mathfrak{b} - 1, & \text{if } \mathfrak{g} + \mathfrak{b} \geq 1 \\ 0, & \text{if } \mathfrak{g} + \mathfrak{b} < 1 \end{cases}$$

$$\mathfrak{g} \rightarrow \mathfrak{b} = \begin{cases} 1, & \text{if } \mathfrak{b} - \mathfrak{g} \geq 0 \\ 1 - \mathfrak{g} + \mathfrak{b}, & \text{if } \mathfrak{b} - \mathfrak{g} < 0 \end{cases}$$

$\mathbb{V}_1 = \{t_1, t_2, t_3, t_4\}, \mathbb{E}_1 = \{t_1t_2, t_1t_3, t_2t_4, t_3t_4\}, \vartheta_1(t_i, t_j) = \vartheta_1^\star(t_i) \boxtimes \vartheta_1^\star(t_j)$, for every $(t_it_j) \in \mathbb{E}_1, \vartheta_1^\star(t_1) = 0.7, \vartheta_1^\star(t_2) = 0.8, \vartheta_1^\star(t_3) = 0.6, \vartheta_1^\star(t_4) = 0.5$, $\tau_1^\star(t_1t_2) = 0.5, \tau_1^\star(t_1t_3) = 0.3, \tau_1^\star(t_2t_4) = 0.3, \tau_1^\star(t_3t_4) = 0.1, \tau_1^\star(t_3t_2) = 0.6$, $\tau_1^\star(t_1t_4) = 0.4, \mathbb{V}_2 = \{t'_1, t'_2, t'_3, t'_4\}, \mathbb{E}_2 = \{t'_1t'_2, t'_2t'_3, t'_1t'_3, t'_1t'_4\}$, $\tau_2^\star(t'_mt'_n) = (\vartheta_2^\star(t'_m) \vee \vartheta_2^\star(t'_n)) \boxtimes (\vartheta_2^\star(t'_m) \vee \vartheta_2^\star(t'_n))$, for every $(t'_mt'_n) \epsilon \mathbb{E}_2, \vartheta_2^\star(t'_1) = 0.8$, $\vartheta_2^\star(t'_2) = 0.6, \vartheta_2^\star(t'_3) = 0.6, \vartheta_2^\star(t_4) = 0.5, \tau_2^\star(t'_1t'_2) = 0.8, \tau_2^\star(t'_2t'_3) = 0.8, \tau_2^\star(t'_1t'_3) = 0.6$, $\tau_2^\star(t'_1t'_4) = 0.6, \tau_2^\star(t'_3t'_4) = 0.2$. Then $\mathcal{M} \boxtimes \mathcal{P}$ is the modular product of $\mathcal{M}$ and $\mathcal{P}$ (figure 5) where, $\mathbb{V} = (t_i, t'_j) \mid 1 \leq \mathfrak{i}, \mathfrak{j} \leq 4\}$, $\mathbb{E} = \{(t_i, t'_1(t_i, t'_2), (t_it'_1)(t_it'_3), (t_it'_2)(t_it'_3), (t_it'_3)(t_it_1) \mid 1 \leq \mathfrak{i} \leq 4\}$, $\vartheta^\star(t_1, t'_1) = 0.7, \vartheta^\star(t_2, t'_1) = 0.8, \vartheta^\star(t_3, t'_1) = 0.6, \vartheta^\star(t_4, t'_1) = 0.5, \vartheta^\star(t_1, t'_2) = 0.6$, $\vartheta^\star(t_2, t'_2) = 0.6, \vartheta^\star(t_3, t'_2) = 0.6, \vartheta^\star(t_4, t'_2) = 0.5, \vartheta^\star(t_1, t'_3) = 0.6, \vartheta^\star(t_2, t'_3) = 0.6$, $\vartheta^\star(t_3, t'_3) = 0.6, \vartheta^\star(t_4, t'_3) = 0.5, \vartheta^\star(t_1, t'_4) = 0.5, \vartheta^\star(t_2, t'_4) = 0.5, \vartheta^\star(t_4, t'_4) = 0.5$, $\vartheta^\star(t_3, t'_4) = 0.5, \tau^\star(t_1t'_1)(t_2t'_2) = 0.3$.

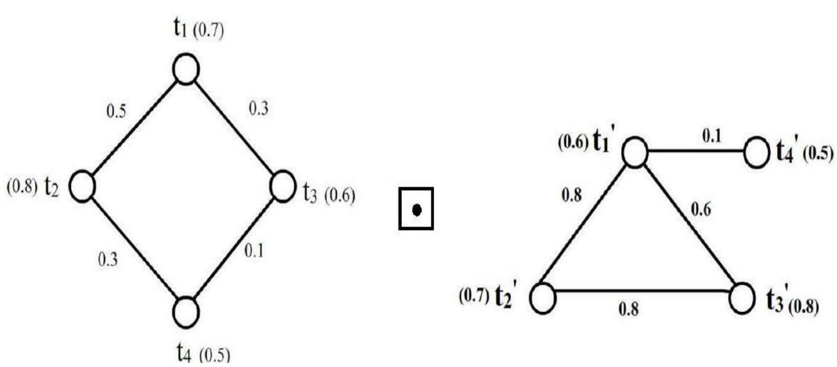

Figure 4: The $\mathfrak{L}$ - Graphs $\mathcal{M}$ and $\mathcal{P}$

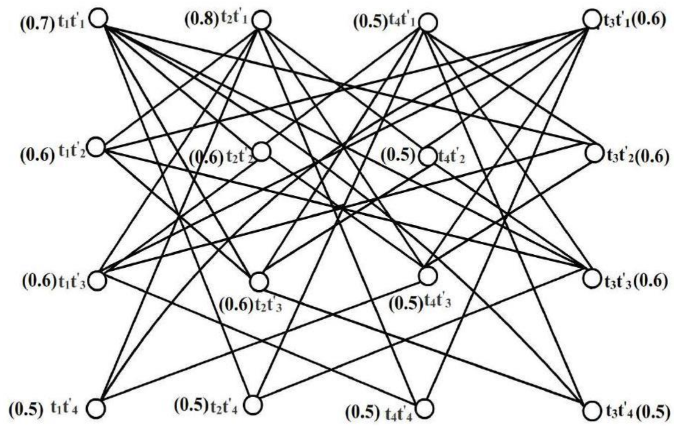

Figure 5: $\mathcal{M} \boxtimes \mathcal{P}$ modular product of two $\mathcal{L}$ - Graphs $\mathcal{M}$ and $\mathcal{P}$

Example 4.4. Suppose $\mathfrak{L} = (\mathfrak{L}, \vee, \wedge, \backsim, \rightarrow, 0, 1)$; then $\mathcal{P} \backsim \mathcal{M}$ be the isomorphic modular product of $\mathcal{P}$ and $\mathcal{M}$, where $\mathbb{V} = \{(t_i, t'_j) \mid 1 \le i, j \le 4\}$, $\mathbb{E} =$
$\{(t'_1 t_i)(t'_2 t_i), (t'_1 t_i)(t'_3 t_i), (t'_2 t_i)(t'_3 t_i),$
$(t'_3 t_i)(t'_1 t_i) \mid 1 \le i \le 4\}$,
$\vartheta^\star(t'_1 t_1) = 0.7, \vartheta^\star(t'_1 t_2) = 0.8, \vartheta^\star(t'_1 t_3) = 0.6, \vartheta^\star(t'_1, t_4) = 0.5, \vartheta^\star(t'_2 t_1) = 0.6,$
$\vartheta^\star(t'_2 t_2) = 0.6, \vartheta^\star(t'_2, t_3) = 0.6, \vartheta^\star(t'_2, t_4) = 0.5, \vartheta^\star(t'_3, t_1) = 0.6, \vartheta^\star(t'_3 t_2) = 0.6,$
$\vartheta^\star(t'_3 t_3) = 0.6, \vartheta^\star(t'_3, t_4) = 0.5, \vartheta^\star(t'_4, t_4) = 0.5,$

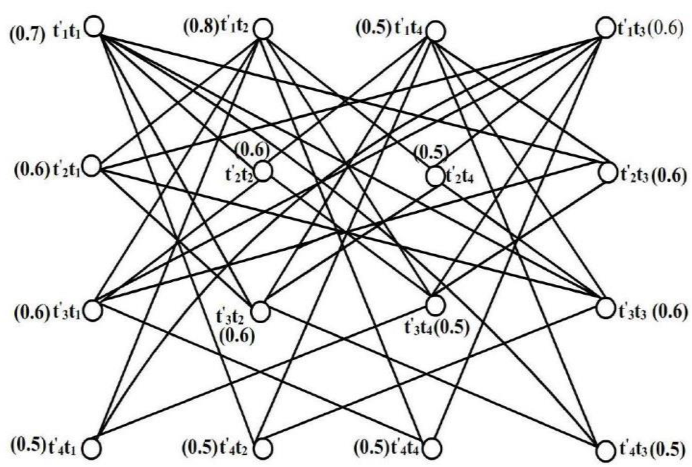

Figure 6: $\mathcal{P} \boxtimes \mathcal{M}$ isomorphic modular product of two $\mathfrak{L}$ – Graphs $\mathcal{M}$ and $\mathcal{P}$

5. Applications

The Internet of Things is a field that has grown rapidly with millions of interconnected devices communicating with each other and central systems to perform several tasks. Such networks are usually hierarchical, where devices at the lowest level connect to gateways, which further communicate with central servers. Efficient communication is paramount in IoT networks to ensure timely data transfer without delay and power consumption, given that most IoT devices are powered through limited sources of power. 𝔏 - Graphs, are very powerful in modeling hierarchical communication structures like this. It captures the ordered relationships between the devices, gateways, and servers and can capture the dependency and flow of communication between various levels of the network. Application of strong products to 𝔏-graphs will allow analyzing all possible routes of communication among devices, gateways, and servers, thus determining efficient pathways for the transfer of data, which ensures minimal communication delays and smooth running. Other than strong products, the modular product is used to study intersection nodes in the network where different devices or pathways share a point. These nodes typically are crucial intersection points within a network; an example includes the shared gateway and router. The optimization of shared pathways means the network has lesser congestion hence improving the data transfer rate and lowering the latency time. This strategy works well for networks carrying high volumes of traffic; an attribute most IoT networks are associated with. Energy efficiency is an important advantage of the use of such graph products, as most of the IoT devices are battery-based, and energy-efficient communication will be necessary for them. These strong and modular products will find the best route to minimize useless data transmission and processing, conserving energy on the network at large. 𝔏 - Graphs with strong and modular products, therefore, are an enormous power in terms of optimizing IoT communication. Such methods will be used to find efficient and reliable routes for easy communication. It helps with congestion at shared points, conserves energy, and supports the scalability of the network. As it undergoes IoT expansion, it will give networks enough reserve to remain efficient and robust enough to meet the demands of modern applications.

5. Conclusion

The study of graph products for 𝔏-graphs highlights their importance in understanding complex relationships inside graph networks. These basic features, as well as their isomorphisms, determine the theoretical and practical value of these products. Strong products enable hierarchical communication pathways, whereas modular devices use common paths to unblock and conserve energy. Integrating graph product operations with machine learning algorithms may enable increased predictive capabilities during IoT network optimization and anomaly identification. Further research on how such an operation scales and works computationally is needed to achieve scalability in such large-scale IoT systems.

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