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A Mathematical and Statistical Study on HIV: A Control Theoretic Approach

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ABSTRACT

In this paper we have considered an optimal control problem on the use of condoms, taken as control, so as to attain a specified reduced value of the infected population of HIV within a certain stipulated time under the minimization of the cost of the control with the aid of 'Maximal Principle' by incorporating statistical knowledge on VCTC report in the mathematical model. Simulated part of this model will be considered in the next phase.

Notation

x, y, z	=	active population of the susceptible class, infected class and condom supplied class respectively.
R	=	rate of recruitment.
β	=	probability of transmission per partnership.
c	=	rate at which an infected individual acquires new sexual partners per unit time.
ε	=	condom efficiency.
π	=	condom compliance.
$\varepsilon\pi$	=	condom induced efficiency.
$\beta(1-\varepsilon\pi)$	=	per exposure risk of infection of individuals not using condoms.
$u(t)$	=	rate of condom supplied to susceptible population, $0 \le u(t) \le A_2$ (say)
w	=	desired weight on the cost of control.
α	=	the proportion of recruits those who earlier supplied with condoms.

1. Introduction

The devastating global impact on HIV have been growing in so much dynamic pace that a broad based research study on the concept of reduction of progression and transmission of HIV will still be made to achieve our successful goal. In developing countries like India, the vaccination as a control strategy is not yet readily available but condom and anti-retroviral drugs (ARV) may play an important role in this regard. The use of condoms may be considered as the most feasible preventive strategy for HIV. Besides, numerous experimental and clinical studies regarding the estimation of condom effectiveness, the most conclusive evidence has gained importance in the study[1]. Countries such as Ethiopia, Uganda, Thailand and Vietnam, specially have given such stress on the use of condoms.

However, HIV models to assess the preventivity of transmission with the use of condoms have been studied [1,2,3,4]. It is to note that one of the interesting study for assessing the effect of condom use based on efficacy and compliance in controlling HIV has been made in [1]. Granting the controlling efficiency of condoms in this respect, the use of condoms consistently from the standpoint of feasible and affording cost of using condoms has been taken as the best measure of prevention.

This study constitutes the formation of a mathematical model as the basic concept of scientific knowledge as in [1]. Besides, we would like to incorporate the collected data from VCTC report on the standpoint of statistical mode. However, though a lot of studies regarding the transmission of HIV have been made [5,6,7,8,9,10,11], yet the incorporation of VCTC report which reveals the reality of the transmission, for estimation of parameters of this mathematical model in different locations/districts/states will clearly specify the formulation of the mathematical model with much real sense. Here we would like to study the optimal control problem with a view to achieve our estimated goal on transmission of HIV within a certain stipulated period where the use of condoms find ways to resist HIV transmission among the active population under the minimization of the cost of the control and thus this mathematical study can be signified as **'Cost minimization optimal control problem'.**

2. Mathematical Model

In the mathematical model, we would like to classify the heterosexually active population into four classes : Counselled (s), Susceptible (x), Injected in HIV (y) and Individuals supplied with condoms (z).

However, it may be assumed from the practical standpoint that new recruits enter in the sexually susceptible population at a rate (R) within which some individuals earlier supplied with condoms may exist by the proportion (α) and this may be obtained from the knowledge of counseling on VCTC. Among the susceptible population a portion may be infected by HIV at a time dependent rate which can be obtained by the way of testing and another part those who are supplied with adequate condoms at a certain rate by u(t). As it is presumed from several experiences that use of condoms is virtually one of important feasible measures to control the transmission of HIV, one can take it into granted that u(t) will be an important control function.

Thus finally the infected part of population moves to the infected class and u(t)x moves to condom class both from susceptible class of population.

Mathematical model thus stands,

$$\frac{d}{dt}x = (1-\alpha)R - \beta c(1-\varepsilon\pi)\left(\frac{y}{s}\right)x - u(t)x \tag{1}$$

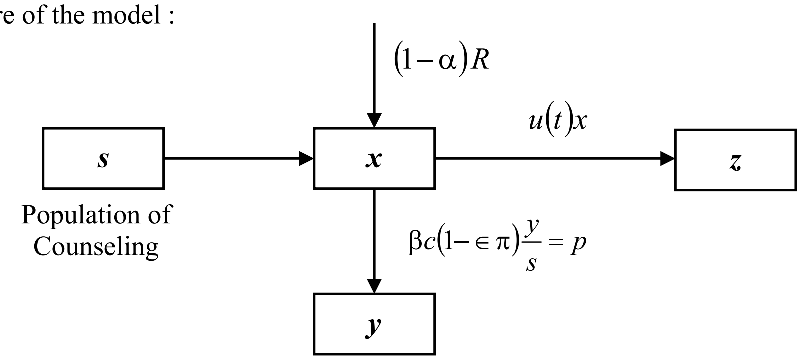
$$\frac{d}{dt}y = \beta c(1-\varepsilon\pi)\left(\frac{y}{s}\right)x \tag{2}$$

$$\frac{d}{dt}z = \alpha R + u(t)x \tag{3}$$

Initial values of the variables x, y, z can be obtained from VCTC reports taking a particular year as a year of reference.

Let us take $x = x_0, y = y_0, z = z_0$ on $t = 0$ (3a)

Structure of the model :



It can be presumed, so far as the practical scheme of prevention of transmission of HIV is concerned, that Voluntary Counseling and Testing Centre (VCTC) plays a vital role . It is located firstly in easily accessible areas. It provides people with an opportunity to learn about HIV and take preventive measures to cope with the stress arising out of HIV infection. Supply of condoms are made from these centres and make people teach to use condoms in compliance with HIV prevention and other aspects.

In this respect, one can think of a selection of susceptible population infected in HIV and those who are not at all susceptible in HIV by means of counseling.

Now we would like to incorporate 'Data' from VCTC reports in different years available from 'State AIDS Control Society' of different states in Indian Context, particularly in West Bengal, where there exists data on population counselled (s), tested taken as susceptible (x) and infected in HIV *(y)* are specified.

Considering the 'principal of least squares ' as the most popular and widely used method of fitting mathematical functions to a given set of data, one can obtain for time series :

$$\frac{y}{s} = B_0 + B_1(t) \tag{4}$$

where the constants B_0 and B_1 can be obtained.

Again, by regression analysis, the regression equation on *y* on *s* stands,

$$\frac{dy}{dt} = A_0 + A_1 \frac{ds}{dt} \tag{5}$$

where A_0 and A_1 are obtainable constants; which is more adjustable in this problem than obtained from equation (4).

Further it is assumed that the rate of population selected for counselling can be taken as the rate of recruitment (R) for the sake of generality with the help of equations (2) and (5);

Thus, $$\frac{ds}{dt} = R = \left(A_0 - \frac{dy}{dt} \right) \Big/ A_1$$

$$= \{A_0 - \beta c(1 - \in \pi)(B_0 + B_1 t)x\} / A_1 \tag{6}$$

Finally with the help of equation (4), (5) and (6)
We have,

$$\frac{dx}{dt} = Qx - u(t)x - \frac{A_0}{A_1}(1 - \alpha) \tag{7}$$

$$\frac{dy}{dt} = qx \tag{8}$$

$$\frac{dz}{dt} = \frac{\alpha}{A_1}[qx - A_0] + u(t)x \tag{9}$$

where,

$$q = \beta c(1 - \varepsilon\pi)(B_0 + B_1 t) \quad \text{and} \quad Q = \frac{q(1 - \alpha - A_1)}{A_1}$$

3. Formation of the problem

According to the practical need in the Indian context, we would like to formulate the optimal control problem stated as follows :

To find the optimal control *u(t)*, the distribution of condoms among the susceptible population, such that one can reach a reduced value of infected population of HIV through the attainment of expected population using condoms under the minimization of the cost on the aspect of distribution of condoms within a stipulated time period T.

Thus the functional stands as,

$$J(u)=\frac{1}{2}\int_0^T\left[\left(y^*-y\right)^2+A_2\left(z^*-z\right)^2+wu^2(t)\right]dt$$

which is to be minimized where w is the weight on cost of control and A_2 is a constant to be specified.

Under the consideration of 'Maximal Principle' [12, 13, 14]
Hamiltonian can be written as :

$$H=-\frac{1}{2}\left[\left(y^*-y\right)^2+A_2\left(z^*-z\right)^2+wu^2(t)\right]+\Psi_1\left[Qx-u(t)x-\frac{A_0}{A_1}(1-\alpha)\right]$$
$$+\Psi_2 qx+\Psi_3\left\{\frac{\alpha}{A_1}\left(qx-A_0\right)+u(t)x\right\} \quad (11)$$

Then the auxiliary functions Ψ_1,Ψ_2,Ψ_3 are given by,

$$\frac{d\Psi_1}{dt}=-\Psi_1 Q+\Psi_1 u(t)+\Psi_2 q+\frac{\alpha}{A_1}q\Psi_3$$

$$\frac{d\Psi_2}{dt}=-\left(y^*-y\right)$$

$$\frac{d\Psi_3}{dt}=-A_2\left(z^*-z\right)$$

and $\frac{\partial H}{\partial u}=0$ gives, $u(t)=-\frac{1}{w}\left(\Psi_1+\Psi_3\right)x$ (13)

and $\Psi_1(T)=\Psi_2(T)=\psi_3(T)=0$ (13a)

Here y* and z* are desired value to be achieved for y and z respectively.

4. Method of Numerical Calculation

Let us take some desired values y^*, z^* in compliance with VCTC report and stipulated time T.

Now we just proceed by the method of iteration :

(i) Take *u(t)* = 0, find *x(t)* and *x(T)* from equation (1) under the condition (3a).

(ii) Find *y(t)* and *y(T)* substituting *x(t)* under the condition (3a)

(iii) Find *z(t)* and z(T) substituting *u(t)* and *x(t)* under the condition (3a)

(iv) Find $\psi_3(t)$ in *(T,t)* by substituting *z(t)* with obtained *z(T)* and $\psi_3(T)$ given in (13a)

(v) Find $\psi_2(t)$ in *(T,t)* by substituting in the same manner as $\psi_3(t)$ with condition (13a)

(vi) Find $\psi_1(t)$in *(T,t)* by substituting *u(t)* = 0 , $\psi_3(t)$ and $\psi_2(t)$.

(vii) Find *u(t)* from equation (13) by ψ_1,ψ_3 and *x*.

(viii) Substitute *y(t), z(t)* and *u(t)* in *J(u)* by equation (10)

(ix) Repeat this operation until the value of *J(u)* diminishes on the error value of 10^{-3} .

It is to note that all values of parameters stated earlier, if not available in the Indian context, the values given in [1] can be approximated in course of computerized scheme.

The integration can however be performed by Simpson's method .

4. Conclusion

It is to note from equation (1) and (2) that the population of susceptibility *(x)* and infected population (*y)* increase with time unless condoms are used which is the case in the practical sense and knowledge.

On the basis of some standard data and taking u(t)=1 it was observed that possible value of *u(t)* was obtained when $\psi_1(t)$ and $\psi_3(t)$ were both negative.

Thus, $\psi_1(t) < 0, \psi_3(t) < 0$ in *(T,t)* can be taken as granted from standpoint of generality since $\psi_1(T) = 0$ and $\psi_3(T) = 0$ and the control.

$u(t) = -\frac{1}{w}(\psi_1 + \psi_3)x$ is positive.

It indicates that the susceptible population (*x)* and infected population *(y)* decrease with time after the practice of the use of condom is administered, though the rate of decrease, by which one can attain the desired goal within the stipulated time T , cannot be ascertain at present due to the lack of availability of certain parameters in Indian situation. It is to note from the solution of the problem that it considers some important parameters [1] according to practical situation that stands on the way of broad based discussion, rather study, in the transmission of HIV. All those parameters, so far as the authors' knowledge are concerned, are not yet available in the Indian context. However, on the basis of nonavailability of those, some parameters can be considered from other spheres [1]. Yet the authors are in search of those parameters in our situation. In this respect, the numerical calculations on broad based spectrum on transmission of HIV in different states/districts will be studied in the next phase which will, not doubt, clearly reveal a clear knowledge on the use and distribution of condoms as the resistance of transmission of HIV in different situations.

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REFERENCES

[1] Mukandavire, Z. et al. : HIV/AIDS model for assessing the effects *of* prophylactic sterilizing vaccines, condoms and treatment with amelioration, journal of Biological System, Vol. 14, 323-335 (2007).

[2] Me Cluskey, CC : A model of HIV/AIDS with staged progression and amelioration, Math. Biosci, 181, 1-16 (2003).

[3] Del Valle S et al : Effects of education, vaccination and treatment on HIV transmission in homosexuals with genetic heterogeneity, Math. Biosci., 187, 111-133 (2004).

[4] Kgosimore M et al : The effects of vaccination and treatment on the spread of HIV/AIDS, J. Bio. Syst., 12, 399 - 417 (2004).

[5] Hyman JM et al : Using mathematical models to understand the AIDS epidemic, Math. Biosci., 90, 415-473 (1998).

[6] Anderson RM et al : Potential of community - wide chemotherapy or immunotherapy to control the spread of HIV, Nature, 350, 356-359 (1991).

[7] Hsich YH et al : The effects of density - dependent treatment and behaviour changes on the dynamics of HIV transmission, J. Math. Biol., 43, 69-80 (2001).

[8] Kirchner, D et al : Optimal Controi of the chemotherapy of HIV, J. Math. Biol, 35, 775 - 792 (1997).

[9] Wein LM et la : Dynamic multidrug therapies for HIV : A control theory approach, J. Theo. Biol. 185, 15 - 29 (1997).

[10] Putter H et al : A Bayesian approach to parameter estimation in HIV dynamic models, Statistic in Medicine, 21, 2199 -2214 (2002).

[11] Huang, Y. et. al : Modeling HIV dynamics and antiviral responses with consideration of time varying drug exposures, sensitivities and adherence, Math. Biosci, 184, 165 - 186 (2003).

[12] Golub, N. N. : Optimal Control of linear and non – linear systems having distributed parameters.
Avtomatika; telemekhanika , No. 9, 16-28 (1969)

[13] Pontryagin, L. S. etal : The mathematical theory of optimal processes, Interscience Publishers , New York (1962)

[14] Lee. I, . B. and Markus, L : Foundations of optimal control theory, John Wiley & sons, New York (1967)