M.Sc. 4th Semester Examination, 2024

APPLIED MATHEMATICS

(*Functional Analysis*)

PAPER – MTM-401

Full Marks : 50

Time : 2 hours

Answer **all** questions

The figures in the right hand margin indicate marks

Candidates are required to give their answers in their own words as far as practicable

GROUP – A

1. Answer any *four* questions : 2×4

   (*a*) Show that the norm function is continuous.

   (*b*) What do you mean by best approximation in an inner product space ?

(*Turn Over*)

(*c*) If in an inner product space, $\langle x,u\rangle = \langle x,v\rangle$ for all x, show that $u = v$.

(*d*) Give an example of an unbounded linear operator.

(*e*) Let $T \in BL(H)$ and $T \geq 0$ where H is a Hilbert space. Show that $\|Tx\|^2 \leq \|T\| < Tx, x >$ for all H.

(*f*) Let X, Y, Z be Banach spaces and F, $F_n \in BL(X, Y)$ and G, $G_n \in BL\ (Z, X)$. Let $F_n(x) \to F(x), x \in X$ and $G_n(z) \to G(z), z \in Z$, as $n \to \infty$, Show that $(F_n G_n)(z) \to (FG)(z)$, when $n \to \infty$.

GROUP – B

2. Answer any *four* questions : 4 × 4

(*a*) Show that every finite dimensional sub-space of a normed space is closed.

(*b*) Let V be an infinite dimensional normed space and W be a non-zero normed space. Then show that there exists a linear operator which is not continuous.

(*c*) If $T \in BL(H,Y)$ where H and Y are simply inner product spaces, then show that T may not have an adjoint.

(*d*) Let H be a Hilbert space and $E \subset H$. Prove that $\overline{span(E)} = E^{\perp\perp}$.

(*e*) State and prove Riesz-Fischer theorem.

(*f*) Let $S \in BL(H)$, where H is a Hilbert space. Prove that for all $x,y \in H$,

$$< Sx, y > = \frac{1}{4}\sum_{n=0}^{3} i^n < S(x+i^n y),(x+i^n y) > .$$

GROUP – C

3. Answer any *two* questions : 8×2

(*a*) (*i*) Let X be a normed space and M be a subspace of X. If $\phi \in M^*$ then show that there exists $\psi \in X^*$ such that $\psi|_M = \phi$ and $\|\psi\| = \|\phi\|$. 6

(*ii*) Define positive and strictly positive operators. 2

(*b*) (*i*) Let H be a Hilbert space and $A \in BL(H)$. If A is self-adjoint, then prove that $\| A \| = \sup\{|< Ax, x >| : x \in H, \| x \| = 1\}$. 5

(*ii*) Let X and Y be normed spaces and $\psi : X \to Y$ be linear. Show that ψ is continuous if and only if for every Cauchy sequence $\{x_n\}$ in X, the sequence $\{\psi(x_n)\}$ is Cauchy in Y. 3

(*c*) (*i*) Let X and Y be Banach spaces and $F : X \to Y$ be linear. Let $\{g_s\} \subset Y^*$ be such that for every nonzero y in Y, there is some s with $g_s(y) \neq 0$. Prove that F is continuous if and only if $g_s \circ F$ is continuous for every s. 4

(*ii*) Let $P \in BL(\mathscr{H})$ be a nonzero projection on a Hilbert space $\mathscr{H}$ and $\|P\| = 1$.

Then show that P is an orthogonal projection. 4

(*d*) (*i*) Let $T : l^2 \to l^2$ be given by

$$T(x_1, x_2, \ldots, x_n, \ldots) = (x_1, \frac{1}{2} x_2, \ldots, \frac{1}{n} x_n, \ldots).$$

Is T bounded ? 2

(*ii*) State and prove the Uniform Boundedness Principle. 6

[Internal Assessment — 10 Marks]