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Application of Maximum Likelihood Estimation Model in X - MR Control Charts

Ying Li, Yi-min Shi and Liang Wang

Department of Applied Mathematics, Northwestern Polytechnical University,
Xi'an 710072, P. R. China
Email: jupipi2008@eyou.com

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ABSTRACT

Based on the individual and moving range control charts, the maximum likelihood estimation model is proposed to estimate the time of a step change in process variance. It is manifested by Monte Carlo simulation that the estimator coming from this model approaches the actual time very well. We compare the estimator with the first signal time, and come to the conclusion that the estimator detects the process variation more quickly and exactly. So the proposed change point estimator can be easily implemented to improve production efficiency.

Key words **:** Individual and moving range control charts (X -MR charts); Maximum likelihood estimation (MLE); Process variance; Step change

1. Introduction

Control charts are used to determine whether or not a process is in control. In many industrial applications, engineers should search for the special cause immediately when a control chart issues a signal. Knowing when a process changed would simplify the search for the special cause. If the time of the change could be determined, process engineers would have a smaller search window within which to look for the special cause. Consequently, the special cause could be identified more quickly, and appropriate actions needed to improve quality could be implemented sooner. So identifying the time of the process change has received considerable attention. Samuel [1] identified the time of a step change in the mean of a normal process using an *X* chart based on *MLE* method. Similarly, Samuel [2] discussed the time of a step change in process variance of a normal distribution with *R* chart and *S* chart, and the correlative characteristics about *C* chart and *P* chart have been studied by Pignatiello [3,4]. However, there are many situations in which the sample used for process monitoring is an individual unit. At the same time, it is necessary to monitor the process variance. So here we apply *MLE* method to estimate the actual time of a step change in process variance with the *X-MR* charts. Then the performance of the estimator is analyzed by Monte Carlo simulation. At the end of this paper, a numerical example shows that this method is effective and provides process engineers with an estimator of the time of the process change.

2. Individual and Moving Range Control Charts

During the application of control chart in the production process, there are many situations in which the sample size used for process monitoring is $n = 1$; that is, the sample consists of an individual unit. Some examples of these situations are as follows:

(1) Automated inspection and measurement technology is used, and every unit manufactured is analyzed. So there is no basis for rational sub grouping.

(2) The production rate is very slow, and it is inconvenient to allow sample sizes of $n > 1$ to accumulate before analysis. The long interval between observations will cause problems with rational sub grouping.

In such situations, the control chart of individual units is useful. Furthermore, in many applications of the individuals control chart we use the moving range of two successive observations as the basis of estimating the process variance. The moving range is defined as

$$MR_i = |x_i - x_{i-1}|$$

Under circumstances in which individual measurements are taken, a combination of a chart for individual measurements and a moving range chart based on two consecutive observations can be used to simultaneously monitor a process mean and standard deviation. That is to say, the *X-MR* charts can monitor both process mean and variance.

The control limits of the *X -MR* charts are as follows:

(i) X chart : $UCL_x = \mu + M\sigma,$

$LCL_x = \mu - M\sigma$

(ii) MR chart : $UCL_{MR} = R\sigma,$

where μ and σ are process mean and variance, respectively, and both M and R are constants. The different selections of M and R will get corresponding different average run length *(ARL)*. The data should be plotted in turn when the *X-MR* charts are used to monitor process. If any observation is beyond the limits, we consider the process is out of control.

3. Maximum Likelihood Estimation Model for the Step Change Time

3.1. Establishment of the Model

Suppose that the process is initially in control and observations follow a normal distribution $N\left(\mu_0, \sigma_0{}^2\right)$ with known mean μ_0 and known variance σ_0^2. We assume that the process is initially in control. After an unknown point in time τ, the process variance changes from σ_0^2 to $\sigma_1^2 = \delta^2\sigma_0^2$, where δ is the unknown magnitude of the shift. We also assume that once this step change in the process variance occurs, the variance remains at the new level of σ_1^2 until the special cause has been removed. $\hat{\tau}$ is the *MLE* of the change point τ .

Here we take independent observations $X_1, X_2, ..., X_\tau, X_{\tau+1}, ..., X_T$, which follow a normal distribution. Concretely, $X_1, X_2, ..., X_\tau$ come from the in control process with mean μ_0 and variance σ_0^2. After an unknown point in time τ , the following observations $X_{\tau+1}, X_{\tau+2}, ..., X_T$ come from the changed process with mean $\boldsymbol{\mu_0}$ and variance σ_1^2. Let T be the time point which issues the out of control signal. The estimator $\hat{\tau}$ is the value of t that maximizes the logarithm likelihood function. The likelihood function is defined as follows :

$$L\left(\tau, \sigma_1^2 \middle| x\right) = \prod_{i=1}^{\mathrm{T}} f\left(x_i, \sigma^2\right)$$

$$= \prod_{i=1}^{\tau} \frac{1}{\sqrt{2\pi}\sigma_0} \exp\left[-\frac{(x_i - \mu_0)^2}{2\sigma_0^2}\right] \times \prod_{i=\tau+1}^{T} \frac{1}{\sqrt{2\pi}\sigma_1} \exp\left[-\frac{(x_i - \mu_0)^2}{2\sigma_1^2}\right]$$

$$= \left(\frac{1}{\sqrt{2\pi}\sigma_0}\right)^{\tau} \exp\left[-\frac{\sum_{i=1}^{\tau}(x_i - \mu_0)^2}{2\sigma_0^2}\right] \times \left(\frac{1}{\sqrt{2\pi}\sigma_1}\right)^{T-\tau} \exp\left[-\frac{\sum_{i=\tau+1}^{T}(x_i - \mu_0)^2}{2\sigma_1^2}\right]$$

The logarithm of the likelihood function is

$$\log L\left(\tau, \sigma_1^2 \middle| x\right) = -\tau \log\left(\sqrt{2\pi}\sigma_0\right) - (T - \tau)\log\left(\sqrt{2\pi}\sigma_1\right) - \frac{\sum_{i=1}^{\tau}(x_i - \mu_0)^2}{2\sigma_0^2} - \frac{\sum_{i=\tau+1}^{T}(x_i - \mu_0)^2}{2\sigma_1^2} \tag{1}$$

If

$$\frac{\partial}{\partial\sigma_1^2} \log L\left(\tau, \sigma_1^2 \middle| x\right) = 0$$

Then

$$\frac{\sum_{i=\tau+1}^{T}(x_i-\mu_0)^2}{2(\sigma_1^2)^2}-\frac{T-\tau}{2\sigma_1^2}=0$$

it follows that $$\sigma_1^2=\frac{\sum_{i=\tau+1}^{T}(x_i-\mu_0)^2}{T-\tau}$$

For any given $\tau = t$, the logarithm likelihood function (1) is maximized by

$$\hat{\sigma}_1^2=\frac{\sum_{i=t+1}^{T}(x_i-\mu_0)^2}{T-t}$$

Replacing σ_1^2 with $\hat{\sigma}_1^2$ and setting $\tau = t$, simultaneously, we get

$$\log L(\tau|x)=-t\log\left(\sqrt{2\pi}\sigma_0\right)-(T-t)\log\left(\sqrt{2\pi}\hat{\sigma}_1\right)-\frac{\sum_{i=1}^{t}(x_i-\mu_0)^2}{2\sigma_0^2}-\frac{\sum_{i=t+1}^{T}(x_i-\mu_0)^2}{2\hat{\sigma}_1^2}$$

$$=-\frac{\sum_{i=1}^{t}(x_i-\mu_0)^2}{2\sigma_0^2}-\frac{T-t}{2}-t\log\left(\sqrt{2\pi}\sigma_0\right)-(T-t)\log\sqrt{\frac{2\pi\sum_{i=t+1}^{T}(x_i-\mu_0)^2}{T-t}}$$

Thus, the *MLE* of τ is the value of t that maximizes the logarithm likelihood function

$$\hat{\tau}=\arg\max_{0<t<T}\left[-\frac{\sum_{i=1}^{t}(x_i-\mu_0)^2}{2\sigma_0^2}-\frac{T-t}{2}-t\log\left(\sqrt{2\pi}\sigma_0\right)-(T-t)\log\sqrt{\frac{2\pi\sum_{i=t+1}^{T}(x_i-\mu_0)^2}{T-t}}\right] \quad (2)$$

If A_t represents the value of logarithm likelihood function at time t, that is

$$A_t=\frac{\sum_{i=1}^{t}(x_i-\mu_0)^2}{2\sigma_0^2}-\frac{T-t}{2}-t\log\left(\sqrt{2\pi}\sigma_0\right)-(T-t)\log\sqrt{\frac{2\pi\sum_{i=t+1}^{T}(x_i-\mu_0)^2}{T-t}}, \quad (3)$$

Then

$$\hat{\tau}=\arg\max_{0<t<T}(A_t) \quad (4)$$

3.2. Performance Analysis

Here, we use Matlab programs to analyze the performance of the estimator by Monte Carlo simulation. Observations are generated randomly in subgroups of size n = 1 from a normal distribution with mean μ_0 = 5 and variance σ_0^2=1, i.e., N (5, 1^2). For subgroups 1, 2, …, 100, observations were generated with σ_0^2. However, there is a step change in process variance from observation 101. Observations are generated randomly from a normal distribution with the same mean μ_0=5 , but the process variance has shifted from σ_0^2 to $\sigma_1^2 = \delta^2 \sigma_0^2 = \delta^2 x 1^2$.i.e., N (5, δ^2). We assume the variance remains at the new level of σ_1^2 until actions have been taken to identify and remove the special cause. Thus, $X_1, X_2, \ldots, X_{100}$ are the observations from in control process, whereas $X_{101}, X_{102}, \ldots, X_T$ are the observations from the changed process. T denotes the time point where the first out of control signal is detected. The estimator $\bar{\hat{\tau}}$, which we are concerned about, is actually close to the real change time τ.

We now study the performance of our estimator for both increases and decreases in the process variance. Table 1 summarizes the performance of the estimator when there is an increase in the process variance. It includes the situations of $\delta = (\sigma_1/\sigma_0) \in \{1.10, 1.20, 1.30, 1.40, 1.50, 2.00, 2.50\}$. Table 2 shows the performance for identifying the time of the change when there has been a decrease in the process variance. We consider the situation of $\delta \in \{0.80, 0.60, 0.50, 0.40, 0.30, 0.20, 0.10\}$. Here the constants M and R should be chosen such that $M = 3$,$R = 3$. From [5], the *ARL of* different magnitude of shifts for the *X-MR* charts can be determined in the case of M = 3 , R = 3.

The data used in these analysis are in table 1 and table 2, where E(T) and $\bar{\hat{\tau}}$ are the expected run length and the average estimated time of the process change, respectively. We should notice that E(T) is the expected time at which the control chart signals a change in process variance that actually occurred following subgroup 100. Thus, *E(T) = ARL + 100.* Since the actual process change is at time 100 by simulation, the average estimator of change point $\bar{\hat{\tau}}$ should approach to 100. By the way, the average estimator $\bar{\hat{\tau}}$ for each size of change comes from 5,000 simulation runs.

Table 1: Expected Run Length and the Average Change Point Estimator for Different Magnitudes of Shift ($\delta > 1$)

δ	1.10	1.20	1.30	1.40	1.50	2.00	2.50
E(T)	120.21	119.60	116.58	114.72	113.39	106.43	103.34
$\bar{\hat{\tau}}$	119.09	116.42	103.05	102.76	101.80	100.33	100.21

Table 2: Expected Run Length and the Average Change Point Estimator for Different Magnitudes of Shift ($\delta < 1$)

δ	0.80	0.60	0.50	0.40	0.30	0.20	0.10
E(T)	126.01	130.20	131.92	131.98	132.76	133.54	133.90
$\bar{\hat{\tau}}$	101.70	97.36	99.12	99.33	99.57	99.71	99.89

From table 1, we can see that $\bar{\hat{\tau}}$ is relatively close to the actual change point of 100 with δ increasing from 1.10 to 2.50. For example, for a step change in the process variance of magnitude $\delta = 1.30$, the control chart issues a signal at time 116.58 on average. In this case, the average estimated time of process change is 103.05, which is relatively close to the actual change point of 100.

Similarly, in table 2, $\bar{\hat{\tau}}$ is also close to 100 with δ decreasing from 0.80 to 0.10. On average, our proposed *MLE* of the time of the process change is fairly close to the actual time of the change. It is not difficult to see that the average estimator of change point has become exact results in the case of $\delta \geq 1.50$ or $\delta \leq 0.50$.

The observed frequency with which the proposed estimator of the time is within *m* subgroups of the actual time of the change, for $m \in \{0,1,...,10\}$, is summarized in table 3. This information illustrates the precision of the proposed estimator from *MLE* model. With the increase of the magnitude of shift in the case of $\delta > 1$, or decrease in the case of $\delta < 1$, the probability of identifying the process change within *m* subgroups increases. For example, in the case of *m*=1 and $\delta = (\sigma_1 / \sigma_0) \in \{1.25,1.50,1.75,2.00\}$, the probability increases from 0.26 to 0.53. Likewise, for a fixed δ, the probability of identifying the process change within *m* subgroups increases with *m* as well. For example, if $\delta = 1.75$ and $m \in \{0,1,...,10\}$, the probability also increases from 0.24 to 0.98. The data show that the estimators coming from this proposed model have better precision.

Table 3: Simulation Results for Different Magnitudes of Shift When τ = 100

	δ						
	0.25	0.50	0.75	1.25	1.50	1.75	2.00
$\hat{p}(\hat{\tau}=\tau)$	0.42	0.33	0.12	0.18	0.21	0.24	0.29
$\hat{p}(\lvert\hat{\tau}=\tau\rvert \le 1)$	0.51	0.46	0.21	0.26	0.29	0.45	0.53
$\hat{p}(\lvert\hat{\tau}=\tau\rvert \le 2)$	0.64	0.60	0.33	0.37	0.41	0.55	0.61
$\hat{p}(\lvert\hat{\tau}=\tau\rvert \le 3)$	0.75	0.69	0.38	0.46	0.50	0.64	0.67
$\hat{p}(\lvert\hat{\tau}=\tau\rvert \le 4)$	0.79	0.71	0.49	0.53	0.59	0.67	0.72
$\hat{p}(\lvert\hat{\tau}=\tau\rvert \le 5)$	0.89	0.82	0.60	0.59	0.64	0.70	0.74
$\hat{p}(\lvert\hat{\tau}=\tau\rvert \le 6)$	0.91	0.89	0.67	0.66	0.69	0.73	0.79
$\hat{p}(\lvert\hat{\tau}=\tau\rvert \le 7)$	0.95	0.93	0.74	0.74	0.78	0.85	0.88
$\hat{p}(\lvert\hat{\tau}=\tau\rvert \le 8)$	0.98	0.97	0.79	0.79	0.84	0.97	0.98
$\hat{p}(\lvert\hat{\tau}=\tau\rvert \le 9)$	0.99	0.98	0.85	0.85	0.90	0.98	0.98
$\hat{p}(\lvert\hat{\tau}=\tau\rvert \le 10)$	0.99	0.98	0.92	0.89	0.96	0.98	0.99

4. A Numeral Example

In the end, an example is presented to show that the proposed *MLE* model is effective. First, suppose that the process is subject to a $N(5, 1^2)$ distribution for the first 10 observations, but between observation 10 and 11 an assignable cause occurs that results in a sustained shift in process variance to a new level $\sigma_1^2 = 1.50^2$. The variance remains at this new level for the remaining 15 observations. Then we denote these observations as x_i, i =l, …, 25, and compute the corresponding MR_i, where $MR_i = |x_i - x_{i-1}|$. If α = 0.0027, M = 3.40 and R = 4.29 are the ideal choice in literature [6]. Finally, the control limits of the X–MR charts are as follows: the control limits of the *X* chart are $UCL_x = 8.40$ and $LCL_x = 1.60$, and the control limit of the *MR* chart is $UCL_{MR} = 4.29$. Our study focuses on estimating τ, which is the last subgroup from the in control process. Thus, the estimator $\hat{\tau}$ should be close to 10.

Table 4 shows that the *MR* chart issues an out of control signal at sample 19 because of $MR_{19} > UCL$, unlike the individual *X* chart, which does not signal as would be expected. To apply the *MLE* model, we should find the value of i ($1 \le i \le 19$), which maximizes A_t in (4). It is easy to see that A_t reaches maximum when t = 10. Since the maximum A_t corresponds to observation 11, the *MLE* model identifies observation 11 as the first observation from the changed process. So engineers could investigate the cause of step change between observation 10 and 11. This example confirms that the application of the *MLE* model exactly estimates the step change time of process variance in *X*-*MR* charts. Therefore, the results are quite satisfactory.

Table 4: Application of MLE Model in Identifying the Time of a Step Change in Process Variance

i th Observation	x_i	MR_i	t	A_t
1	3.9806		0	-33.0485
2	6.0338	2.0532	1	-32.9711
3	6.0008	0.0330	2	-32.8951
4	5.0706	0.9302	3	-32.7955
5	3.5178	1.5528	4	-32.4196
6	5.9012	2.3834	5	-32.6246
7	3.9142	1.9870	6	-32.4449
8	4.0720	0.1578	7	-32.3564
9	5.9126	1.8406	8	-32.1612
10	5.6555	0.2571	9	-31.9374
11	3.2463	2.4092	10	-31.5565
12	7.3597	4.1134	11	-32.0324
13	3.5443	3.8154	12	-33.2997
14	6.3689	2.8246	13	-33.4737
15	6.3900	0.0211	14	-33.5659
16	6.1889	0.2011	15	-33.6664
17	6.1226	0.0663	16	-33.5657
18	3.0435	3.0791	17	-33.3416
19	7.3599	4.3164	18	-34.0669
20	5.0070	2.3529		
21	3.7296	1.2774		
22	3.8916	0.1620		
23	4.7424	0.8508		
24	3.8814	0.8610		
25	4.4795	0.5981		

5. Conclusion

Generally, given a signal from a control chart, process engineers do not generally know what caused the change nor when the process changed. Knowing the time of the process change would simplify the search for the special cause. If the process engineers know when the process has changed, the search would simply reduce to discovering the aspect of the process that changed at that time. Thus, process engineers would increase their chances of identifying correctly the special cause quickly. This would allow them to take appropriate actions for improving quality.

On the basis of *X–MR* charts, this paper suggests the *MLE* model which is applied to estimate the actual time when a step change is taken place in process variance. The results indicate that it helps engineers to identify the special cause in time and enables the production process to return to normal. Our proposed change point estimator can be easily implemented using a spreadsheet. The estimator of the time of the process dispersion change will be useful to process engineers who will be able to identify variables that might cause a change in process variance more easily. It has been tested that the model is simple and practical.

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