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General Solutions for the Space-and Time-fractional Diffusion-wave Equation

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ABSTRACT

This paper presents a general solution for a space-and time-fractional diffusion-wave equation defined in a bounded space domain. The space-and time-fractional derivatives are described in the Caputo sense. The application of Adomian decomposition method, developed for differential equations of integer order, is extended to derive a general solution of the space-and time-fractional diffusion-wave equation. The solutions of our model equation are calculated in the form of convergent series with easily computable components. Two examples are presented to show the application of the present technique. The effect to varying the order of the time-and space-fractional derivatives on the behaviour of solutions has been investigated. Results show the transition from a pure diffusion process to a pure wave process and the solution continuously depends on the space-fractional derivative.

Keywords : *Fractional diffusion-wave equation, Caputo fractional derivative, decomposition method.*

A fractional

1. Introduction

A fractional diffusion-wave equation is a linear integro partial differential equation obtained from the classical diffusion or wave equation by replacing the first or second-order time derivative term by a fractional derivative of order α, $0 < \alpha \leq 2$, and the second space derivative by a fractional derivative of order, β, $1 < \beta \leq 2$. There has been a great deal of interest in fractional diffusion equations. These equations arise in continuous-time random walks [24], modeling of anomalous diffusive and subdiffusive systems [13], unification of diffusion and wave propagation phenomenon [18], and simplification of the results. The nature of the diffusion is characterized by the temporal scaling of the mean-square displacement

$< r^2(t) > \sim t^{\alpha}$. For standard diffusion $\alpha = 1$, whereas in anomalous sub-diffusion $\alpha < 1$, and in anomalous super-diffusion $\alpha > 1$. Both types of anomalous diffusion have been unified in continuous time random walk models with spatial and temporal memories [13, 14].

Oldham and Spanier [21] considered a fractional diffusion equation that contains first order derivative in space and half order derivative in time. Nigmatullin [20] pointed out that many of the universal electromagnetic, and mechanical responses can be modeled accurately using the fractional diffusion-wave equaitons.

Fujita [10] presented the existence and uniquencess of the solution of the Cauchy problem of the fullowing type

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = \frac{\partial^{\beta} u(x,t)}{\partial x^{\beta}}, 1 \le \alpha, \beta \le 2$$

The results presented offer an interpretation to phenomena between the heat equation ($\alpha = 1, \beta = 2$) and the wave equation ($\alpha = \beta = 2$). In [11, 12], Fujita considered integro-differential equations which exhibit heat diffusion and wave propagation properties.

Mainardi [16, 17] presented analytical investigation of the time-fractional diffusion-wave equations. Using Laplace transform method, he obtained the fundamental solutions of the basic Cauchy and signalling problems and expressed them in terms of an auxiliary function $M(z; \gamma)$, where $z = |x|/t^{\gamma}$ is the similarity variable. He further showed that such a function is an entire function of Wright type. Mainardi [18] provided a comprehensive review of research on the application of calculus in continuum and statistical mechanics including research on fractional diffusion-wave solutions.

Agarwal [3] presented a general solution for a time-fractional diffusion-wave equation defined in a bounded space domain. His solution depends upon using the finite sine transform technique to convert fractional diffusion-wave equation from a space domain to a wave number domain, then the Laplace transform is used to reduce the resulting equation to an ordinary algebraic equation, finally, the inverse Laplace and inverse sine transforms are used to obtain the desired solutions. In [4], Agarwal used the same technique to obtain a general solution for a fourth-order fractional diffusion-wave equation.

Al-Khaled and Momani [5] used the decomposition method to obtain an approximate solution for the generalized time-fractional diffusion-wave equation. Their results showed the transition from a pure diffusion process ($\alpha = 1$) to a pure wave process ($\alpha = 2$).

This brief review of fractional diffusion-wave equations and their applications is by no means complete. References to other papers akin to fractional diffusion-wave equaitons can be found in Refs. [18-22]

In this paper, we consider the following space-and time-fractional diffusion-wave equation

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = b^2 \frac{\partial^{\beta} u}{\partial x^{\beta}}, 0 < \alpha \le 2, 1 < \beta \le 2, \tag{1.1}$$

subject to the boundary and initial conditions

$$u(0,t)=h_1(t),\ u(l,t)=h_2(t),\ t\geq 0, \qquad (1.2)$$

$$u(x,0)=f(x),\ 0<x<l, \qquad (1.3)$$

$$\frac{\partial u(x,0)}{\partial t}=g(x),\ 0<x<l,\ for\ 1<\alpha\leq 2, \qquad (1.4)$$

where α and β are parameters describing the order of time and space fractional derivatives, respectively, *b* denotes a constant coefficient, and *u(t, x)* is the field defined in the space domain [0, *l*]. Following the terminology, introduced by Mainardi [16, 17, 18], we refer to Equation (1.1) as to the space-and time-fractional diffusion and to the space- and time-fractional wave equation in the cases $\{0 < \alpha \leq 1, 1 < \beta \leq 2\}$ and $\{1 < \alpha \leq 2, 1 < \beta \leq 2\}$, respectively.

Most of the fractional diffusion-wave equations discussed in the references above contain second order space derivative terms and allow the order of the time derivative to vary between 0 and 2. In this paper, we use the Adomian decomposition method to solve a fractional diffusion-wave equation that contains time and space fractional derivatives (with the Caputo space-fractional derivative of order $1 < \beta \leq 2$ instead of the second-order space derivative in Eq. (1.1). The Adomian method assumes a series solution for the unknown function u(x, t). Unlike the method of separation of variables that require initial and boundary conditions, the decomposition method may provide an analytic solution by using the initial conditions only. The boundary conditions can be used only to justify the obtained result. The Adomian decomposition method has many advantages over the classical techniques mainly, it avoids discretization and provides an efficient numerical solution with high accuracy, minimal calculations.

The structure of this paper is as follows. We begin by introducing some necessary definitions and mathematical preliminarirs of the fractional calculus theory which are required for establishing our results. In section 3 we extend application of the decomposition method to construct our numerical solutions for the fractional diffusion-wave equation. In section 4 we present two examples to show the efficiency and simplicity of the method and to demonstrate the behaviour of the solution of the fractional diffusion-wave equation as the the order of the time and space-fractional derivatives are changes.

2. Basic Definitions

We give some basic definitions and properties of the fractional calculus theiry which are used further in this paper.

Definition 2.1 *The Riemann-Liouville fractional integral operator of order* $\alpha \geq 0$, *of a function* $\alpha \in 0$, $\alpha \geq -1$, *is defined as*

$$J^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\, dt, \ \alpha > 1, x > 0,$$

$$J^0 f(x) = f(x).$$

Properties of the operator J^{α} can be found in [15], we mention only the following :

For $f \in C_{\mu}$, $\mu \geq -1$, α, $\beta \geq 0$ and $\gamma > -1$:

$$J^{\alpha} x^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}.$$

The Riemann-Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, we shall introduce now a modified fractional differential operator D^{α}_{*} proposed by M. Caputo in his work on the theory of viscoelasticity [6].

Definition 2.2. *The fractional derivative of f(x) in the Caputo sense is defined as*

$$D_*^{\alpha} f(x) = J^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t)\, dt, \tag{2.1}$$

for $m-1 < \alpha \leq m, m \in N, x > 0, f \in C_{-1}^m$.

Also, we need here two of its basic properties.

Lemma 2.1. *If* $m-1 < \alpha \leq m, m \in N$ *and* $f \in C_{\mu}^m$, $\mu \geq -1$, *then*

$$D_*^{\alpha} J^{\alpha} f(x) = f(x),$$

and,

$$J^{\alpha} D_*^{\alpha} f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{x^k}{k!}, \ x > 0.$$

In this paper, the fractional derivatives are considered in the Caputo sense. The reason for adopting the Caputo definition is as follows [9] : to solve defferential equations (both classical and fractional), we need to specify additional conditions in order to produce a unique solution. For the case of Caputo fractional differential equations, these additional conditions are just the traditional conditions, which are akin to those of classical differential equations, and are hterefore familiar to us. In contrast, for Riemann-Liouville fractional differential equations, these additional conditions constitute certain fractional derivatives (and/or integrals) of the unknown solution at the initial point $x = 0$, which are functions of x. These initial conditions are not physical; furthermore, it is not clear how such quatities are to be measured from experiment, say, so that they can be appropriately assigned in an analysis. For more details of the geometric and physical interpretation for fractional derivatives of both the Riemann-Liouville and Caputo types see [23].

3. Analysis of the Numerical Method

The standard form of the fractional diffusion equation (1.1) in an operator from

$$D_{*t}^{\alpha} u = b^2 D_{*x}^{\beta} u \text{ , } 0 < x < l, \; t > 0, \tag{3.1}$$

whee the time and space-fractional differential operators $D^{\alpha}{}_{*x}$ and $D^{\beta}{}_{*x}$ are defined as in (2.1), denoted by :

$$D_{*t}^{\alpha} = \frac{\partial^{\alpha}}{\partial t^{\alpha}} \text{ , } D_{*x}^{\beta} = \frac{\partial^{\beta}}{\partial x^{\beta}}$$

The method is based on applying the operator $J^{\alpha} = J^{\alpha}{}_{0}$, the inverse of the operator $D^{\alpha}{}_{*t}$, on both sides of equation (3.1) to obtain

$$u(x,t) = \sum_{k=0}^{m-1} \frac{\partial^k u}{\partial t^k}\left(x, 0^+\right)\frac{t^k}{k!} + J^{\alpha}\left(b^2 D_{*x}^{\beta} u\right) \tag{3.2}$$

The Adomian's decomposition method [1, 2] assumes a series solution for $u(x, t)$ given by

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \tag{3.3}$$

where the components $u_n(x, t)$ will be determined recursively. Substituting (3.3) into both sides of (3.2) gives

$$\sum_{n=0}^{\infty} u_n(x,t) = \sum_{k=0}^{m-1} \frac{\partial^k u}{\partial t^k}\left(x, 0^+\right)\frac{t^k}{k!} + J^{\alpha}\left(b^2 D_{*x}^{\beta}\left(\sum_{n=0}^{\infty} u_n(x,t)\right)\right) \tag{3.4}$$

Following the decomposition method, we introduce the recursive relations as

$$u_0(x,t) = \sum_{k=0}^{m-1} \frac{\partial^k u}{\partial t^k}\left(x, 0^+\right)\frac{t^k}{k!}, \tag{3.5}$$

$$u_{n+1}(x,t) = b^2 J^{\alpha}\left(D_{*x}^{\beta} u_n(x,t)\right), \; n \geq 0. \tag{3.6}$$

It is worth noting that if the zeroth component u_0 is defined than the remaining components u_n, $n \geq 1$, can be completely determined such that each terms are determined by using the previous terms, and the series solutions are thus entirely determined. Finally, we approximate the solution $u(x, t)$ by the truncated series

$$\phi N(x,t) = \sum_{n=0}^{N-1} u_n(x,t) \text{ and } \lim_{N \to \infty} \phi N(x,t) = u(x,t). \tag{3.7}$$

However, the inclusion of boundary conditons in fractional differential equations introduces additional difficulties. The Adomain decomposition method can handle these difficulties by using the time-fractional operator $D^{\alpha}_{*_t}$ and the initial conditions only. The method provides the solution in the form of a rapidly convergent series that may lead to the exact solution in the case of integer space derivative (β = 1 and 2) and to an efficient numerical solution with high accuracy for $1 < \beta < 2$ on the interval $0 \le x \le l$. The convergence of the decomposition series has been investigated by several authors [7, 8].

4. Applications and Results

In this section we present two examples to demonstrate the behavious of the solution of a fractional diffusion-wave equation as the order of the time and space-fractional derivatives are changed. In both examples, we take $b = l = 1$. For numerical computation, the series in Equation (3.7) is truncated after 30 terms and all the results are calculated by using the symbolic calculus software Mathematica.

Example 4.1. *Consider the following space- and time-fractional diffusion-wave equation*

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{\partial^{\beta} u}{\partial x^{\beta}}, \ 0 < \alpha \le 2,\ 1 < \beta \le 2,\ 0 < x < 1,\ t > 0, \tag{4.1}$$

subject to the initial conditions

$$u(x,0) = \sin(2\pi x),$$

$$\frac{\partial u(x,0)}{\partial t} = 2\pi \sin(2\pi x), \tag{4.2}$$

and the boundary conditions

$$u(0, t) = u(1, t) = 0,\ t \ge 0. \tag{4.3}$$

The last initial condition is assumed to ensure continuous dependence of the solution on the parameter α in in the transition from $\alpha = 1^-$ to $\alpha = 1^+$. Therefore in (4.1) we need to distinguish two cases :

Case I : for $0 < \alpha \le 1$. In this case we choose $m = 1$ in equation (3.5), and so upon using *Mathematica,* the solutions can be obtained as

$$u_0(x,t) = \sum_{k=0}^{0} \frac{\partial^k}{\partial t^k} u(x,0) \frac{t^k}{k!} = \sin(2\pi x),$$

$$u_1(x,t) = J^{\alpha}\left[D^{\beta}_{*x} u_0\right] = \frac{(2\pi)^{\beta} \sin\left(2\pi x + \frac{\pi\beta}{2}\right)}{\Gamma(\alpha+1)} t^{\alpha},$$

$$u_2(x,t) = J^{\alpha}\left[D_{*x}^{\beta}u_1\right] = \frac{(2\pi)^{2\beta}\sin\left(2\pi x + \frac{2\pi\beta}{2}\right)}{\Gamma(2\alpha+1)}t^{2\alpha},$$

$$u_3(x,t) = J^{\alpha}\left[D_{*x}^{\beta}u_2\right] = \frac{(2\pi)^{3\beta}\sin\left(2\pi x + \frac{3\pi\beta}{2}\right)}{\Gamma(3\alpha+1)}t^{3\alpha}$$

$$\vdots$$

$$u_n(x,t) = J^{\alpha}\left[D_{*x}^{\beta}u_{n-1}\right] = \frac{(2\pi)^{n\beta}\sin\left(2\pi x + \frac{n\pi\beta}{2}\right)}{\Gamma(n\alpha+1)}t^{n\alpha}$$

Substituting u_0, u_1, u_2, u_3, ... into (3.7) gives the solution $u(x, t)$ in a series form soltion by

$$u(x,t) = \sum_{k=0}^{n}\frac{(2\pi)^{k\beta}\sin\left(2\pi x + \frac{k\pi\beta}{2}\right)}{\Gamma(k\alpha+1)}t^{k\alpha}, \tag{4.4}$$

and therefore, the solution for the second-order heat equation (when $\alpha = 1$ and $\beta = 2$) is given by

$$u(x,t) = \sum_{k=0}^{n}\frac{(2\pi)^{2k}\sin(2\pi x + k\pi)}{\Gamma(k+1)}t^{k}. \tag{4.5}$$

The solution (4.5) can be written in a closed form solution given by $u(x,t) = e^{-4\pi^2 t}\sin(2\pi x)$, which can be verified through substitution.

Case II : for $1 < \alpha \le 2$. In this case we choose $m = 2$ in equation (3.5), and so upon using *Mathematica,* the solutions can be obtained as

$$u_0(x,t) = \sum_{k=0}^{1}\frac{\partial^k}{\partial t^k}u(x,0)\frac{t^k}{k!} = \sin(2\pi x) + 2\pi\sin(2\pi x)t,$$

$$u_1(x,t) = J^{\alpha}\left[D_{*x}^{\beta}u_0\right] = \frac{(2\pi)^{\beta}\sin\left(2\pi x + \frac{\pi\beta}{2}\right)}{\Gamma(\alpha+1)}t^{\alpha} + \frac{(2\pi)^{\beta+1}\sin\left(2\pi x + \frac{\pi\beta}{2}\right)}{\Gamma(\alpha+2)}t^{\alpha+1},$$

$$u_2(x,t) = J^{\alpha}\left[D_{*x}^{\beta}u_1\right] = \frac{(2\pi)^{2\beta}\sin\left(2\pi x + \frac{2\pi\beta}{2}\right)}{\Gamma(2\alpha+1)}t^{2\alpha} + \frac{(2\pi)^{2\beta+1}\sin\left(2\pi x + \frac{2\pi\beta}{2}\right)}{\Gamma(2\alpha+2)}t^{2\alpha+1},$$

$$u_3(x,t) = J^{\alpha}\left[D_{*x}^{\beta}u_2\right] = \frac{(2\pi)^{3\beta}\sin\left(2\pi x + \frac{3\pi\beta}{2}\right)}{\Gamma(3\alpha+1)}t^{3\alpha} + \frac{(2\pi)^{3\beta+1}\sin\left(2\pi x + \frac{3\pi\beta}{2}\right)}{\Gamma(3\alpha+2)}t^{3\alpha+1}$$

$$\vdots$$

$$u_n(x,t) = J^{\alpha}\left[D_{*x}^{\beta}u_{n-1}\right] = \frac{(2\pi)^{n\beta}\sin\left(2\pi x + \frac{n\pi\beta}{2}\right)}{\Gamma(n\alpha+1)}t^{n\alpha} + \frac{(2\pi)^{n\beta+1}\sin\left(2\pi x + \frac{n\pi\beta}{2}\right)}{\Gamma(n\alpha+2)}t^{n\alpha+1}$$

and so on. Substituting the above components into (3.7), we obtain the solution in a series form as

$$u(x,t) = \sum_{k=0}^{n}\sin\left(2\pi x + \frac{k\pi\beta}{2}\right)\left[\frac{(2\pi)^{k\beta}}{\Gamma(k\alpha+1)}t^{k\alpha} + \frac{(2\pi)^{k\beta+1}}{\Gamma(k\alpha+2)}t^{k\alpha+1}\right], \tag{4.6}$$

and therefore, the solution for the second-order wave equation (when $\alpha = 2$ and $\beta = 2$) is given by

$$u(x,t) = \sum_{k=0}^{n}\sin(2\pi x + k\pi)\left[\frac{(2\pi)^{2k}}{\Gamma(2k+1)}t^{2k} + \frac{(2\pi)^{2k+1}}{\Gamma(2k+2)}t^{2k+1}\right] \tag{4.7}$$

and in this case the solution can be written in a closed form

$$u(x,t) = \sin(2\pi x)\left(\sin(2\pi t) + \cos(2\pi t)\right).$$

In order to examine the effect of varying the order of the time-fractional derivative on the behaviour of the solution, we take the space derivative $\beta = 2$ and vary the order of the time-fractional derivative α. Figures 1 – 4 show the evolution results for $\alpha = 1, 2, \frac{3}{4}$, and $\frac{3}{2}$, respectively. Figures 1 and 2 ($\alpha = 1$ and 2) show the diffusion and the wave solutions, respectively. Figures 1 and 3 shows that compared to the 1-order time derivative system, the $\frac{3}{4}$-order time fractional derivative system exhibits fast diffusion in the beginning and slow diffusion later. Similar behaviour was observed in [3, 5]. Figure 4 shows that for $1 < \alpha < 2$, the system exhibits the combined diffusion and wave behaviour.

Next, we examine the effect of varying the space-fractional derivative on the behaviour of solution. Therefore, we set the time-fractional derivative $\alpha = 1$ and vary the space-fractional derivative β. Figure 5 and 6 show the evolution results for $\beta = \frac{3}{4}$ and $\frac{7}{4}$, respectively. It can be seen from Figures 5 and 6 that the solution continuously depends on the space-fractional derivative.

Example 4.2. *In this example we consider the following space- and time-fractional diffusion equation*

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{\partial^{\beta} u}{\partial x^{\beta}}, \; 0 < \alpha \le 1, 1 < \beta \le 2, \; t > 0, \tag{4.8}$$

subject to the initial condition

$$u(x,0) = f(x).$$

and the boundary conditions

$$u(0,t) = u(1,t) = 0, \; t \ge 0. \tag{4.10}$$

Setting m = 1 and substituting the initial conditions (4.9) into equation (3.5), yields the following recursive relations :

$$u_0 = \sum_{k=0}^{0} \frac{\partial^k}{\partial t^k} u(x,0) \frac{t^k}{k!} = f(x),$$

$$u_{n+1}(x,t) = J^{\alpha}\left[D_{*x}^{\beta} u_{n-1}\right].$$

Using the above recursive relationship and Mathematica, the first few terms of the decomposition series are give by

$$u_0(x,t) = \sum_{k=0}^{0} \frac{\partial^k}{\partial t^k} u(x,0) \frac{t^k}{k!} = f(x)$$

$$u_1(x,t) = J^{a}\left[D_{*x}^{\beta} u_0\right] = \frac{t^{\alpha}}{\Gamma(\alpha+1)} D_{*x}^{\beta} f(x)$$

$$u_2(x,t) = J^{a}\left[D_{*x}^{\beta} u_1\right] = \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} D_{*x}^{2\beta} f(x)$$

$$u_3(x,t) = J^{a}\left[D_{*x}^{\beta} u_2\right] = \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} D_{*x}^{3\beta} f(x)$$

$$\vdots$$

$$u_n(x,t) = J^{a}\left[D_{*x}^{\beta} u_{n-1}\right] = \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} D_{*x}^{n\beta} f(x)$$

The solution is series form is given by

$$u(x,t)=\sum_{k=0}^{n}\frac{t^{ka}}{\Gamma(k\alpha+1)}D_{*x}^{k\beta}f(x) \tag{4.11}$$

To show the efficiency of the present method for solving the fractional diffusion equation (4.8), we choose the initial condition (4.9) to be the simple function $f(x)=x^2-x$. Then the series solution (4.11) becomes

$$u(x,t)=\sum_{k=0}^{n}\frac{t^{ka}}{\Gamma(k\alpha+1)}\left(\frac{2x^{2-k\beta}}{\Gamma(3-k\beta)}-\frac{x^{1-k\beta}}{\Gamma(2-k\beta)}\right) \tag{4.12}$$

It is clear that the general solution obtained in the above example can be used for numerical purposes only because a closed form solution of the above equation is not available. However, more terms can be determined to improve the accuracy level.

5. Conclusions

The main concern of this work has been to construct a general solution for space- and time-fractional diffusion-wave equation defined in finite space domians. The goal has been achieved by applying Adomian decomposition method and by using the initial conditions only. The analytical results have been given in terms of a power series with easily computed terms. The fractional derivatives were defined in Caputo sense. In special cases of α = 1 and α = 2, the general solution reduces to the diffusion and wave solutions, respectively.

The effect of varying the time and space-fractional derivatives on the behaviour of solutions has been investigated. Numerical results (when $\alpha=\frac{3}{4}$ and β = 2) leads to fast diffusion in the beginning and slow diffusion later, and when $\alpha=\frac{3}{2}$ and β = 2 leads a process between diffusion and wave propagation. The results show that the solution continuously depends on the space-fraction derivative.

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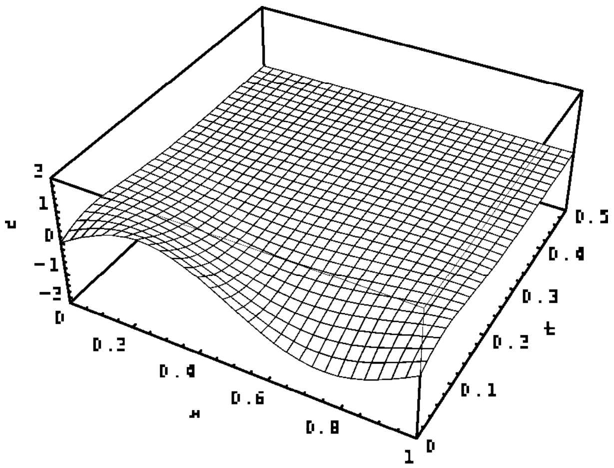

Figure 1 : Evolution of the initial state ($\alpha = 1$ and $\beta = 2$)

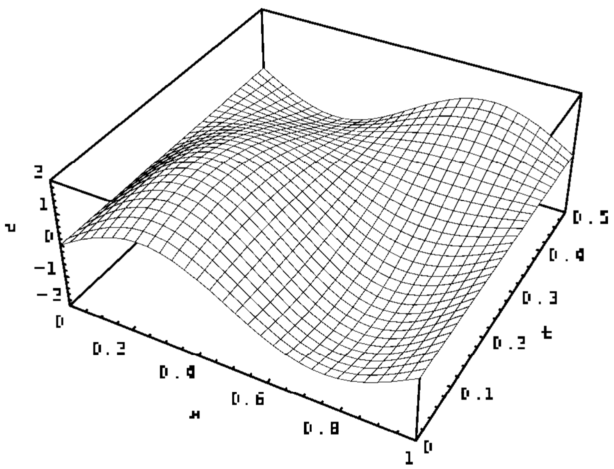

Figure 2 : Evolution of the initial state ($\alpha = 2$ and $\beta = 2$)

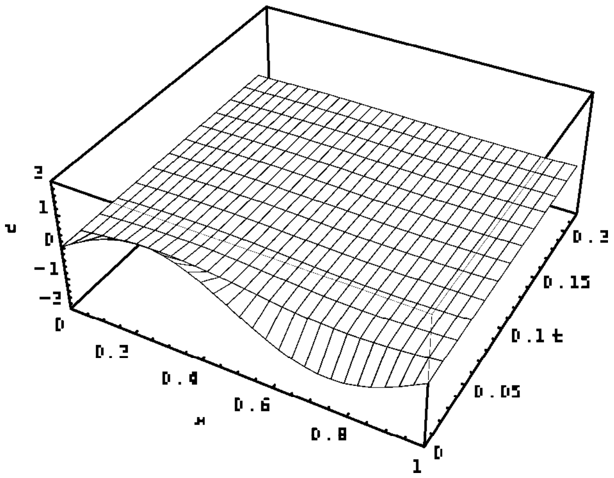

Figure 3 : Evolution of the initial state ($\alpha = \frac{3}{4}$ and $\beta = 2$)

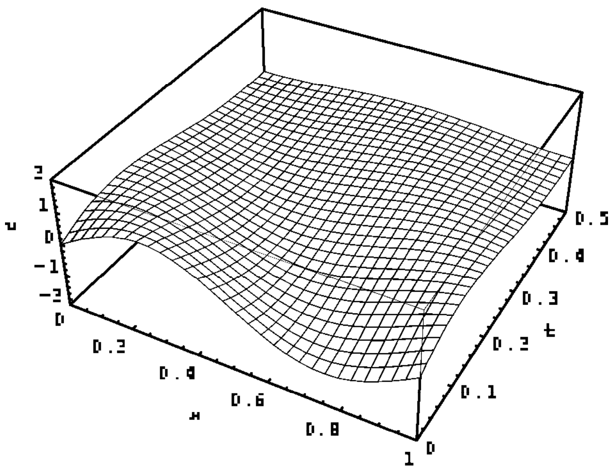

Figure 4 : Evolution of the initial state ($\alpha = \frac{3}{2}$ and $\beta = 2$)

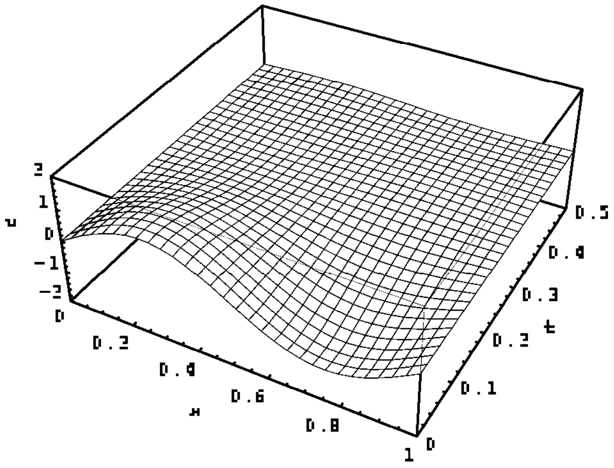

Figure 5 : Evolution of the initial state ($\alpha = 1$ and $\beta = \frac{5}{4}$)

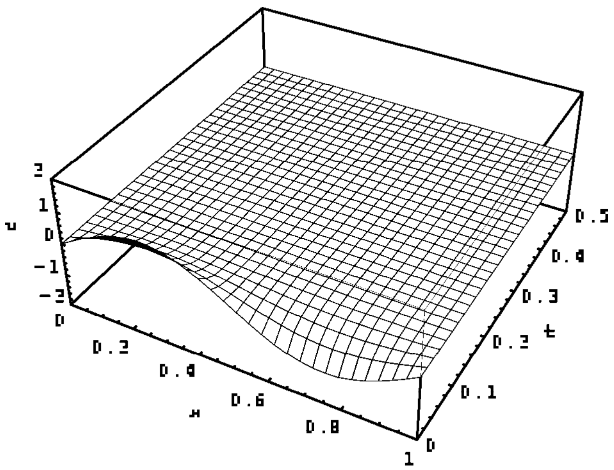

Figure 6 : Evolution of the initial state ($\alpha = 1$ and $\beta = \frac{7}{4}$)