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A Fixed Point Result Using a Function of 5-Variables

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ABSTRACT

Here we prove two fixed point theorems in metric spaces. The results are obtained for mapping which satisfy certain inequalities through a function of five variables.

1. Introduction

The Banach contraction mapping theorem is a pivotal result in mathematical analysis [1]. After that efforts have been made by mathematicians to obtain fixed point theorems of mappings satisfying several contractive inequalities [2],[3]. As a result, a large number of research articles on the existence of fixed points of several types of mappings satisfying different contractive inequalities have appeared in literature. Some of such works deal with one or more mappings which are assumed to satisfy inequalities through given functions. In the present work we assume such an inequality with the help of a five variable function. We have two theorems in one of which we prove a common fixed point result in a metric space which is not necessarily complete. In the other theorem we prove the existence of a unique fixed point of a self-mapping defined on a complete metric space. We support our result by an example.

The following is the definition of a class of functions which we call G-functions.

Definition 1.1.

$g: [0,\infty)^5 \to [0,\infty)^5$ is said to be a G-type mapping if

(i) g is continuous
(ii) g is nondecreasing in each variable
(iii) if $h(r) = g(r, r, r, r, r)$.

then $r \to r - h(r)$ is strictly increasing and positive in $(0, \infty)$.

Examples of G-type mapping are the following

(i) $g(r_1, r_2, r_3, r_4, r_5) = \alpha_1 r_1 + \alpha_2 r_2 + \alpha_3 r_3 + \alpha_4 r_4 + \alpha_5 r_5$

where $\alpha_i' s$ are non-negative and $0 < \sum_1^5 \alpha_i < 1$.

(ii) $g(r_1, r_2, r_3, r_4, r_5) = \lambda \max\{r_1, r_2, r_3, r_4, r_5\}$ where $0 < \lambda < 1$

(iii) $g(r_1, r_2, r_3, r_4, r_5) = \ln[1 + \max\{r_1, r_2, r_3, r_4, r_5\}]$

2. Main Results

Theorem 2.1.

Let (X,d) be a metric space and $A, B : X \to X$ be two self-mappings satisfying the inequality

$d(Ax, By) \le g(d(x,y), d(Ax,x), d(By,y), d(Ax,y), d(By,x))$ where g belongs to the class of G-type functions. (2.1)

Let $\{x_n\}$ be any sequence in X which satisfies $d(x_n, Ax_n) \to 0$ as $n \to \infty$ (2.2)

If x_n converges to a point x then any other sequence y_n having the property that $d(y_n, By_n) \to 0$ as $n \to \infty$ will also converge to x and x is a common fixed point of A and B.

Proof.

We assume that $x_n \to x$ as $n \to \infty$ (2.3)

Let $\varepsilon > 0$ be given. We choose $\delta > 0$ as $\delta = \frac{1}{3}(\varepsilon - h(\varepsilon))$ (2.4)

This choice of δ is possible in view of definition 1.1. As (2.2) is true, corresponding to the choice of $\delta > 0$ which by (2.4), depends also on $\varepsilon > 0$, a positive integer n_0 can be found out such that for $n > n_0$.

$d(x_n, Ax_n) < \delta$ and $d(y_n, By_n) < \delta$ (2.5)

Consequently for $n > n_0$

$$d(x_n, y_n) \le d(x_n, Ax_n) + d(Ax_n, By_n) + d(By_n, y_n)$$
$$\le 2\delta + g(d(x_n, y_n), d(Ax_n, x_n), d(By_n, y_n), d(Ax_n, y_n), d(By_n, x_n))$$

(using (2.1) and (2.5))

As g is non-decreasing in each variable, we have

$$d(x_n, y_n) \le 2\delta + g(d(x_n, y_n), d(Ax_n, x_n), d(By_n, y_n),$$
$$d(Ax_n, x_n) + d(x_n, y_n), d(By_n, y_n) + d(y_n, x_n))$$
$$\le 2\delta + g(d(x_n, y_n), \delta, \delta, \delta + d(x_n, y_n), \delta + d(x_n, y_n),$$

$\leq 2\delta + h(\delta + d(x_n, y_n))$

or, $d(x_n, y_n) + \delta \leq 3\delta + h(\delta + d(x_n, y_n))$

or, $(d(x_n, y_n) + \delta) - h(\delta + d(x_n, y_n)) \leq 3\delta = \varepsilon - h(\varepsilon) \quad (by (2.4))$

Since *r–h(r)* is strictly increasing and positive in $(0, \infty,)$ the above inequality implies

$$d(x_n, y_n) + \delta < \varepsilon \; for \;\; n > n_0$$

Consequently $d(x_n, y_n) < \varepsilon$ for $n > n_0$ (2.6)

(2.2), (2.3) and (2.6) jointly imply that if $x_n \to x$

then

$$y_n \to x, \; Ax_n \to x, \; By_n \to x \text{ as } n \to \infty \quad (2.7)$$

Again, using (2.1)

$$d(Ax, By_n) \leq g(d(x, y_n), d(Ax, x), d(By_n, y_n), d(Ax, y_n), d(By_n, x))$$

Making $n \to \infty$ and noting that g is continuous, we obtain from (2.7) that

$$d(Ax, x) \leq g(0, d(Ax, x), 0, d(Ax, x), 0)$$

$$\leq h(d(Ax, x))$$

which implies

$$d(Ax, x) - h(d(Ax, x)) \leq 0$$

using the property of *h* in definition 1.1, it follows that

$$d(Ax, x) = 0$$

or equivalently $Ax = x$

Similarly it can be proved that $Bx = x$

This completes the proof.

Theorem 2.2.

Let $A : X \to X$ be a self mapping in a complete metric space *(X, d)* satisfying

$$d(Ax, Ay) \leq g(d(x, y), d(Ax, x), d(Ay, y), d(Ax, y), k\, d(Ay, x)) \quad (2.8)$$

Where g belongs to the class *G* and $0 < k \leq \frac{1}{2}$ (2.9)

Then for any $x \in X$, the sequence $\{A^n x\}$ is such that

$$d(A^n x, A^{n+1} x) \to 0 \text{ as } n \to \infty$$

Further if $\{A^n x\}$ is convergent then it converges to the unique fixed point of *A*.

Also in that case any other sequence $\{y_n\}$ satisfying $d(Ay_n, y_n) \to 0$ as $n \to \infty$ will also converge to the unique fixed point.

Proof.

We construct $\alpha_n = d(A^{n-1}x, A^n x), n = 1,2,...............$

If $\alpha_n = 0$ for some n, then A has a fixed point at y= $A^{n-1}x$

We assume that $\alpha_n \neq 0$ for all n = 1,2 ,…………… (2.10)

Replacing x, y with $A^{n-1}x, A^n x$ respectively in (2.8) and using definition 1.1
We obtain

$$d(A^n x, A^{n+1}x,) \le g(d(A^{n-1}x, A^n x), d(A^n x, A^{n-1}x), d(A^{n+1}x, A^n x),$$
$$d(A^n x, A^n x,), kd(A^{n+1}x, A^{n-1}x))$$
$$\le g(d(A^{n-1}x, A^n x), d(A^n x, A^{n-1}x), d(A^{n+1}x, A^n x),$$
$$0, \mathrm{k}(d(A^{n+1}x, A^n x) + d(A^n x, A^{n-1}x))$$

which implies

$$\alpha_{n+1} \le g(\alpha_n, \alpha_n, \alpha_{n+1}, 0, K(\alpha_n + \alpha_{n+1})) \qquad (2.11)$$

If possible, let $\alpha_{n+1} \ge \alpha_n$

Since $0 < k \le \frac{1}{2}$, it follows that

α_{n+1} - h d$(\alpha_{n+1}) \le 0$ or $\alpha_{n+1} = 0$ which contradicts (2.10)

Consequently, $0 < \alpha_{n+1} < \alpha_n$ for n = 1,2,………………

which implies $\{\alpha_n\}$,being a decreasing sequence ,is convergent.

Let $\alpha_n \to \alpha$ as $n \to \infty$

As $\alpha_{n+1} < \alpha_n$, using (2.9) and (2.11), we obtain

$\alpha_{n+1} \le h(\alpha_n)$

Making $n \to \infty$, it follows that

$\alpha \le h(\alpha)$ or $\alpha = 0$ (2.12)

Let $x_n = A^n x, n = 1,2,..............$

Then $d(A^n x, x_n) = d(A^{n+1}x, A^n x) = \alpha_{n+1} \to 0 \; as \; n \to \infty$

Let $A^n x \to z \; as \; n \to \infty$. We observe that when the assume A=B, we obtain (2.8) from (2.1) as a special case. Then by the application of theorem -1 the result of the present theorem follows except for the uniqueness of fixed point.

To prove the uniqueness, we suppose x and y as two fixed points of A.

Then from (2.8)

$d(x, y) \leq g(d(x, y),0,0, d(x, y), d(x, y)) \leq h(d(x, y)$

or $d(x, y) - h(d(x, y)) \leq 0$ or $d(x, y) = 0$

or x = y

This completes the proof.

Example 2.1.

Let $X = R$, $d(x, y) = |x - y|$ and $\mathrm{Ax} = \frac{x+1}{4}$

If $g(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = \frac{1}{3}\alpha_1 + \frac{1}{8}\alpha_2 + \frac{1}{8}\alpha_3 + \frac{1}{8}\alpha_4 + \frac{1}{8}\alpha_5$

Then A satisfies the condition of theorem 2.2. It is seen that $x = \frac{1}{3}$ is the unique fixed point of A.

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