(ii) Transform the basis

$$\{(0,1,1,0),(-1,1,0,0),(1,2,0,-1),(-1,0,0,-1)\}$$

of $\mathbb{R}^4$ into an orthonormal basis by using Gram-Schmidt orthogonalization process. 4

(iii) Show that a non-zero nilpotent operator is not diagonalizable. 2

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2024

M.Sc. 2nd Semester Examination

APPLIED MATHEMATICS

PAPER : MTM-203

Full Marks : 40

Time : 2 hours

The figures in the right-hand margin indicate marks.

UNIT—I (20 Marks)
(Abstract Algebra)

A. Answer any **two** questions : 2×2=4

1. Show that any finite extension is algebraic.

2. Are the fields $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(i)$ isomorphic? Justify your answer.

3. Is $\mathbb{C}/\mathbb{R}$ an algebraic field extension? Justify.

4. Is every finite field perfect? Justify.

B. Answer any **two** questions : 4×2=8

5. Prove that it is impossible to construct the regular 9-gon, by using straight edge and compass only. 4

6. Let $F \subset K$ be a field extension and $\alpha \in K$ be algebraic over F and $f(x) = irr(\alpha, F)$.

Let $n = \deg f$. Then show that —

(i) $F[\alpha] = F(\alpha) \simeq F[x]/(f(x))$;

(ii) $\dim_F F(\alpha) = n$ and $\{1, \alpha, \alpha^2, \ldots, \alpha^{n-1}\}$ is an F- basis of $F(\alpha)$. 4

7. Let $f(x) = x^4 + x^2 + 1 \in \mathbb{Q}[x]$. Find the splitting field S of $f(x)$ over $\mathbb{Q}$. Also find $[S : \mathbb{Q}]$. (2+2)

8. Show that no finite field is algebraically closed. Show that the field $\mathbb{C}$ of complex numbers is an algebraic closure of the field $\mathbb{R}$ of real numbers. (2+2)

C. Answer *any* **one** question : 8×1=8

9. *(i)* A regular polygon of n sides is constructible if and only if $n = 2^r p_1 p_2 \ldots p_s$, where $r \in \mathbb{N}$ and $p_1, p_2, \ldots, p_s$ are distinct Fermat primes. 6

8. What do you mean by dual space of vector space? Find the dual basis of the basis $\mathcal{B} = \{(1,-1,3), (0,1,-1), (0,3,-2)\}$ of $\mathbb{R}^3$. (2+2)

C. Answer any **one** questions : 8×1=8

9. *(i)* If V be a finite dimensional vector space over a field F, then prove that a bilinear form H on V is symmetric or asymmetric according as the matrix of H in any ordered basis is symmetric or asymmetric.

(ii) Define quadratic form. Find rank, index and signature of the quadratic form $x^2 + y^2 + z^2 + xy - yz - zx$. (4+4)

10. *(i)* What do you mean by adjoint T^* of a linear operator T on an inner product space V? Let T_1 and T_2 be two linear operators on an inner product space V and assume that T_1^*, T_2^* exist. Show that

$$(T_1 + T_2)^* = T_1^* + T_2^*$$ (1+1)

(ii) Let ζ be a complex primitive p^{th} root of unity where p is a prime number. Then show that $G(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ is a cyclic group of order $p-1$. 2

10. *(i)* Use the fact that $\alpha = 2\cos\left(\frac{2\pi}{5}\right)$ satisfies the equation $x^2 + x - 1 = 0$ to conclude that the regular 5-gon is constructible by ruler and compass. 5

(ii) Show that the polynomial $f(x) = x^4 + x^2 + [1]$ is separable over $\mathbb{Z}_2$. 3

UNIT—II (20 Marks)

(Linear Algebra)

A. Answer any **two** questions : 2×2=4

1. Define rank of a bilinear form. When is a bilinear form said to be degenerate or non-degenerate?

2. Show that the quadratic form

$$Q(x,y,z) = x^2 + 2y^2 + 2z^2 + 2xy - 2xz - 4yz$$

on $\mathbb{R}^3$ is positive semidefinite.

3. State Sylvester's Law of Inertia.

4. Let $T:\mathbb{R}^2 \to \mathbb{R}^2$ be a linear operator defined by $T(x,y) = (x,0)$ for all $(x,y) \in \mathbb{R}^2$. Find the minimal polynomial of T.

B. Answer any **two** questions : 4×2=8

5. Let T be the linear operator on $P_2(R)$ defined by

$$T(f(x)) = f(1) + f'(0)x + (f'(0) + f''(0))x^2.$$

Test the diagonalizability of T. Find the eigen space and then find the coordinate vectors in relate to eigen space. 4

6. When is a linear operator T on an inner product space V said to be normal? Show that T is normal if and only if $\|T(v)\| = \|T^*(v)\|$ for all $v \in V$. (1+3)

7. Let V be a finite-dimensional inner product space and let T be a linear operator on V. Then, show that there exists a unique function $T^*: V \to V$ such that $\langle T(x), y\rangle = \langle x, T^*(y)\rangle$ for all $x, y \in V$. Furthermore, prove that T^* is linear. 4