Journal of Physical Sciences, Vol. 11, 2007, 62-67

The Generalised Inverse of Intuitionistic Fuzzy Matrices

Susanta K. Khan* and Anita Pal**

*Department of Applied Mathematics with Oceanology and Computer Programming
Vidyasagar University, Midnapur - 721102, India
**Department of Mathematics
National Institute of Technology Durgapur
Durgapur, West Bengal, India
e-mail: anita.buie@gmail.com

Received September 9, 2007; accepted November 11, 2007

ABSTRACT

In this paper, we introduce the concept of generalised inverse (g-inverse) for intuitionistic fuzzy matrices (IFMs). We define minus partial ordering and studied several properties of it. Here we have shown that the minus partial ordering is a partial ordering for IFMs.

Key words : *Intuitionistic fuzzy matrix (IFM), generalised inverse (g-inverse), minus partial order.*

1. Introduction

Atanassov introduced the idea of intuitionistic fuzzy sets [1] in the year 1983. Later on much fundamental works have been done with this concept by Atanassov and others. Several authors presented a number of results on fuzzy matrices. Kim and Roush [4] studied the canonical form of an idempotent matrix. Hashimoto [2] studied the canonical form of a transitive fuzzy matrix. Xin [9, 10] studied controllable fuzzy matrices. Hemasinha et al. [3] investigated iterates of fuzzy circulants matrices. Thomson [8] and Kim [5] defined the adjoint of square fuzzy matrix. Pal [6] introduced intuitionistic fuzzy determinant. Pal, Khan and Shaymal [7] studied intuitionistic fuzzy matrices (IFMs). Khan and Pal [11] studied intuitionistic fuzzy tautological matrices and also studied interval-valued intuitionistic fuzzy matrices [13].

Now, we define fuzzy matrices (FMs) and the intuitionistic fuzzy matrices (IMFs) in the following.

Definition 1. ***A* fuzzy matrix (FM)** *A of order $m \times n$ is defined as $A=[a_{ij}, a_{ij\mu}]$ where $a_{ij\mu}$ is the membership value of the element a_{ij} in A. For simplicity, we write A as* $A = [a_{ij\mu}]$.

Definition 2. *[7]* ***An* intuitionistic fuzzy matrix (IFM)** *A of order $m\times n$ is defined as $A=[x_{ij\mu}, <a_{ij\mu}, a_{ij\nu}>]$ where $a_{ij\mu}$ and $a_{ij\nu}$ are called membership and non-membership values of x_{ij} in A, maintaining the condition* $0 \le a_{ij\mu} + a_{ij\nu} \le 1$.

For simplicity, we may write $A = [x_{ij}, a_{ij}]$ or $[a_{ij}]_{m \times n}$ where $a_{ij} = < a_{ij\mu}, b_{ij\mu}>$.

Let a and b be two elements of an IFM A then $a + b = <\max\{a_{ij\mu}, b_{ij\mu}\}, \min\{a_{ijv}, b_{ijv}\}>$ and $a.b = <\min\{ a_{ij\mu}, b_{ij\mu}\},\max\{a_{ijv}, b_{ijv}\}>$. For simplicity, we denote $a.b$ as ab.

Definition 3. *An IFM A of order $m\times n$ is said to be* **regular** *if there exists X which is also an IFM such that $AXA = A$.*

Definition 4. *If A and X are two IFMs of order $m\times n$ satisfies the relation $AXA = A$ then X is called a* **generalised inverse** (g-inverse) *of A which is denoted by A^-. The g-inverse of an IFM (also FM) is not necessarily unique. We denote the set of all g-inverses by A{1}.*

Definition 5. If *two IFMs A and X of order $m\times n$ satisfies the following equations*

$$AXA = A,\ XAX = X,\ (AX)^T = AX \text{ and } (XA)^T = XA$$

then X is called Moore-Penrose inverse of A which is denoted by A^+.

For FM A^+ need not exist. But, if Moore-Penrose inverse of an FM A exists, then it is unique and coincides with its transpose A^T [4].

Definition 6. *Let A and B be two IFMs of order $m \times n$. The minus ordering is denoted by $A \le B$. If $A^- A = A^- B$ and $AA^- = BA^-$ then we say $A \le B$.*

Conversely, if $A \le B$ then $A^- A = A^- B$ and $AA^- = BA^-$.

Definition 7. *The row space R(A) is the subspace of the set of all FMs of order $m \times n$, generated by the rows of A. Similarly, column space of A is denoted by C(A) and is generated by the columns of A.*

2. Properties of Minus Ordering

Property 1. *The following are equivalent for the IFMs A and B*

(i) $A \le B$
(ii) $A = AA^- B = BA^- A = BA^- B$.

Proof. (i) $\Rightarrow$ (ii).
$A \le B$
$AA^- = BA^-$ and $A^- A = A^- B$ for some $A^- \in A\{1\}$.

Now, $A = A(A^- A) = AA^- B = (AA^-)A = BA^- A = B(A^- A) = BA^- B$.

(ii) $\Rightarrow$ (i).
Let $X = A^{-}AA^{-}$

$AXA = A(A^{-}AA^{-})A = (AA^{-}A)A^{-}A = A.$

$\Rightarrow$ X is g-inverse.

Now

$XA = (A^{-}AA^{-})AA^{-}B = A^{-}(AA^{-}A)A^{-}B = (A^{-}AA^{-})B = XB.$

Similarly, $AX = BX$.

Hence $A \le B$ for $X \in A\{1\}$.

Property 2. *Let A, B be two IFMs. If $A \le B$ then $B\{1\} \subseteq A\{1\}$.*

Proof. $A \le B \Rightarrow A = AA^{-}B = BA^{-}A$ (by above property)

For $B^{-} \in B\{1\}$.

$AB^{-}A = (AA^{-}B)B^{-}(BA^{-}A) = AA^{-}\{BB^{-}B)A^{-}A$

$= (AA^{-}B)A^{-}A = AA^{-}A = A.$

Hence, $AB^{-}A = A$ for each $B^{-} \in B\{1\}$.

Therefore, $B\{1\} \subseteq A\{1\}$.

Property 3. *If $A \le B$ and B is idempotent then B is a g-inverse of A.*

Proof. Since B is idempotent, B is regular and B itself is a g-inverse of B. Here $B \in B\{1\}$. Then by above property $B\{1\} \subseteq$ A{1}. Hence B is a g-inverse of A.

Example. Consider

A=$\begin{bmatrix} <1,0> & <1,0> \\ <.5,.5> & <0,1> \end{bmatrix}$ and B = $\begin{bmatrix} <1,0> & <1,0> \\ <1,0> & <0,1> \end{bmatrix}$

B is not idempotent.

$$A\{1\} = \{X : X = \begin{bmatrix} <1,0> & <\beta,0> \\ <1,0> & <\alpha,0> \end{bmatrix}, 0.5 \le \beta \le 1 \; and \; 0 \le \alpha \le 1 .$$

Here A $\le$ B for

$$A^{-} = \begin{bmatrix} <0,1> & <.5,0> \\ <1,0> & <1,0> \end{bmatrix} \; but \; \mathrm{B} \notin \mathrm{A}\,\{1\} .$$

Property 4. *For any two IFMs A, B, the following are equivalent*

(i) $A \le B$

(ii) $R(A) \subseteq R(B), C(A) \subseteq C(B)$ and $AB^{-}A = A$.

Proof. (i) $\Rightarrow$ (ii)

$A = BA^{-}B = BA^{-}(BB^{-}B) = (BA^{-}B)B^{-}B = AB^{-}B$. Also, $A = AB^{-}B$ for each $B^{-} \in B\{1\}$

$\Rightarrow R(A) \subseteq R(B)$.

Similarly,
$A = BB^{-}A$ for each $B^{-} \in B\{1\}$
$\Rightarrow C(A) \subseteq C(B)$ and also $A = AB^{-}A$.
(ii) $\Rightarrow$ (i)
Let X = $B^{-}AB^{-}$.
$AXA = A(B^{-}AB^{-})A = (AB^{-}A)B^{-}A$
$= AB^{-}A = A$.
$\Rightarrow X \in A\{l\}$.
Now,
$AX = A(B^{-}AB^{-}) = BB^{-}A(B^{-}AB^{-})$
$= BB^{-}(AB^{-}A)B^{-} = BX$.
Similarly, $XA = XB$.
Hence, $A \leq B$ for $X \in A\{1\}$.

Property 5. *In IFMs, the minus ordering $\leq$ is a partial ordering.*

Proof. (i) $A \leq A$ is obvious.
Hence $\leq$ is reflexive.
(ii) $A \leq B$
$\Rightarrow A = BA^{-}B$.
$B \leq A$.
$\Rightarrow B = B^{-}A$.
$A = BA^{-}B = (BB^{-}A)A(AB^{-}B) = BB^{-}(AB^{-}B)$
$= BB^{-}B = B$.
Thus, $A \leq B$ and $B \leq A$.
$\Rightarrow A = B$.
Hence $\leq$ is anti symmetric,
(iii) $A \leq B \Rightarrow A = AB^{-}A$ and $A = AB^{-}B = BB^{-}A$.
Also, $B \leq C \Rightarrow B = BB^{-}C = CB^{-}B$.
Let $X = B^{-}AB^{-}$
Then $AXA = A(B^{-}AB^{-})A = (AB^{-}A)B^{-}A = AB^{-}A = A$.
$\Rightarrow X \in A\{1\}$.
Since, $A \leq B$ and $B \leq C$, by using above result repeatedly, we have
$AX = A(B^{-}AB^{-}) = BB^{-}A(B^{-}AB^{-}) = BB^{-}(AB^{-}A)B^{-}$

$= BB^{-} AB^{-} = (CB^{-}B) B^{-}AB^{-}$

$= CB^{-}(BB^{-}A)B^{-} = C(B^{-}AB^{-}) = CX.$

Similarly, $XA = XC$. Since $X \in A\{1\}$ with $AX = CX$ and $XA = XC$, it follows that $A \leq C$. Hence $\leq$ is a partial ordering.

3. Properties of Minus Partial Ordering

In the following, we present some results regarding minus partial order. These results are also valid for FMs. Here we shown that the results are also valid for IFMs.

Property 6. *If A and B are two IFMs, then* $A \leq B \Rightarrow A^T \leq B^T$.

Proof. $A \leq B \Rightarrow AA^{-} = BA^{-}$ and $A^{-}A = A^{-}B$.
Here

$AA^{-} = BA^{-} \Leftrightarrow (AA^{-})^T = (BA^{-})^T$

$\Leftrightarrow (A^{-})^T A^T = (A^{-})^T B^T$

$\Leftrightarrow (A^T)^{-} A^T = (A^T)^{-} B^T$.

Hence, $AA^{-} = BA^{-} \Leftrightarrow (A^T)^{-} A^T = (A^T)^{-} B^T$.

Similarly, $A^{-}A = A^{-}B$

$\Leftrightarrow A^T (A^T)^{-} = B^T. (A^T)^{-}$

Hence $A \leq B \Rightarrow A^T \leq B^T$.

Property 7. *Let A and B be two IFMs. If A≤ B and B is idempotent then A is idempotent.*

Proof.

$A^2 = A.A = (AA^{-}B)(BA^{-}A) = AA^{-}B^2A^{-}A$

$= (AA^{-}B)A^{-}A$ (since B is idempotent)

$= AA^{-}A = A.$

Remark: It can be shown by example that if $A \leq B$ and A is idempotent then B is not necessarily idempotent.

Property 8. *Let A and B be two IFMs.*
If $A \leq B$ *then* $B^2 = 0 \Rightarrow A^2 = 0$.

Proof.

$A^2 = A.A = (AA^{-}B)(BA^{-}A)$

$= AA^{-}(B^2)A^{-}A = AA^{-}(0)A^{-}A = 0.$

By using the above results one can prove the following result.

Property 9. *Let A and B be two IFMs. If* $A \leq B$*, then* $A + B$ *is regular and* $(A^{-} + B^{-})$ *is a g-inverse of* $A + B$.

REFERENCES

[1] K.Atanassov, Intuitionistic fuzzy sets, in: V.Sgurev, Ed., VII ITKR's session, Sofia, June 1983 (Central Sci. and Techn. Library, Bulg. Academy of Sciences, 1984).

[2] H.Hasimoto, Canonical form of a transitive fuzzy matrix, *Fuzzy Sets and Systems,* 11 (1983) 157-162.

[3] R.Hemasinha, N.R.Pal and J.C.Bezdek, Iterates of fuzzy circulant matrices, *Fuzzy Sets and Systems,* 60 (1993) 199-206.

[4] K.H.Kim and F.W.Roush, Generalized fuzzy matrices, *Fuzzy Sets and Systems,* 4 (1980) 293-315.

[5] J.B.Kim, Determinant theory for fuzzy and Boolean matrices, *Congressus Numerantium* (1988) 273-276.

[6] M.Pal, Intuitionistic fuzzy determinant, *V.U.J.Physical Sciences, 7* (2001) 87-93.

[7] M.Pal, S.Khan and A.K.Shyamal, Intuitionistic fuzzy matrices, *Notes on Intuitionistic Fuzzy Sets,* 8 (2) (2002) 51-62.

[8] M.G.Thomason, Convergence of powers of a fuzzy matrix, *J. Math. Anal. Appl.,* 57 (1977) 476-480.

[9] L.J.-Xin, Controllable fuzzy matrices, *Fuzzy Sets and Systems,* 45 (1992) 313-319.

[10] L.J.-Xin, Convergence of powers of controllable fuzzy matrices, *Fuzzy Sets and Systems,* 62 (1994) 83-88.

[11] S.K.Khan and M.Pal, Intuitionistic fuzzy tautological matrices, *V.U. Journal of Physical Sciences ,* 8 (2002-2003) 92-100.

[12] S.K.Khan and M.Pal, Some operations on intuitionistic fuzzy matrices, *Acta Ciencia Indica,* XXXII M (2006) 515-524.

[13] S.K.Khan and M.Pal, Interval valued intuitionistic fuzzy matrices, *Communicated.*