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# Novel Spectral Conditions for Diagonalizability and Connectivity in Spectral Fuzzy Graph Theory

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**ABSTRACT**

This paper explores the properties of fuzzy matrices in fuzzy graphs and the conditions for the diagonalizability of fuzzy matrices. Necessary and sufficient conditions for fuzzy graphs to have non-negative and distinct eigenvalues are provided, and the existence of orthogonal eigenvectors corresponding to distinct eigenvalues in fuzzy matrices are discussed. Also, conditions for the second smallest eigenvalue of the Laplacian matrix are established to ensure connectivity in fuzzy graphs.

***Keywords:*** Spectral fuzzy graph, adjacency matrix, laplacian matrix, eigenvalue, eigenvector, connected fuzzy graphs

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## 1. Introduction

A graph $G = (V, E)$ is a mathematical construct comprising vertices interconnected by edges. The study of graphs, known as graph theory, traces its origins to Euler's resolution of the Königsberg bridge problem, which laid the groundwork for this discipline. Since that seminal event, graphs have become indispensable tools across various fields, facilitating the modeling of networks, relationships, and interactions. Bondy and Murty offered an exhaustive treatment of graph theory [4], blending both foundational theories and practical applications, and their work continues to serve as a pivotal reference in the exploration and application of graph-theoretical concepts.

Spectral graph theory, a specialized branch within graph theory, investigates the attributes of graphs through the eigenvalues and eigenvectors of matrices associated with them, such as the adjacency matrix and Laplacian matrix [5]. This domain provides profound insights into the structural characteristics of graphs [12], including aspects such as connectivity, bipartiteness, and community structure. Fiedler's seminal work on algebraic connectivity [7] established a crucial link between spectral properties and graph connectivity and robustness. Chung provided a comprehensive analysis of the field, encompassing a wide array of theoretical results and their applications [6]. Brouwer and Haemers delved into the intricate relationship between graph structures and their spectral properties [5], while Godsil and Royle integrated spectral methods with broader algebraic techniques [8], offering a panoramic view of graph spectral methods. Horn and Johnson laid the mathematical foundations essential for the comprehension of spectral graph theory [11], whereas Hogben addressed significant combinatorial issues [10] related to spectral properties.

Fuzzy graph theory $G = (\sigma, \mu)$ extends conventional graph theory by incorporating the notion of fuzzy sets [24], a concept introduced by Zadeh, which allows for degrees of membership for vertices and edges. This framework is particularly advantageous for modeling scenarios characterized by uncertainty and imprecision. Rosenfeld was a trailblazer in this area, establishing the fundamental definitions and operations for fuzzy graphs [17]. Bhattacharya further refined this theoretical framework [3], probing into structural properties and operations. Mordeson and Nair synthesized various theories and applications, rendering their work an invaluable resource for researchers [13]. Narayanan and Mathew extended the concept of graph energy into the fuzzy domain [14], thereby connecting fuzzy graph theory with spectral properties. Gutman and Zhou examined the energy associated with the Laplacian matrix of a graph [9], paving the way for future research endeavors. Jiang explored how traditional spectral methods [12] could be adapted to fuzzy contexts, thereby bridging these two fields and illuminating the potential for further interdisciplinary exploration and innovation.

Recent studies in fuzzy graph theory have explored domination sets in vague graphs [23], vertex connectivity [2], and interval-valued fuzzy graph properties [15]. Complex Pythagorean fuzzy graphs [21] and neutrosophic graphs [19] have extended these models, while vague graphs aid in medical diagnosis [18]. Research on picture fuzzy graph energy [20] and signless Laplacian energy [16] has advanced spectral analysis, with isomorphism studies further refining structural insights [22].

Spectral fuzzy graph theory amalgamates the principles of spectral graph theory and fuzzy graph theory by scrutinizing the spectral properties of fuzzy graphs. This hybrid framework facilitates the analysis of complex systems marked by uncertainty and intricate structural features. By merging the strengths of both spectral and fuzzy graph theories, this approach offers novel insights and applications, especially in domains where uncertainty is a significant factor.

This paper aims to elucidate the properties of spectral fuzzy graphs by synthesizing concepts from spectral graph theory and fuzzy graph theory, aspiring to furnish fuzzy graphs' structural and spectral characteristics. The research focuses on the impact of fuzziness in vertices and edges on the spectral properties of fuzzy graphs.

The paper is organized as follows: Section 2 expounds upon foundational concepts and background information. In Section 3, theoretical constructs related to the properties of spectral fuzzy graphs are explored, supplemented by illustrative examples. Section 4 concludes the paper with a summary of the findings.

## 2. Preliminaries

**Definition 2.1.** *[13] A fuzzy graph $G = (V, \sigma, \mu)$ is a triple consisting of a non-empty set*

*$V$ together with a pair of functions $\sigma: V \longrightarrow [0,1]$ is a fuzzy vertex set and $\mu: V \times V \longrightarrow [0,1]$ is a fuzzy edge set such that $\mu_{ij} \leq \sigma_i \wedge \sigma_j$ for all $i, j \in V$.*

**Definition 2.2.** *[14] The adjacency matrix $A$ of a fuzzy graph $G = (V, \sigma, \mu)$ is an $n \times n$ matrix defined as $A = [a_{ij}]$ where $a_{ij} = \mu_{ij}$. The eigenvalues are denoted by $\lambda_i: \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n$ of $A$.*

**Definition 2.3.** *[14] Let $G = (V, \sigma, \mu)$ be a fuzzy graph with $n$ vertices. The Laplacian matrix $L$ of $G$ is defined as,*

$$L = [l_{ij}]_{n\times n} = \begin{cases} d_i & \text{if } v_i = v_j \\ -\mu_{ij} & \text{if } (v_i, v_j) \in \mu \\ 0 & \text{Otherwise} \end{cases}$$

Also, $L = D - A$, where $D$ denotes the diagonal degree matrix $d_i$ in $G$.

**Definition 2.4.** *[1] For a square matrix $M$, the multiset of eigenvalues of $M$ is called the spectrum of $M$ and is denoted by $\Gamma(G) = \{\lambda_1^{(m_1)}, \lambda_2^{(m_2)}, \ldots, \lambda_p^{(m_p)}\}$ where each $\lambda_i$ is a distinct eigenvalue of $M$ with multiplicity $m_i$,* forall $i = 1,2, \ldots, p$.

**Definition 2.5.** *[14] Let $G$ be a fuzzy graph and $A$ be its adjacency matrix. The eigenvalues of $A$ are the eigenvalues of $G$. The adjacency eigenvalues, along with their algebraic multiplicities, collectively constitute the fuzzy spectrum $\Gamma(G)$.*

**Definition 2.6.** *[8] The spectral radius $\rho(G)$ of a fuzzy graph $G$ with an adjacency matrix $A$ is the maximum absolute value of the eigenvalues of $A$. It is given by, $\rho(G) = \max\{|\lambda_1|, |\lambda_2|, \ldots, |\lambda_n|\}$ where $\lambda_i$ are the spectrum of $A$.*

## 3. Main results

**Proposition 3.1.** *Let $D$ be a fuzzy diagonal matrix with entries $\lambda_1, \lambda_2, \ldots, \lambda_n$. The eigenvalues of $D$ are given by $\lambda_1, \lambda_2, \ldots, \lambda_n$, and if $D$ is diagonalizable, then eigenvalues are distinct.*

**Proof:** The eigenvalues of a diagonal matrix are the entries on its diagonal. For the fuzzy diagonal matrix $D$, the characteristic polynomial is formed by $\det(D - \lambda I) = 0$, where $I$ is the identity matrix. Since $D - \lambda I$ is also diagonal, its determinant is the product of the diagonal elements $\lambda_i - \lambda$ for $i = 1,2, \ldots, n$. Therefore,

$$\det(D - \lambda I) = \prod_{i=1}^{n} (\lambda_i - \lambda) = 0.$$

The solutions $\lambda$ are exactly the diagonal elements $\lambda_1, \lambda_2, \ldots, \lambda_n$, which are the eigenvalues of $D$.

If $D$ is fuzzy diagonalizable, the fuzzy eigenvalues must be distinct. Diagonalizability implies that the matrix can be decomposed into a diagonal form using a fuzzy unitary matrix. If the eigenvalues are not distinct, Jordan blocks arises in the decomposition, violating diagonalizability. Thus, $D$ is diagonalizable if and only if the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ are distinct.

**Proposition 3.2.** *A fuzzy unitary matrix $U$ satisfies $UU^{-1} = I$, where $U^{-1}$ is the inverse of $U$, with both $U$ and $U^{-1}$ respecting the fuzzy membership function $\mu$.*

**Proof:** A matrix $U$ is unitary if $U^*U = I$, where $U^*$ is the conjugate transpose and $I$ is the identity. In the fuzzy case, entries of $U$ are associated with fuzzy membership values $\mu_{ij} \in [0,1]$, reflecting the degree of membership. The inverse matrix $U^{-1}$ must satisfy the same fuzzy membership constraints as $U$. Hence, $UU^{-1} = I$ holds in the fuzzy context, ensuring the preservation of the unitary property, with the product yielding the identity matrix while respecting fuzziness.

**Lemma 3.3.** *Let $A$ be a fuzzy matrix. There exists a fuzzy unitary matrix $U$ such that $A = UTU^{-1}$, where $T$ is a fuzzy upper triangular matrix.*

**Proof:** The result follows from Schur decomposition theorem [5]. For any fuzzy matrix $A$, there exists a fuzzy unitary matrix $U$ such that the decomposition

$$A = UTU^{-1}$$

holds, where $T$ is an upper triangular matrix with fuzzy eigenvalues along its diagonal. This decomposition is obtained by unitary transformations on $A$, preserving the fuzzy membership functions associated with the matrix elements. Since every matrix consists of fuzzy membership values, it can be decomposed into an upper triangular form using unitary transformations. The membership function of each element remains intact, ensuring that the fuzziness of $A$ and $U$ does not affect the triangular structure of $T$, and the fuzzy eigenvalues are preserved along the diagonal of $T$.

**Lemma 3.4.** *If a fuzzy matrix $A$ has distinct fuzzy eigenvalues, then $A$ is diagonalizable by a fuzzy unitary matrix.*

**Proof:** The distinctness of the fuzzy eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ implies that there exist $n$ linearly independent fuzzy eigenvectors corresponding to these eigenvalues. The set of these eigenvectors spans the entire space to construct a fuzzy unitary matrix $U$ such that

$$A = UDU^{-1},$$

where $D$ is a diagonal matrix with the distinct fuzzy eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ on its diagonal. Since the eigenvalues are distinct, there are no Jordan blocks in the Jordan canonical form of $A$, and the matrix is diagonalizable. This diagonalization ensures that the fuzzy structure of the matrix is preserved while transforming $A$ into its diagonal form.

**Proposition 3.5.** *If a fuzzy matrix $A$ possesses repeated fuzzy eigenvalues, then $A$ is non-diagonalizable by a fuzzy unitary matrix.*

**Proof:** When a fuzzy matrix $A$ has repeated fuzzy eigenvalues, the eigenspaces corresponding to these eigenvalues doesnot possess linearly independent fuzzy eigenvectors to form a complete basis. In such cases, the Jordan canonical form of $A$ contains non-trivial Jordan blocks, which prevent the matrix from being diagonalized. The presence of such repeated fuzzy eigenvalues implies that $A$ can only be transformed into a block upper triangular matrix. Thus, $A$ can be expressed as

$$A = UTU^{-1},$$

where $T$ is a block triangular matrix, and $U$ is a fuzzy unitary matrix. The lack of a sufficient number of linearly independent eigenvectors due to the repeated fuzzy eigenvalues leads to this restriction, preventing the diagonalization of $A$ and forcing a block triangular form.

**Example 1.** *Consider a fuzzy graph $G$ as shown in Fig. 3.1, which has repeated eigenvalues but still no self-loops.*

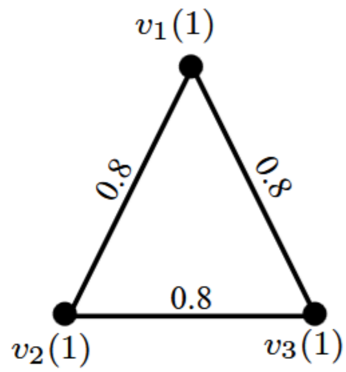


**Figure 3.1:** Fuzzy Graph $G$ with $\mu_{12} = \mu_{13} = \mu_{23} = 0.8$

Now,

$$A = \begin{bmatrix} 0 & 0.8 & 0.8 \\ 0.8 & 0 & 0.8 \\ 0.8 & 0.8 & 0 \end{bmatrix}$$

The eigenvalues of this matrix are: $\lambda = \{1.6, -0.8, -0.8\}$. Here,the repeated eigenvalues are $\lambda_2$ and $\lambda_3$. As per the proposition, this matrix is non- diagonalizable by a fuzzy unitary matrix due to repeated eigenvalues. But $A$ can be triangularized. There exists a fuzzy unitary matrix $U$ such that $A = UTU^{-1}$ where $T$ is a block upper triangular matrix (because of the repeated eigenvalues). In this case:

$$T = \begin{bmatrix} 1.6 & * & * \\ 0 & -0.8 & * \\ 0 & 0 & -0.8 \end{bmatrix}$$

**Theorem 3.6.** ***Fuzzy Spectral Theorem:*** *Let $A$ be an $n \times n$ fuzzy matrix with fuzzy entries, and let $\mu \in [0,1]$ be a fuzzy membership function. Then, there exists a fuzzy unitary matrix $U$ and a fuzzy diagonal matrix $D$ such that*

$$A = UDU^{-1}$$

if and only if $A$ is fuzzy diagonalizable, and the fuzzy eigenvalues of $A$ are distinct.

**Proof: Necessary Part:** Assume $A = UDU^{-1}$, where $U$ is a fuzzy unitary matrix, and $D$ is a fuzzy diagonal matrix. The decomposition $A = UDU^{-1}$ indicates that $A$ is diagonalizable in the fuzzy sense. Since $U$ is fuzzy unitary, its inverse $U^{-1}$ also exists and is fuzzy unitary. The matrix $D$ being diagonal, contains the fuzzy eigenvalues of $A$, and these eigenvalues must be distinct.

Diagonalizability via a fuzzy unitary matrix implies no Jordan blocks, which arise only with repeated eigenvalues. If $A$ had repeated fuzzy eigenvalues, $D$ would contain off-diagonal blocks in its Jordan form, contradicting diagonalizability. Therefore, distinct fuzzy eigenvalues are necessary for the fuzzy diagonalization of $A$.

**Sufficient Part:** Conversely, assume that $A$ is fuzzy diagonalizable and its fuzzy eigenvalues are distinct. By Schur's theorem, for any matrix $A$, there exists a unitary matrix $V$ such that

$$A = VTV^{-1},$$

where $T$ is an upper triangular matrix. In the fuzzy context, $V$ becomes fuzzy unitary and $T$ is a fuzzy upper triangular matrix. Since the eigenvalues are distinct, $T$ must have zero off-diagonal elements, as non-zero entries indicates Jordan blocks, contradicting the distinctness of the fuzzy eigenvalues. Thus, $T$ is diagonal and $T = D$, where $D$ is a diagonal matrix with distinct fuzzy eigenvalues.

Therefore,

$$A = VDV^{-1}.$$

Since $D$ is diagonal and $V$ is fuzzy unitary, $A$ is diagonalizable by the fuzzy unitary matrix $V$. Let $U = V$. Thus,

$$A = UDU^{-1},$$

where $U$ is fuzzy unitary and $D$ contains the distinct fuzzy eigenvalues of $A$.

Thus, $A$ is diagonalizable by a fuzzy unitary matrix if and only if the fuzzy eigenvalues of $A$ are distinct, ensuring both diagonalizability and the distinctness of eigenvalues.

**Example 2.** *Consider a fuzzy graph $G$ as shown in Fig. 3.2 with $\sigma_1 = \sigma_2 = \sigma_3 = 1$ and $\mu_{12} = 0.6, \mu_{23} = 0.5, \mu_{13} = 0.3$.*

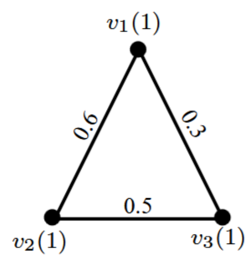


**Figure 3.2:** Fuzzy Graph $G$ with no self-loops

The Adjacency matrix is given by

$$A = \begin{bmatrix} 0 & 0.6 & 0.3 \\ 0.6 & 0 & 0.5 \\ 0.3 & 0.5 & 0 \end{bmatrix}$$

The eigenvalues of $A$ are $\lambda = \{0.94, -0.29, -0.65\}$.

From the Fuzzy Spectral Theorem, since $A$ has distinct eigenvalues, it is diagonalizable. There exists a fuzzy unitary matrix $U$ such that $A = UDU^{-1}$ where $D$ is given by,

$$D = \begin{bmatrix} 0.94 & 0 & 0 \\ 0 & -0.29 & 0 \\ 0 & 0 & -0.65 \end{bmatrix}$$

Since all the eigenvalues are distinct, this matrix is diagonalizable using a fuzzy unitary matrix.

**Proposition 3.7.** *For any simple, undirected fuzzy graph $G$, the adjacency spectrum $A(G)$ consists of real numbers.*

**Proof:** Consider a simple undirected fuzzy graph $G = (\sigma, \mu)$. The adjacency matrix $A(G)$ is an $n \times n$ matrix (where $n = |V|$ ) with entries $A_{ij}$ corresponding to the edge membership values $\mu_{ij}$.

Since $G$ is undirected, the membership function satisfies $\mu_{ij} = \mu_{ji}$ for all $i, j \in V$, which implies that $A(G)$ is symmetric, i.e., $A_{ij} = A_{ji}$. According to linear algebra, the eigenvalues of any real symmetric matrix are guaranteed to be real. Therefore, the adjacency matrix $A(G)$ has real eigenvalues due to its symmetry.

More specifically, there exists an orthogonal matrix $P$ such that:

$$A(G) = PDP^T,$$

where $D$ is a diagonal matrix containing the eigenvalues of $A(G)$. Since $P$ is orthogonal ($P^T = P^{-1}$), this diagonalization shows that all eigenvalues of $A(G)$ must be real.

Hence, the adjacency spectrum of the fuzzy graph $G$ consists entirely of real numbers.

**Theorem 3.8.** *For a fuzzy graph $G$ with $n$ vertices, the following properties hold:*

1. $\sum_{i=1}^{n} \lambda_i = 0$,
2. $\sum_{i=1}^{n} \lambda_i^2 \le \sum_{i=1}^{n} \deg(v_i)$,
3. If $\lambda_1 = \lambda_2 = \cdots = \lambda_n$, then $\mu_{ij} = 0$ for all $i \ne j$.

**Proof:** Let $G = (\sigma, \mu)$ be a fuzzy graph with vertex set $\sigma$ and edge set $\mu$, where $\mu: \sigma \times \sigma \to [0,1]$ is the fuzzy membership function representing the strength of the connection between any two vertices $i$ and $j$. The adjacency matrix $A(G) = [a_{ij}]$ of the graph $G$ has entries $a_{ij} = \mu_{ij}$, the fuzzy membership value between vertices $i$ and $j$. The matrix $A(G)$ is symmetric, meaning $a_{ij} = a_{ji}$ for all $i, j$.

The eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ of $A(G)$, which form the adjacency spectrum of $G$, satisfy the following conditions:

**Part i)** The sum of the eigenvalues of any matrix equals its trace, which is the sum of its diagonal elements. Since $G$ is a simple graph without self-loops, all diagonal entries $a_{ii} = 0$. Therefore, the trace of $A(G)$ is zero:

$$\mathrm{Tr}(A(G)) = \sum_{i=1}^{n} a_{ii} = 0.$$

Thus, the sum of the eigenvalues is:

$$\sum_{i=1}^{n} \lambda_i = 0.$$

**Part ii)** The sum of the squares of the eigenvalues is equal to the Frobenius norm of the adjacency matrix. The Frobenius norm is the square root of the sum of the squares of the matrix entries:

$$\| A(G) \|_F^2 = \sum_{i,j} a_{ij}^2.$$

In a fuzzy graph, the degree of a vertex $v_i$ is given by:

$$\deg(v_i) = \sum_{j=1}^{n} \mu_{ij} = \sum_{j=1}^{n} a_{ij}.$$

Hence, the total degree sum is:

$$\sum_{i=1}^{n} \deg(v_i) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}.$$

Since $a_{ij}^2 \le a_{ij}$ for all $a_{ij} \in [0,1]$, it follows that:

$$\sum_{i=1}^{n} \lambda_i^2 = \| A(G) \|_F^2 = \sum_{i,j} a_{ij}^2 \le \sum_{i=1}^{n} \deg(v_i).$$

**Part iii)** If all eigenvalues of $A(G)$ are equal, say $\lambda_1 = \lambda_2 = \cdots = \lambda_n = \lambda$, then the matrix $A(G)$ must be a scalar multiple of the identity matrix:

$$A(G) = \lambda I,$$

where $I$ is the identity matrix. This implies that $a_{ij} = 0$ for all $i \neq j$, since the identity matrix has zero off-diagonal elements. Therefore, the fuzzy membership function $\mu_{ij}$ must be zero for all $i \neq j$, indicating that there are no edges between distinct vertices in the fuzzy graph.

Thus, the conditions of the theorem are satisfied.

**Example 3.** *Consider a fuzzy graph $G$ as shown in Fig. 3.3 with $\sigma_1 = \sigma_2 = \cdots = \sigma_5 = 1$ and $\mu_{12} = 0.7, \mu_{23} = 0.6, \mu_{13} = 0.5, \mu_{24} = 0.4, \mu_{35} = 0.8, \mu_{45} = 0.3$.*

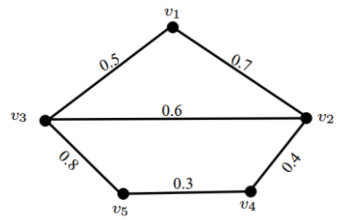


**Figure 3.3:** Fuzzy Graph $G$

The eigenvalues are $\lambda = \{1.442, 0.3355, 0.0739, -0.7405, -1.1109\}$. These eigenvalues confirm that all are real numbers.

The sum of the eigenvalues is: $\sum_{i=1}^{5} = 1.442 + 0.3355 + 0.0739 - 0.7405 - 1.1109 = 0$, satisfying the first condition of the theorem.

Now, $deg(v_1) = 1.2$, $deg(v_2) = 1.7$, $deg(v_3) = 1.9$, $deg(v_4) = 0.7$, $deg(v_5) = 1.1$.

The total degree sum is: $\sum_{i=1}^{5} deg(v_i) = 6.6$ and the sum of the squares of the eigenvalues is: $\sum_{i=1}^{5} \lambda_i^2 = 3.98$. Thus, the sum of the squares of the eigenvalues is less than the total degree sum.

Since the eigenvalues are distinct, third condition does not apply for this example and holds only for null graphs.

**Lemma 3.9.** *A fuzzy graph $G$ with $\vartheta_n = 0$ as the smallest eigenvalue of the Laplacian matrix $L(G)$ always possesses non-negative eigenvalues.*

**Proof:** The Laplacian matrix $L(G) = D(G) - A(G)$ is symmetric because both the diagonal matrix $D(G)$ and the adjacency matrix $A(G)$ are symmetric, with $\mu_{ij} = \mu_{ji}$. Therefore, the eigenvalues of $L(G)$ are real.

To prove that the eigenvalues are non-negative, consider the Rayleigh quotient for $L(G)$:

$$\vartheta = \frac{u^T L u}{u^T u},$$

where $u$ is any non-zero vector. Expanding $u^T L u$, we obtain:

$$u^T L u = u^T D u - u^T A u,$$

which simplifies to:

$$u^T L u = \sum_{i=1}^{n} D_{ii} u_i^2 - \sum_{i,j} \mu_{ij} u_i u_j.$$

Substituting $D_{ii} = \sum_j \mu_{ij}$, we get:

$$u^T L u = \sum_{i=1}^{n} \sum_{j=1}^{n} \mu_{ij} u_i^2 - \sum_{i,j} \mu_{ij} u_i u_j,$$

Equivalently:

$$u^T L u = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \mu_{ij} (u_i - u_j)^2.$$

Since $\mu_{ij} \geq 0$ and $(u_i - u_j)^2 \geq 0$, we have $u^T L u \geq 0$, and thus $\vartheta \geq 0$. This shows that the eigenvalues of $L(G)$ are non-negative.

To show that $\vartheta_n = 0$, consider the all-ones vector $1 = (1,1, \ldots, 1)^T$. Then, $L(G) \cdot 1 = (D(G) - A(G)) \cdot 1 = D(G) \cdot 1 - A(G) \cdot 1$. Both $D(G) \cdot 1$ and $A(G) \cdot 1$ yield the same result (the row sums of $A(G)$), so we have $L(G) \cdot 1 = 0$. This shows that $\vartheta_n = 0$, confirming the result.

**Lemma 3.10.** *The null space of any Laplacian matrix $L(G)$ has dimension $1$ if and only if the fuzzy graph $G$ is connected.*

**Proof: Necessary Condition:** If $G$ is connected, then for any non-zero vector $x$ in the null space of $L(G)$, we have $L(G) \cdot x = 0$. On Expanding $x^T L(G) x$,

$$x^T L(G) x = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \mu_{ij} (x_i - x_j)^2 = 0.$$

Since $\mu_{ij} \geq 0$ and $(x_i - x_j)^2 \geq 0$, equality holds only if $x_i = x_j$ for all $i, j$. Therefore, $x$ is proportional to the all-ones vector, and the null space of $L(G)$ is spanned by $1$, implying its dimension is $1$.

**Sufficient Condition:** If $G$ is disconnected, $L(G)$ can be permuted into a block diagonal matrix, where each block corresponds to a connected component of $G$. Each block has its own Laplacian matrix with a zero eigenvalue corresponding to the all-ones vector of that component. Thus, the dimension of the null space is greater than $1$ if $G$ is disconnected. Therefore, if the null space has dimension $1$, $G$ must be connected.

**Proposition 3.11.** *The second smallest eigenvalue $\vartheta_{n-1}$ of the Laplacian matrix $L(G)$ is positive if and only if the fuzzy graph $G$ is connected.*

**Proof:** If $G$ is connected, the null space of $L(G)$ is spanned only by the all-ones vector, which corresponds to $\vartheta_n = 0$. The second smallest eigenvalue $\vartheta_{n-1}$ must be positive because there is no other independent eigenvector corresponding to the eigenvalue $0$.

Conversely, assume $\vartheta_{n-1} > 0$, but $G$ is disconnected. In this case, $L(G)$ must have at least two $0$ eigenvalues, corresponding to each connected component of $G$, which contradicts the assumption that $\vartheta_{n-1} > 0$. Therefore, $G$ is connected if and only if $\vartheta_{n-1} > 0$.

**Theorem 3.12.** *A fuzzy graph $G = (\sigma, \mu)$ is connected if and only if $\vartheta_{n-1} > 0$.*

**Proof: Part (i):** If $G$ is connected, then for any two vertices $\sigma_i$ and $\sigma_j$, there is a path between them. The Laplacian matrix $L(G)$ has a smallest eigenvalue $\vartheta_n = 0$, with the corresponding eigenvector being the all-ones vector. The null space is spanned only by $1$, and therefore, $\vartheta_{n-1} > 0$.

**Part (ii):** If $G$ is disconnected, the Laplacian matrix can be permuted into a block diagonal matrix, with each block corresponding to a connected component. The null space of each block has a dimension corresponding to the number of connected components, implying that $\vartheta_{n-1} = 0$. Thus, if $\vartheta_{n-1} > 0$, $G$ must be connected.

**Example 4.** *Consider a fuzzy graph $G$ with five vertices $v_1, v_2, v_3, v_4, v_5$, where $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = \sigma_5 = 1$, and the fuzzy membership values are given as: $\mu_{12} = 0.8, \mu_{13} = 0.7, \mu_{23} = 0.6, \mu_{34} = 0.5, \mu_{45} = 0.4$.*

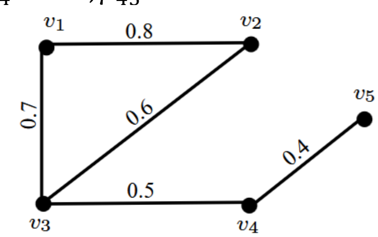


**Figure 3.4:** Fuzzy Graph G forming $L(G)$

The corresponding Laplacian matrix $L(G) = D(G) - A(G)$ is computed as follows:

$$L(G) = \begin{bmatrix} 1.5 & -0.8 & -0.7 & 0 & 0 \\ -0.8 & 1.4 & -0.6 & -0.5 & 0 \\ -0.7 & -0.6 & 1.8 & -0.5 & 0 \\ 0 & -0.5 & -0.5 & 0.9 & -0.4 \\ 0 & 0 & 0 & -0.4 & 0.4 \end{bmatrix}$$

The eigenvalues of the Laplacian matrix are $\vartheta = \{0, 0.432, 1.101, 1.667, 2.4\}$. These eigenvalues are non-negative, confirming the lemma.

The second smallest eigenvalue is $\vartheta_{n-1} = 0.432$, which is positive, confirming that the fuzzy graph $G$ is connected, as stated in the theorem.

Thus, the graph satisfies the condition that $\vartheta_{n-1} > 0$, indicating connectivity.

**Lemma 3.13.** *If $A$ is a positive semi-definite matrix, there exists a matrix $B$ such that $A = B^T B$.*

**Proof:** Let $A \in \mathbb{R}^{n\times n}$ be a real symmetric positive semi-definite matrix. By definition, for any vector $x \in \mathbb{R}^n$, it holds that

$$x^T A x \geq 0.$$

Since $A$ is real and symmetric, the Fuzzy Spectral Theorem implies that $A$ is diagonalizable. Specifically, an orthogonal matrix $Q \in \mathbb{R}^{n\times n}$ exists such that

$$A = QDQ^T,$$

where $D$ is a diagonal matrix whose entries are the eigenvalues of $A$. Since $A$ is positive semi-definite, all its eigenvalues are non-negative. Let the eigenvalues of $A$ be denoted $\lambda_1, \lambda_2, \ldots, \lambda_n$, with $\lambda_i \geq 0$ for each $i$.

Each eigenvalue $\lambda_i$ can be expressed as the square of a non-negative number. Define

$$\sigma_i = \sqrt{\lambda_i},$$

for each $i = 1,2,\ldots,n$, such that $\sigma_i \geq 0$ and $\sigma_i^2 = \lambda_i$. A diagonal matrix $\Sigma$ can be constructed as:

$$\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n),$$

so that $\Sigma^2 = D$.

Now, define $B$ as $B = \Sigma Q^T$. It follows that
$$B^T B = (Q\Sigma)^T (Q\Sigma) = \Sigma^T Q^T Q \Sigma.$$
Since $Q$ is orthogonal, $Q^T Q = I$, the identity matrix, leading to
$$B^T B = \Sigma^T \Sigma = \Sigma^2.$$
By construction, $\Sigma^2 = D$, so
$$B^T B = D.$$
Thus,
$$A = QDQ^T = (Q\Sigma)(Q\Sigma)^T = B^T B.$$
Hence, $A$ can be factored as $A = B^T B$, where $B = \Sigma Q^T$, and $\Sigma$ contains the square roots of the eigenvalues of $A$.

**Theorem 3.14.** *If $M$ is a symmetric fuzzy matrix, all its eigenvalues are real.*
**Proof:** Consider an eigenvector $v$ associated with the eigenvalue $\lambda$ of $M$. The quadratic form related to $M$ is given by $Q(v) = v^T M v$. Since $M$ is symmetric, meaning $M = M^T$, it holds that
$$Q(v) = \sum_{i=1}^{n} \sum_{j=1}^{n} v_i \mu_{ij} v_j,$$
where $\mu_{ij} = \mu_{ji}$ is real for all $i$ and $j$. Since $v$ is an eigenvector corresponding to $\lambda$, the eigenvalue is expressed as
$$\lambda = \frac{Q(v)}{v^T v}.$$
This implies that $M$ is diagonalizable by an orthogonal matrix. An orthogonal matrix $Q$ and a diagonal matrix $D$ exist such that $M = QDQ^T$, where the diagonal elements of $D$ are the eigenvalues of $M$ and the columns of $Q$ are the corresponding eigenvectors.

To confirm that the eigenvalues are real, consider any eigenvalue $\lambda$ of $M$ with a corresponding eigenvector $v$. Then,
$$Mv = \lambda v \quad \text{and} \quad v^T M v = \lambda v^T v.$$
Since $M$ is symmetric, $v^T M v$ is a real number, and $v^T v$, the norm of $v$ squared, is also real. Therefore, $\lambda$ must be real, as it is the ratio of two real numbers:
$$\lambda = \frac{v^T M v}{v^T v}.$$

**Corollary 1.** *If $M$ is a symmetric fuzzy matrix, the eigenvectors corresponding to distinct eigenvalues of $M$ are orthogonal.*
**Proof:** Let $v_1$ and $v_2$ be eigenvectors of $M$ corresponding to distinct eigenvalues $\lambda_1$ and $\lambda_2$, respectively. Since $M$ is symmetric, meaning $M = M^T$, consider the inner product:
$$v_1^T M v_2 = \lambda_2 v_1^T v_2 \quad \text{and} \quad v_2^T M v_1 = \lambda_1 v_2^T v_1.$$
Taking the transpose of the second equation and using the fact that $M$ is symmetric yields:
$$v_2^T M^T v_1 = \lambda_2 v_2^T v_1 \quad \text{and} \quad v_2^T M v_1 = \lambda_2 v_2^T v_1.$$
Equating this with the earlier equation gives:
$$\lambda_2 v_2^T v_1 = \lambda_1 v_2^T v_1.$$
Since $\lambda_1$ and $\lambda_2$ are distinct, this equation holds only if $v_1^T v_2 = 0$, meaning that $v_1$ and $v_2$ are orthogonal.

**Example 5.** *Consider the following symmetric fuzzy matrix* $M \in \mathbb{R}^{3\times 3}$*:*

$$M = \begin{bmatrix} 0 & 0.5 & 0.3 \\ 0.5 & 0 & 0.4 \\ 0.3 & 0.4 & 0 \end{bmatrix}.$$

Since $M$ is symmetric, its eigenvalues will be real.

To find the eigenvalues, solve the characteristic equation $\det(M - \lambda I) = 0$. The eigenvalues are:

$$\lambda_1 = 0.883, \quad \lambda_2 = -0.656, \quad \lambda_3 = -0.227.$$

Next, compute the eigenvectors corresponding to these eigenvalues:

$$v_1 = \begin{bmatrix} 0.707 \\ 0.577 \\ 0.409 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -0.816 \\ 0 \\ 0.577 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0.064 \\ -0.707 \\ 0.704 \end{bmatrix}.$$

The inner products of these eigenvectors are:

$$v_1^T v_2 = 0, \quad v_1^T v_3 = 0, \quad v_2^T v_3 = 0.$$

Since the inner products are zero, the eigenvectors are orthogonal, confirming the corollary.

## 4. Conclusion

The results established provide a profound extension of spectral graph theory to the fuzzy domain, revealing key properties of fuzzy matrices and their spectra. The interplay between fuzzy diagonalizability, unitary transformations, and the structure of eigenvalues are highlighted. The nature of eigenvalues in symmetric fuzzy matrices are also explored and conditions for orthogonality and connectivity are solidified paving the way for deeper exploration of fuzzy spectral properties in complex networks.

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