Journal of Physical Sciences, Vol. 12, 2008, 51-64

Unsteady Viscous Incompressible Flow Due to an Oscillating Plate in a Rotating Fluid

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Received November 25, 2008; accepted December 10, 2008

ABSTRACT

An analytical solution is obtained for the flow of a viscous incompressible fluid due to an oscillating plate in a rotating system. An exact solution of the governing equations has been obtained by using Laplace transform technique. The velocity distribution and the shear stresses at the plate have been obtained for cosine as well as sine oscillations of the plate. The steady-state solution as well as the transient solution have also been derived. It is observed that for large time the transient solution tends to zero. It is also found that the steady-state solution does not exist when the rotation parameter is equal to the frequency parameter.

1. Introduction

The unsteady flow of the viscous incompressible fluid due to an oscillation of plane wall was studied by Erdogan[1]. He provides the steady-state solution as well as transient solution for both cosine and sine oscillations of the plate. Penton [2] has discussed the transient solution for the flow due to an oscillating plate. He has assumed that for large times steady-state flow is set-up with the same frequency as the velocity of the plane boundary. Tokuda [3] has studied the impulsive motion of a flat plate in a viscous fluid. Zeng and Weinbaum[4] has investigated the Stokes' problem for moving plane. There is another class of problem where both the fluid and the plate rotate in unison with uniform angular velocity. It has many applications in cosmical and geophysical fluid dynamics. Other possible applications of this problem are in acoustics and optics. The unsteady flow of a viscous incompressible fluid in a rotating system have been studied by Thornley [5], Pop and Soundalgekar[6], Puri [7], Gupta and Gupta[8], Deka et al. [9] and many other researchers. Flow in the Ekman layer on a oscillating plate have been studied by Gupta et al. [10]. On the other hand, hydromagnetic flow in the Ekman layer on an oscillating porous plate have been studied by Guria and Jana [11].

In this paper, we have considered the unsteady flow of a viscous incompressible fluid due to an oscilletion of a plate in a rotating system where both the fluid and the plate rotate in unison with uniform angular velocity as well as the plate oscillates non-torsionally. The fluid and the plate rotate in unison with uniform angular velocity Ω about an axis perpendicular to the plate. It is found that for large times the starting solution tends to the steady-state solution. The steady-state solution does not exist when the frequency parameter is equal to the two times rotation parameter. It is also found that for large time the transient solution vanishes.

2. Mathematical formulation and its solution

Consider the unsteady flow of a viscous incompressible fluid, occupying the region $z > 0$, rotating with uniform angular velocity Ω about the z-axis normal to the plate. The plate is oscillating in its own plane with the velocity $U(t)$. At time $t = 0$, the fluid is at rest. At time $t > 0$, the plate starts to oscillate in its own plane with sinusoidal variation of velocity. The velocity components are (u, v, w) relative to a rotating frame of reference. Since the plate is infinitely long, all physical quantities will be function of z and t only. The equation of continuity $\nabla.\vec{q} = 0$ gives $\frac{\partial w}{\partial z} = 0$ which on integration yields $w = \text{constant} = 0$, everywhere in the flow. The Navier- Stokes' equations of motion in a rotating frame of reference yields

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial z^2} + 2\Omega v, \tag{1}$$

$$\frac{\partial v}{\partial t} = \nu \frac{\partial^2 v}{\partial z^2} - 2\Omega u, \tag{2}$$

where ν is the kinematic coefficient of viscosity.

The initial and the boundary conditions for u and v are

$$u = v = 0 \text{ at } t = 0 \text{ for } z > 0, \tag{3}$$

$$u = U(t), v = 0 \text{ at } z = 0 \text{ for } t > 0; \tag{4}$$

$$u \to 0, v \to 0 \text{ as } z \to \infty \text{ for } t > 0$$

Introduce the non-dimensional variables

$$\eta = \frac{U_0 z}{\nu}, \tau = \frac{U_0{}^2 t}{\nu}, u_1 = \frac{u}{U_0}, v_1 = \frac{v}{U_0}, U(t) = U_0 G(\tau), \tag{5}$$

where U_0 being a constant mean velocity in the x-direction and $G(\tau)$ the non-dimensional oscillatory velocity of the plate.

On the use of (5), equations (1) and (2) become

$$\frac{\partial u_1}{\partial \tau} = \frac{\partial^2 u_1}{\partial \eta^2} + 2K^2 v_1, \tag{6}$$

$$\frac{\partial v_1}{\partial \tau} = \frac{\partial^2 v_1}{\partial \eta^2} - 2K^2 u_1, \tag{7}$$

where $K^2 = \frac{\Omega \nu}{U_0^2}$ is the rotation parameter.

The initial and boundary conditions (3) and (4) become

$$u_1 = v_1 = 0 \text{ at } \tau = 0 \text{ for } \eta > 0, \tag{8}$$

$$u_1 = G(\tau), v_1 = 0 \text{ at } \eta = 0 \text{ for } \tau > 0;$$

$$u_1 \to 0, v_1 \to 0 \text{ as } \eta \to \infty \text{ for } \tau > 0, \tag{9}$$

Equations (6) and (7) can be written in combined form as

$$\frac{\partial F}{\partial \tau} = \frac{\partial^2 F}{\partial \eta^2} - 2iK^2 F, \tag{10}$$

where

$$F = u_1 + iv_1, \quad i = \sqrt{-1}. \tag{11}$$

Assuming

$$G(\tau) = ae^{i\sigma\tau} + be^{-i\sigma\tau}, \tag{12}$$

the corresponding initial and the boundary conditions (8) and (9) become

$$F(\eta, 0) = 0, \; F(0, \tau) = ae^{i\sigma\tau} + be^{-i\sigma\tau}, \; F(\infty, \tau) = 0, \tag{13}$$

where a and b are complex constants and $\sigma = \frac{\omega\nu}{U_0^{\,2}}$, is the non-dimensional frequency of the oscillation.

To solve the equation (10), we assume

$$F(\eta, \tau) = H(\eta, \tau) e^{-2iK^2\tau}. \tag{14}$$

On the use of (14), equation (10) becomes

$$\frac{\partial H}{\partial \tau} = \frac{\partial^2 H}{\partial \eta^2}, \tag{15}$$

with the initial and the boundary conditions

$$H(\eta, 0) = 0, \; H(0, \tau) = ae^{i(\sigma + 2K^2)\tau} + be^{-i(\sigma - 2K^2)\tau}, \; H(\infty, \tau) = 0. \tag{16}$$

Taking Laplace transform of (15) and using initial condition (16), we get

$$\frac{d^2 \overline{H}}{d\eta^2} = p\overline{H}, \tag{17}$$

where

$$\overline{H} = \int_0^\infty H(\eta, \tau) e^{-p\tau} d\tau \tag{18}$$

The boundary conditions (16) become

$$\overline{H}(0, \tau) = \frac{a}{p - i(\sigma + 2K^2)} + \frac{b}{p + i(\sigma - 2K^2)}, \; \overline{H}(\infty, \tau) = 0. \tag{19}$$

The solution of (17) subject to the boundary conditions (19) is

$$\overline{H}(\eta, \tau) = \left[\frac{a}{p - i(\sigma + 2K^2)} + \frac{b}{p + i(\sigma - 2K^2)} \right] e^{-\sqrt{p}\,\eta}. \tag{20}$$

The inverse transform of the equation (20) gives

$$H(\eta,\tau)=\frac{1}{2}e^{2iK^2\tau}\left[ae^{i\sigma\tau}\left\{e^{\sqrt{r_1}\eta}erfc\left(\frac{\eta}{2\sqrt{\tau}}+\sqrt{r_1\tau}\right)+e^{-\sqrt{r_1}\eta}erfc\left(\frac{\eta}{2\sqrt{\tau}}-\sqrt{r_1\tau}\right)\right\}\right.$$
$$\left.+be^{-i\sigma\tau}\left\{e^{\sqrt{r_{2,3}}\eta}erfc\left(\frac{\eta}{2\sqrt{\tau}}+\sqrt{r_{2,3}\tau}\right)+e^{-\sqrt{r_{2,3}}\eta}erfc\left(\frac{\eta}{2\sqrt{\tau}}-\sqrt{r_{2,3}\tau}\right)\right\}\right], \quad (21)$$

where

$$r_{1,2}=i(2K^2\pm\sigma),\ r_3=-i(\sigma-2K^2). \quad (22)$$

Substituting the value of $H(\eta,\tau)$ in the equation (14), we get

$$F(\eta,\tau)=\frac{1}{2}ae^{i\sigma\tau}\left[e^{(1+i)\alpha_1\eta}erfc\left(\frac{\eta}{2\sqrt{\tau}}+(1+i)\alpha_1\sqrt{\tau}\right)\right.$$
$$\left.+e^{-(1+i)\alpha_1\eta}erfc\left(\frac{\eta}{2\sqrt{\tau}}-(1+i)\alpha_1\sqrt{\tau}\right)\right]$$
$$+\frac{1}{2}be^{-i\sigma\tau}\left[e^{(1\pm i)\alpha_{2,3}\eta}erfc\left(\frac{\eta}{2\sqrt{\tau}}+(1\pm i)\alpha_{2,3}\sqrt{\tau}\right)\right.$$
$$\left.+e^{-(1\pm i)\alpha_{2,3}\eta}erfc\left(\frac{\eta}{2\sqrt{\tau}}-(1\pm i)\alpha_{2,3}\sqrt{\tau}\right)\right], \quad (23)$$

where

$$\alpha_1=\frac{1}{\sqrt{2}}\left(\sigma+2K^2\right)^{1/2},\ \alpha_2=\frac{1}{\sqrt{2}}\left(2K^2-\sigma\right)^{1/2},\ \alpha_3=\frac{1}{\sqrt{2}}\left(\sigma-2K^2\right)^{1/2}. \quad (24)$$

In the equation (23), we use positive sign and 2 for $2K^2>\sigma$ and negative sign and 3 for $2K^2<\sigma$. For $2K^2=\sigma$, we have

$$F(\eta,\tau)=\frac{1}{2}ae^{i\sigma\tau}\left[e^{(1+i)\alpha_1\eta}erfc\left(\frac{\eta}{2\sqrt{\tau}}+(1+i)\alpha_1\sqrt{\tau}\right)\right.$$
$$\left.+e^{-(1+i)\alpha_1\eta}erfc\left(\frac{\eta}{2\sqrt{\tau}}-(1+i)\alpha_1\sqrt{\tau}\right)\right]+be^{-i\sigma\tau}erfc\left(\frac{\eta}{2\sqrt{\tau}}\right) \quad (25)$$

where α_1 is given by (24). When $a=b=\frac{1}{2}$ (for cosine oscillations of the plate) with $K^2=0$ then $v_1=0$ and the equation (23) coincides with equation (8) of Erdogan [1] with slight change of notation. Further, if $a=\frac{1}{2i}, b=-\frac{1}{2i}$ (for sine oscillations of the plate) with $K^2=0$ then $v_1=0$ and the equation (23) reduces to the equation (15) of Erdogan [1].

As $\tau\to\infty$,

$$erfc\left(\frac{\eta}{2\sqrt{\tau}}+(1+i)\alpha_1\sqrt{\tau}\right)\to 0,\ erfc\left(\frac{\eta}{2\sqrt{\tau}}-(1+i)\alpha_1\sqrt{\tau}\right)\to 2$$
$$erfc\left(\frac{\eta}{2\sqrt{\tau}}+(1\pm i)\alpha_{2,3}\sqrt{\tau}\right)\to 0,\ erfc\left(\frac{\eta}{2\sqrt{\tau}}-(1\pm i)\alpha_{2,3}\sqrt{\tau}\right)\to 2 \quad (26)$$

and using these results (23) becomes

$$F_s(\eta,\tau)=a\,e^{i\sigma\tau-(1+i)\,\alpha_1\eta}+b\,e^{-i\sigma\tau-(1\pm i)\alpha_{2,3}\eta}, \quad (27)$$

where F_s is the steady -state solution. Hence for large times the starting solution tends to the steady-state solution. It is found that the steady-state solution exists for large values of time only, it is independent of the initial conditions given by equation (3). For $2K^2=\sigma$ no steady state solution exists. The transient solution is obtained by the substraction of equation (27) from equation (23) as

$$\begin{aligned}F_\tau(\eta,\tau)=\frac{1}{2}ae^{i\sigma\tau}\Bigg[&e^{(1+i)\alpha_1\eta}erfc\left((1+i)\alpha_1\sqrt{\tau}+\frac{\eta}{2\sqrt{\tau}}\right)\\&-e^{-(1+i)\alpha_1\eta}erfc\left((1+i)\alpha_1\sqrt{\tau}-\frac{\eta}{2\sqrt{\tau}}\right)\Bigg]\\&+\frac{1}{2}be^{-i\sigma\tau}\Bigg[e^{(1\pm i)\alpha_{2,3}\eta}erfc\left((1\pm i)\alpha_{2,3}\sqrt{\tau}+\frac{\eta}{2\sqrt{\tau}}\right)\\&-e^{-(1\pm i)\alpha_{2,3}\eta}erfc\left((1\pm i)\alpha_{2,3}\sqrt{\tau}-\frac{\eta}{2\sqrt{\tau}}\right)\Bigg], \quad (28)\end{aligned}$$

where F_τ denotes the transient solution. It is observed that for large time the transient solution given by (28) vanishes. The plate oscillates with velocities $\cos\sigma\tau$ and $\sin\sigma\tau$ according as $a=b=\frac{1}{2}$ and $a=\frac{1}{2i}, b=-\frac{1}{2i}$ respectively. If $\sigma=0$ and $a=b=\frac{1}{2}$ then the plate starts with the uniform velocity U_0 impulsively.

3. Shear stresses

The shear stresses at the plate $\eta=0$ due to the primary and the secondary flows are given by

$$\begin{aligned}\tau_x+i\tau_y&=\left[\frac{\partial(u_1+iv_1)}{\partial\eta}\right]_{\eta=0}\\&=-\Bigg[ae^{i\sigma\tau}\left\{(1+i)\alpha_1\,erfc(1+i)\alpha_1\sqrt{\tau}+\frac{2}{\sqrt{\pi\tau}}e^{-2i\alpha_1^2\tau}\right\}\\&\quad+b\,e^{-i\sigma\tau}\left\{(1\pm i)\alpha_{2,3}\,erfc(1\pm i)\alpha_{2,3}\sqrt{\tau}+\frac{2}{\sqrt{\pi\tau}}e^{\mp 2i\alpha_{2,3}^2\tau}\right\}\Bigg]. \quad (29)\end{aligned}$$

In the equation (29), we use positive sign and 2 for $2K^2 > \sigma$ and negative sign and 3 for $2K^2 < \sigma$. For $2K^2 = \sigma$, the shear stresses at the plate $\eta = 0$ due to the primary and the secondary flows are given by

$$\tau_x + i\tau_y = \left[\frac{\partial(u_1 + iv_1)}{\partial\eta}\right]_{\eta=0}$$

$$= -\left[ae^{i\sigma\tau}\left\{(1+i)\alpha_1 erfc(1+i)\alpha_1\sqrt{\tau} + \frac{2}{\sqrt{\pi\tau}}e^{-2i\alpha_1^2\tau}\right\} + \frac{b}{\sqrt{\pi\tau}}e^{-i\sigma\tau}\right]. \quad (30)$$

Substituting $a = b = \frac{1}{2}$ in the equations (29) and (30), we obtain shear stress components when the plate oscillates with velocity $\cos\omega\tau$. Similarly, for the sine oscillation $\sin\omega\tau$ of the plate we take $a = \frac{1}{2i}$ and $b = -\frac{1}{2i}$ in the equations (29) and (30).

4. Results and discussion

Now, we discuss the following cases:

Case I: For cosine oscillation of the plate, we substitute $a = b = \frac{1}{2}$ in the equation (23). The numerical values of the velocity components due to cosine oscillation of the plate in a rotating system for different values of rotation parameter K^2, frequency parameter σ, phase angle $\sigma\tau$ and time τ, are plotted against η in Figs.1-4. The primary velocity profile u_1 and the secondary velocity profile v_1 are shown in Fig.1 for several values of K^2 with $\sigma = 2$, $\sigma\tau = \frac{\pi}{2}$ and $\tau = 0.2$. It is observed that the primary velocity decreases and the magnitude of secondary velocity increases with increase in K^2. In Fig.2 the velocity profiles are shown for different values of frequency parameter σ with $K^2 = 3$, $\tau = 0.2$ and $\sigma\tau = \frac{\pi}{2}$. It is seen that both the primary velocity and the secondary velocity increases with increase in σ. Fig.3 indicates the variations of phase angle $\sigma\tau$ on the primary and the secondary flows with $K^2 = 3, \tau = 0.2$ and $\sigma = 2$. It is found that both the primary velocity and the secondary velocity decrease with increase in $\sigma\tau$. Fig.4 shows the effect of time τ on the primary and the secondary flows for $K^2 = 3$, $\sigma = 2$ and $\sigma\tau = \frac{\pi}{2}$. It is seen that both the primary velocity and the magnitude of the secondary velocity increase with increase in time τ.

In Figs.5-6 the non-dimensional shear stresses τ_x and τ_y due to the primary and the secondary flows at the plate $\eta = 0$ are drawn for different values of the phase angle $\sigma\tau$ and the rotation parameter K^2 against frequency parameter σ on taking $a = b = \frac{1}{2}$. Fig.5 shows that for fixed values of $\sigma\tau$ and τ, with increase in σ the shear stress τ_x increases while the magnitude of τ_y decreases. It is also shows that both τ_x and the magnitude of τ_y decrease with increse in K^2. On the other hand, it is found from Fig.6 that for fixed values of K^2 and $\sigma\tau$ the magnitude of both τ_x and τ_y decrease with increase in $\sigma\tau$.

Case II: For sine oscillation of the plate, we substitute $a = \frac{1}{2i}$, $b = -\frac{1}{2i}$ in the equation (23). The numerical values of the velocity components due to sine oscillation of the plate in a rotating system for different values of rotation parameter K^2, frequency parameter σ, phase angle $\sigma\tau$ and time τ are depicted graphically against η in Figs.7-10. The primary velocity u_1 and the secondary velocity v_1 are shown in Fig.7 for several values of K^2 with $\sigma = 2$, $\sigma\tau = \frac{\pi}{2}$ and $\tau = 0.2$. It is observed that both the primary velocity and the magnitude of the secondary velocity decrease with increase in K^2. In Fig.8 the velocity components are shown for different values of frequency parameter σ with $K^2 = 3, \tau = 0.2$ and $\sigma\tau = \frac{\pi}{2}$. It is seen that the primary velocity u_1 decreases whereas the secondary velocity increases with increase in σ. Fig.9 indicates the variations of phase angle $\sigma\tau$ on the primary and secondary velocities for $K^2 = 3$, $\tau = 0.2$ and $\sigma = 2$. It is found that the primary velocity increases but the secondary velocity decreases with increase in $\sigma\tau$. Fig.10 shows the effect of time τ on the velocity components for $K^2 = 3, \sigma = 2$ and $\sigma\tau = \frac{\pi}{2}$. It is seen that both the primary and secondary velocities decrease with increase in time τ.

In figs.11-12 the non-dimensional shear stresses τ_x and τ_y due to the primary and the secondary flows at the plate $\eta = 0$ are drawn for different values of the phase angle $\sigma\tau$ and the rotation parameter K^2 against frequency parameter σ. Fig.11 shows that for fixed values of $\sigma\tau$ and τ, both τ_x and τ_y decreases with increase in K^2. It is also shows that τ_x decreases while τ_y increases with increase in σ. and increases with increase in $\sigma\tau$ for fixed value of K^2 and τ. On the other hand, It is observed from Fig.12 that both the shear stress components decrease with increase in $\sigma\tau$ when K^2, σ and τ are fixed.

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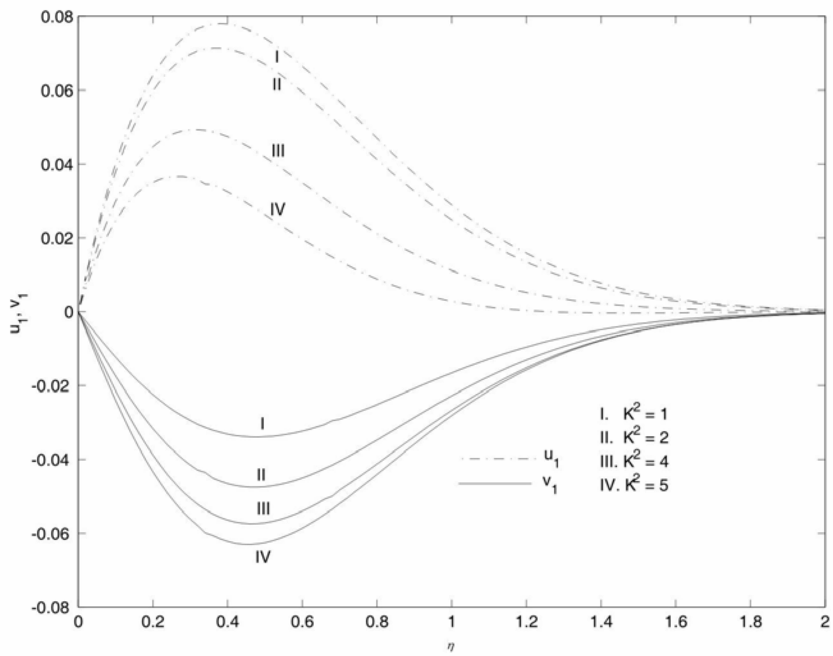

Fig.1: Variations of u_1 and v_1 against η for $\sigma = 2.0$, $\sigma\tau = \dfrac{\pi}{2}$, $\tau = 0.2$.

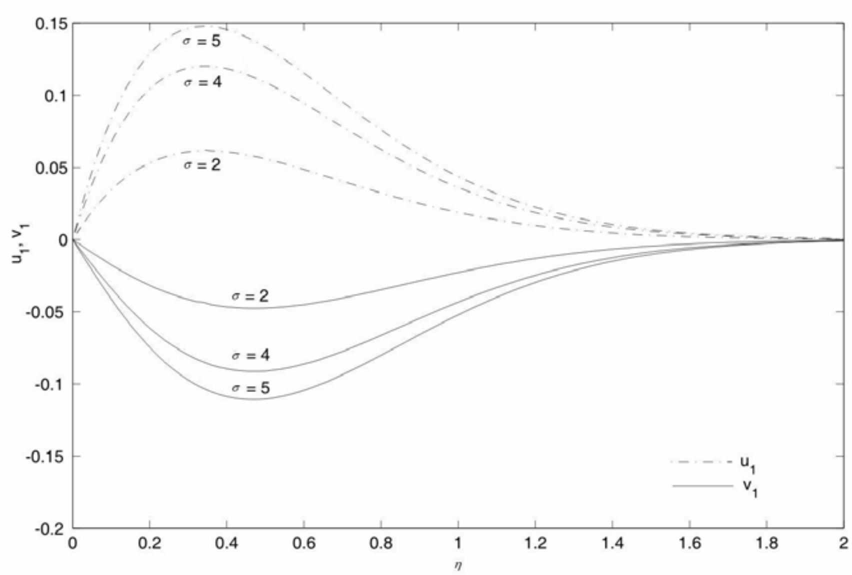

Fig.2: Variations of u_1 and v_1 against η for $K^2 = 3.0$, $\sigma\tau = \dfrac{\pi}{2}$, $\tau = 0.2$.

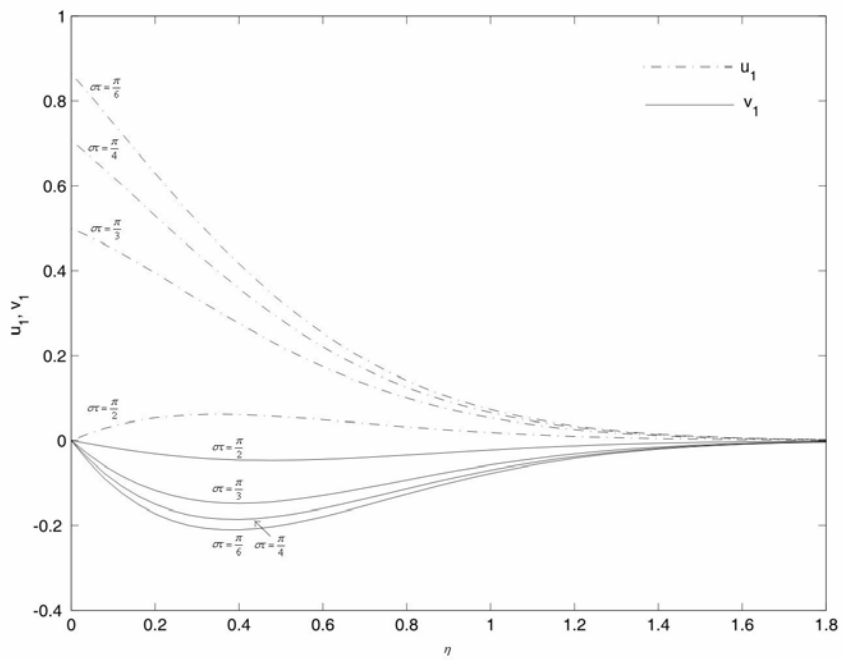

Fig.3: Variations of u_1 and v_1 against η for $K^2 = 3.0$, $\sigma = 2.0$, $\tau = 0.2$

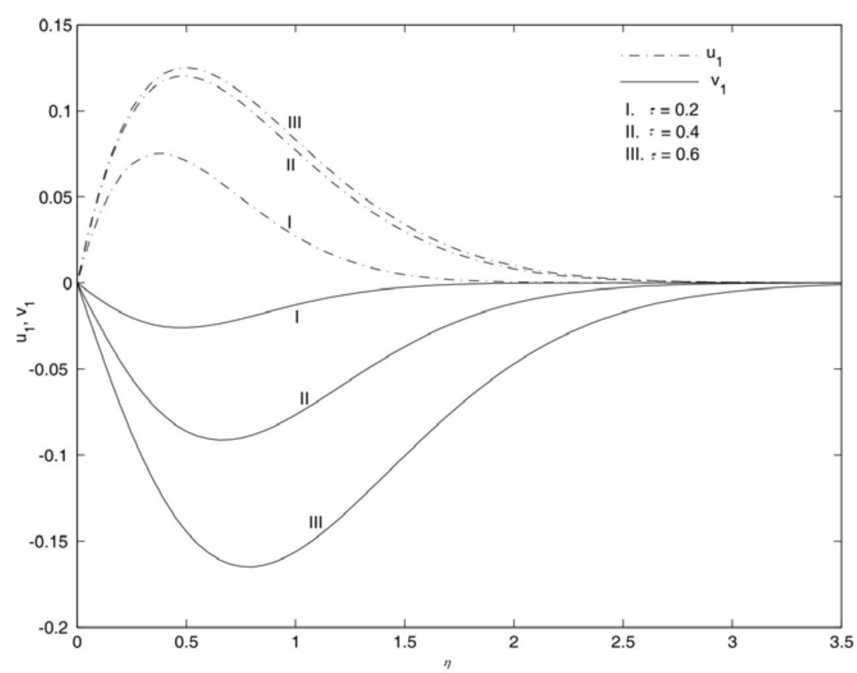

Fig.4: Variations of u_1 and v_1 against η for $K^2 = 3.0$, $\sigma = 2.0$, $\sigma\tau = \dfrac{\pi}{2}$.

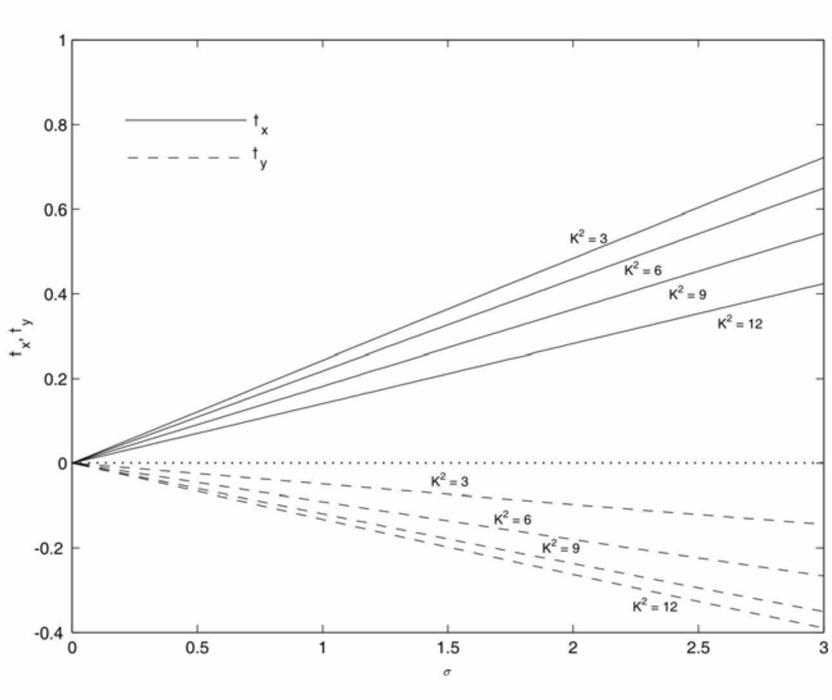

Fig.5: Variations of τ_x and τ_y for $\sigma\tau = \dfrac{\pi}{2}$ and $\tau = 0.2$.

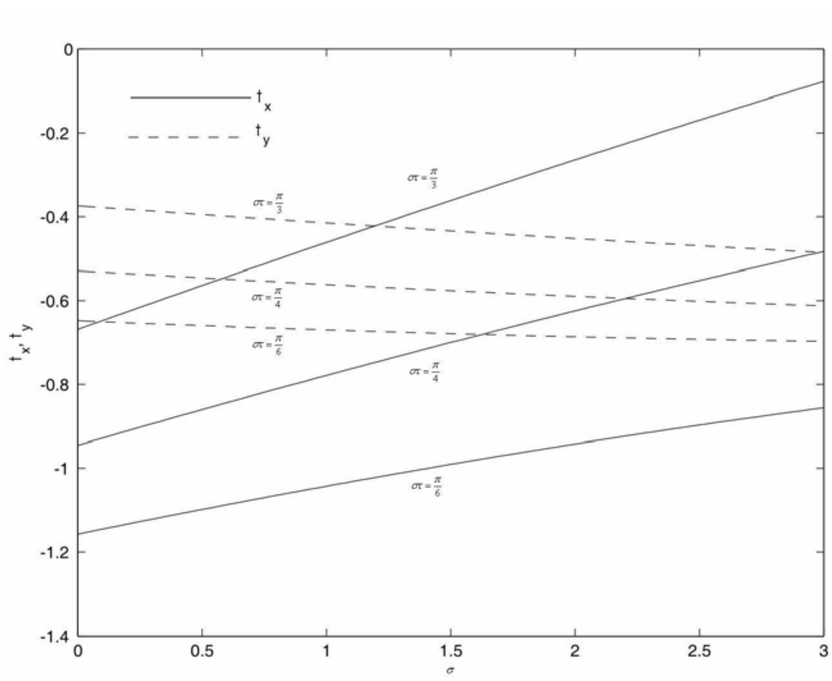

Fig.6: Variations of τ_x and τ_y for $K^2 = 3.0$ and $\tau = 0.2$.

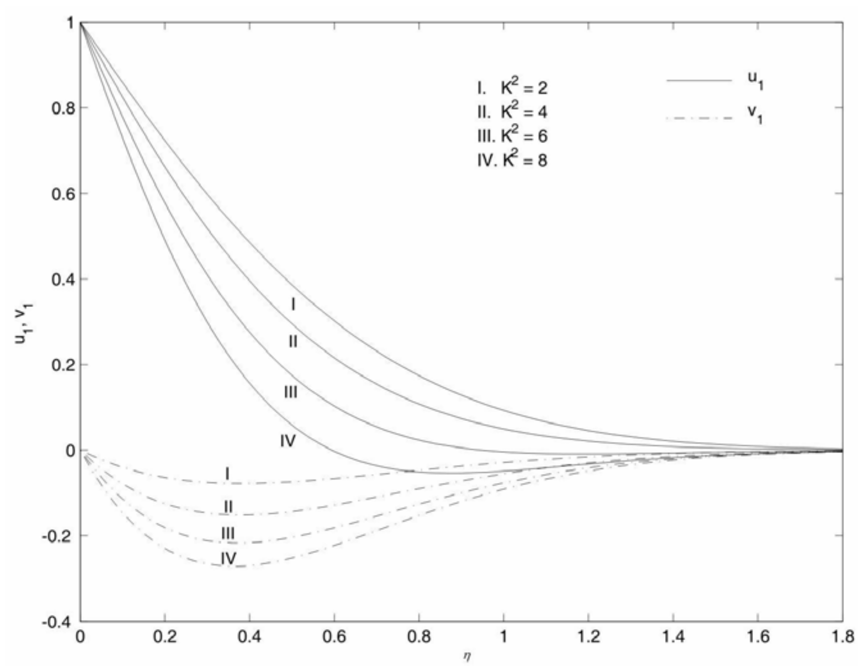

Fig.7: Variations of u_1 and v_1 against η for $\sigma = 2.0$, $\sigma\tau = \dfrac{\pi}{2}$, $\tau = 0.2$.

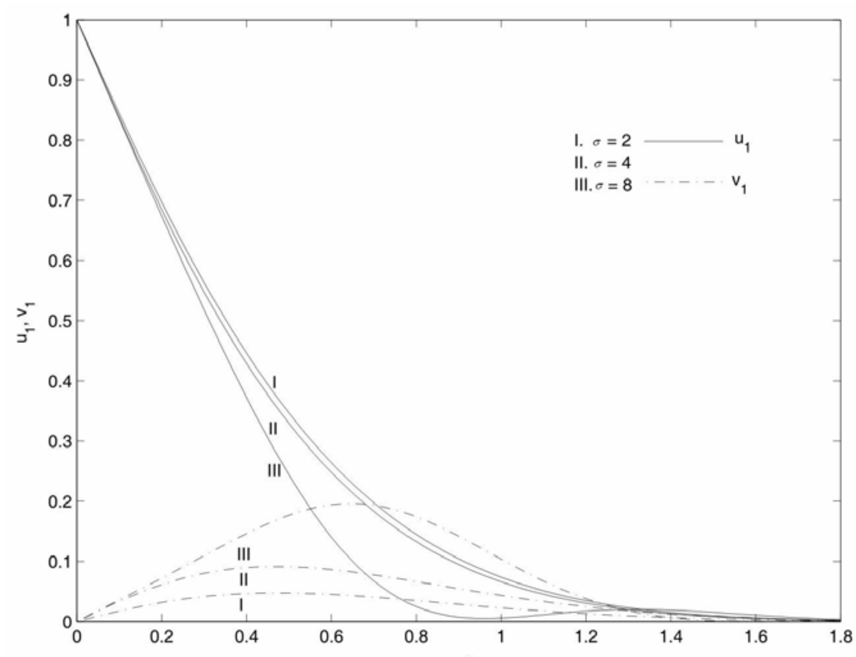

Fig.8: Variations of u_1 and v_1 against η for $K^2 = 3.0$, $\sigma\tau = \dfrac{\pi}{2}$, $\tau = 0.2$.

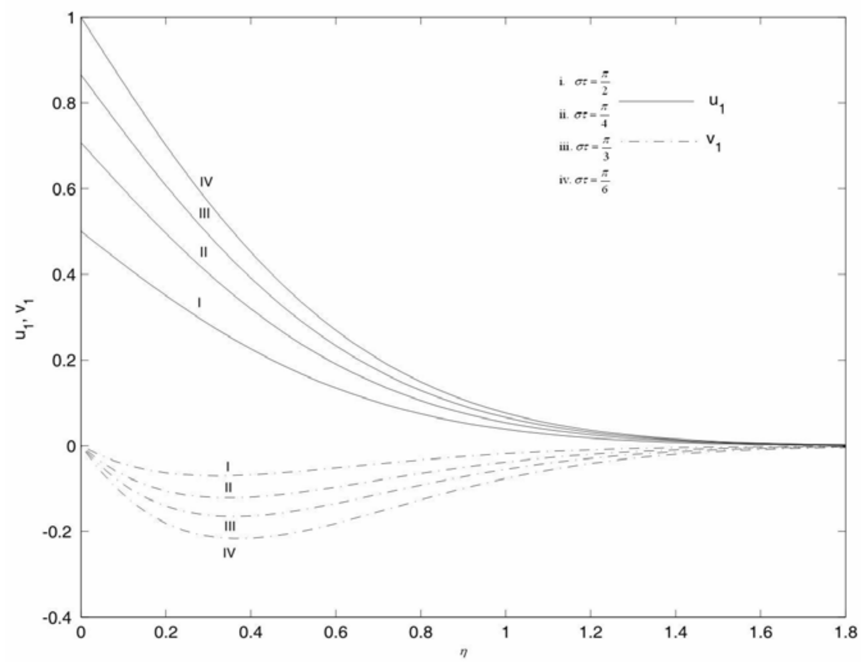

Fig.9: Variations of u_1 and v_1 against η for $K^2 = 3.0$, $\sigma = 2.0$, $\tau = 0.2$.

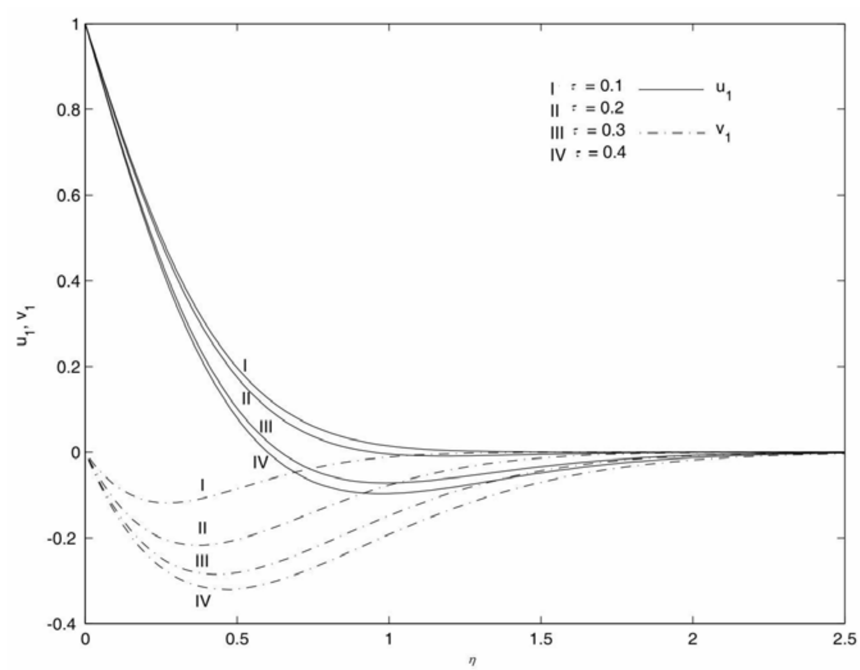

Fig.10: Variations of u_1 and v_1 against η for $K^2 = 3.0$, $\sigma = 2.0$, $\sigma\tau = \frac{\pi}{2}$.

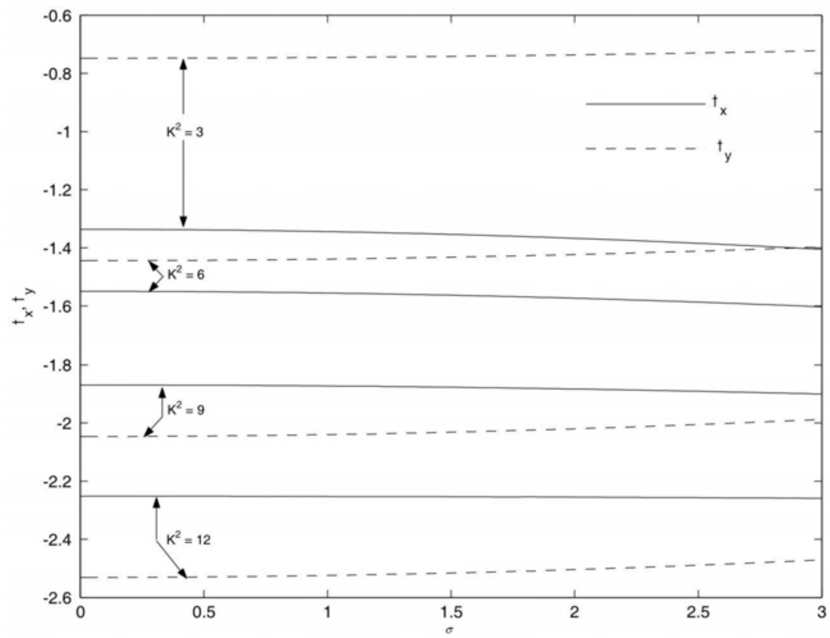

Fig.11: Variations of τ_x and τ_y for $\sigma\tau = \dfrac{\pi}{2}$ and $\tau = 0.2$.

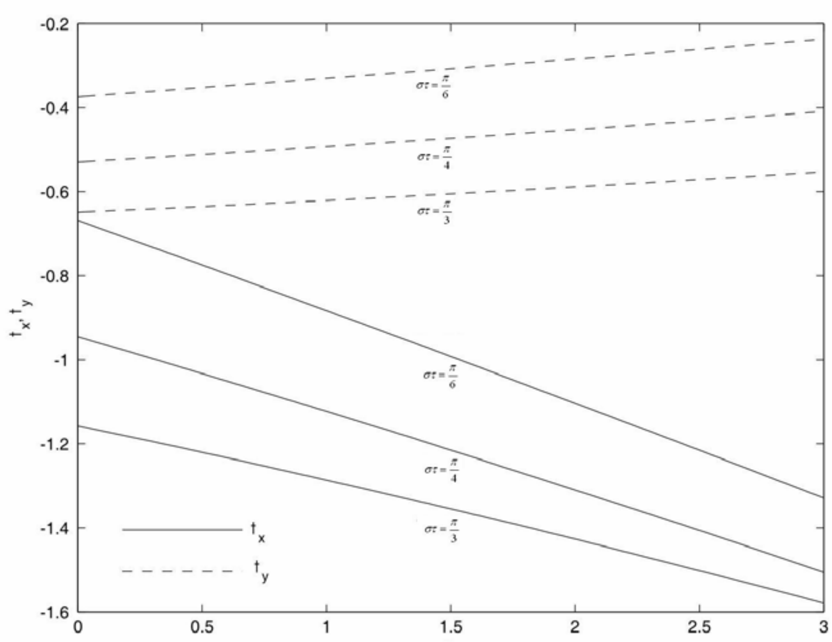

Fig.12: Variations of τ_x and τ_y for $K^2 = 3.0$ and $\tau = 0.2$.