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Existence of Nonoscillatory Solution of High-Order Neutral Difference Equations with Continuous Arguments

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ABSTRACT

In this paper, we consider the following high-order neutral difference equations with continuous arguments

$$\Delta_\tau^n\left(x(t)-p(t)x(t-\sigma)\right)=(-1)^n q(t)x\left(g(t)\right)$$

where $n \geq 1$ is an integer, $\tau,\sigma\in R^+, q,g\in C\left([t_0,\infty],R^+\right)$, $R^+=(0,\infty)$, $q(t)\not\equiv 0, g(t)\le t, \lim_{t\to\infty} g(t)=\infty$. We obtain the existence of nonoscillatory solutions under some conditions and necessary and sufficient conditions by using proper fixed point theorem.

Keywords : Neutral difference equations; Nonoscillatory solutions; Existence; Integral transformations.

1. Introduction

In this paper, we consider the following high-order neutral difference equations with continuous arguments

$$\Delta_\tau^n\left(x(t)-p(t)x(t-\sigma)\right)=(-1)^n q(t)x\left(g(t)\right) \tag{1}$$

where $n \geq 1$ is an integer, $\tau,\sigma\in R^+, q,g\in C\left([t_0,\infty],R^+\right)$, $R^+=(0,\infty)$, $q(t)\not\equiv 0, g(t)\le t$, $\lim_{t\to\infty} g(t)=\infty$. The forward difference Δ_τ is defined as usual, i.e., $\Delta_\tau x(t)=x(t+\tau)-x(t)$.

Throughout the paper we shall define the usual factorial expression

$$j^{(k)} = j(j-1)\ldots(j-k+1),\ j^{(0)} = 1.$$

By a solution of Eq.(1), we mean a function $x(t)$ which is defined for all $t - \sigma \geq t_0$, $g(t) \geq t_0$ and satisfies Eq.(1).

The neutral difference equations are increasingly noticed in many fields. The most important reason is that many problems in other fields such as biology, population dynamics and economics may be translated into models of difference equation.

Oscillation and nonoscillation theories of neutral functional differential equations and difference equations have developed very quickly in recent years. We refer to [1-12]. Zhang, Bi [7] investigated the oscillation of solution of second order neutral difference equation with continuous argument. Existence of nonoscillatory solutions of high-order neutral differential equations with positive and negative coefficients

$$\frac{d^m}{dt^m}\left[x(t)+cx(t-\tau)\right]+(-1)^{n+1}\left[P(t)x(t-\sigma)-Q(t)x(t-\sigma)\right]=0$$

has been investigated by Zhou and Zhang [11]. The higher-order neutral difference equations with continuous arguments have received much less attention, which is mainly due to the technical difficulties arising in its analysis. In particular, there is no nonoscillation result for Eq.(1).

In this paper, we obtain the existence of nonoscillatory solutions under some conditions and necessary and sufficient conditions by using proper fixed point theorem.

A solution $x(t)$ of Eq.(1) is called oscillatory if it is neither eventually positive nor eventually negative, otherwise, it is nonoscillatory. The equation is called oscillatory if all its solutions are oscillatory.

Throughout the paper we will use an equality :

$$\sum_{i=0}^{\infty}\frac{(n+i-2)^{(n-1)}}{(n-1)!}q(t+i\tau)=\sum_{i_{n-1}=0}^{\infty}\sum_{i_{n-2}=i_{n-1}}^{\infty}\cdots\sum_{i=i_1}^{\infty}q(t+i\tau)$$

2. Related Lemma and Main Results

Lemma 2.1. *Suppose that $p(t)$ is bounded and $x(t)$ is a bounded positive solution of Eq.(1). Set $z(t) = x(t) - p(t)x(t-\sigma)$. Then $\Delta_\tau^i z(t)\, i=1,2,...,n$ is constant sign, $(-1)^{(i)}\Delta_\tau^i z(t)>0$, $i = 1, 2, ..., n$, and $\lim_{k\to\infty}\Delta_\tau^i z(t+k\tau)=0$.*

Proof. Due to $x(t)$ and $p(t)$ are bounded, $z(t)$ is bounded. $\Delta_\tau^{n-1}z(t)$ is increasing on t from $\Delta_\tau^n z(t)>0$. We shall show $\Delta_\tau^{n-1}z(t)<0$, If not, there exists $T > t_0$, such that $\Delta_\tau^{n-1}z(T)\geq 0$. Due to $\Delta_\tau^{n-1}z(T)=\Delta_\tau^{n-2}z(T+\tau)-\Delta_\tau^{n-2}z(T)$, there exists $l_1 > 0$ such that $\Delta_\tau^{n-2}z(T+\tau)>\Delta_\tau^{n-2}z(T)+l_1$. So $\lim_{k\to\infty}\Delta_\tau^{n-2}z(T+k\tau)=\infty$, which contradicts the boundedness of $z(t)$. Thus we have shown $\Delta_\tau^n z(t) < 0$ and $\Delta_\tau^n(t)$ is constant sign. By induction we have $(-1)^i \Delta_\tau^i z(t) > 0$ and $(-1)^i \Delta_\tau^i z(t)$ is constant sign for $i = 1, 2, ..., n$. From above we have $\Delta_\tau^i z(t)$ is bounded.

Next we shall show $\lim_{k\to\infty} \Delta_\tau^i z(t+k\tau) = 0$. If not, when i is odd, we know $(-1)^i \Delta_\tau^i z(t) < 0$. Assume there exists $l_2 > 0$ such that $\lim_{k\to\infty} \Delta_\tau^i z(t+k\tau) = -l_2 < 0$. So there exists N such that $\Delta_\tau^i z(t+k\tau) < -l_2/2$ for $k > N$. So $\Delta_\tau^{i-1} z(t+(N-1)\tau) <$ $\Delta_\tau^{i-1} z(t+N\tau) - l_2/2$. We have $\lim_{k\to\infty} \Delta_\tau^{i-1} z(t+k\tau) = -\infty$, which contradicts the boundness of $z(t)$. When i is even, we can get a contradiction similar to the above proof. So $\lim_{k\to\infty} \Delta_\tau^i z(t+k\tau) = 0$.

Theorem 2.1. *Assume there exist positive constants p_1, p_2 such that*

$$1 < p_1 \le p(t) \le p_2, t \ge t_0 \tag{2.1}$$

Then Eq.(1) *has a bounded positive solution between two positive constants if and only if*

$$\sup_{t\in[t_0, t_0+\tau]} \sum_{i=0}^{\infty} \frac{(n+i-2)^{(n-1)}}{(n-1)^{(n-1)}} q(t+i\tau) < \infty \tag{2.2}$$

Proof. *Sufficiency.* Consider the Banach space BC of all continuous bounded functions x defined on $[t_0, \infty]$ with sup norm, i.e., $\|x\| = \sup_{t\ge t_0} |x(t)|$. Let Ω denote the subset of BC defined by

$$\Omega = \left\{ x \in BC : \frac{2p_1}{3(p_2-1)} \le x(t) \le \frac{4p_1}{3(p_1-1)} \right\}$$

then Ω is a bounded, closed and convex subset of BC.

By (2.2) let $\in \frac{p_1-1}{4}$, there exists N, such that

$$\sup_{t\in[t_0, t_0+\tau]} \sum_{i=N=1}^{\infty} \frac{(n+i-2)^{(n-1)}}{(n-1)!} q(t+i\tau) < \frac{p_1-1}{4}$$

for $n \geq N$. So

$$\sup_{t\in[t_0,t_0+\tau]} \sum_{i=N+1}^{\infty} \frac{(n+i-N-1-2)^{(n-1)}}{(n-1)!} q(t+i\tau) < \frac{p_1-1}{4}$$

that is

$$\sup_{t\in[t_0,t_0+\tau]} \sum_{j=0}^{\infty} \frac{(n+j-2)^{(n-1)}}{(n-1)!} q(t+(N+1)\tau+j\tau) < \frac{p_1-1}{4}$$

Let $t_1 = t_0 + (N + 1)\tau$, so

$$\sup_{t\in[t_0(N+1)\tau,t_0+(N+2)\tau]} \sum_{j=0}^{\infty} \frac{(n+j-2)^{(n-1)}}{(n-1)!} q(t_1+j\tau) < \frac{p_1-1}{4}$$

i.e.

$$\sum_{i_{n-1}=0}^{\infty} \sum_{i_{n-2}=i_{n-1}}^{\infty} \ldots \sum_{i=i_1}^{\infty} q(t+\sigma+i\tau) < \frac{p_1-1}{4} \quad \text{for } t > t_1. \tag{2.3}$$

Define a mapping $T:\Omega \to BC$ as follows :

$$(Tx)(t) = \begin{cases} \dfrac{1}{p(t+\sigma)}\left(p_1 + x(t+\sigma) - \displaystyle\sum_{i=0}^{\infty} \frac{(n+i-2)^{(n-1)}}{(n-1)^{(n-1)}} q(t+\sigma+i\tau)x(g(t+\sigma+i\tau)) \right), & t \geq t_1 \\ (Tx)(t_1), & t_0 \leq t \leq t_1 \end{cases}$$

we shall show $T\Omega \subset \Omega$. In fact, for every $x \in \Omega$ and $t \geq t_1$, using (2.1) we get

$$(Tx)(t) \leq \frac{1}{p_1}\left[p_1 + \frac{4p_1}{3(p_1-1)} \right] = \frac{3p_1+1}{3(p_1-1)} \leq \frac{4p_1}{3(p_1-1)}$$

and

$$(Tx)(t) \geq \frac{1}{p_2}\left[p_1 + \frac{2p_1}{3(p_2-1)} - \frac{p_1-1}{4}\frac{4p_1}{3(p_1-1)} \right] = \frac{2p_1}{3(p_2-1)} t \geq t_0$$

Thus we prove that T maps Ω into Ω. Next we shall show T is a contraction on Ω.
In fact, for any x , y ∈ Ω and $t \geq t_0$, using (2.1) (2.3) we have

$$\left|(Tx)(t) - (Ty)(t)\right|$$

$$\geq \frac{1}{p(t+\sigma)}\left(|x(t+\sigma)-y(t+\sigma)| + \sum_{i=0}^{\infty} \frac{(n+i-2)^{(n-1)}}{(n-1)^{(n-1)}} q(t+\sigma+i\tau) |x(g(t+\sigma+\tau)) - y(g(t+\sigma+\tau))| \right)$$

$$\leq \frac{1}{p(t+\sigma)}\left(1 + \sum_{i=0}^{\infty} \frac{(n+i-2)^{(n-1)}}{(n-1)^{(n-1)}} q(t+\sigma+i\tau) \right) \|x-y\|$$

$$\leq \frac{1}{p_1}\left(1 + \frac{p_1-1}{4} \right) \|x-y\|$$

$$= \frac{p_1+3}{4p_1}\|x-y\|$$

where $0 < \frac{p_1+3}{4p_1} < 1$, which shows that T is a contraction on Ω. Then by the Banach contraction principle, T has a fixed point x ∈ Ω, that is, *Tx* = *x*. So *x*(*t*) is a bounded positive solution of Eq. (1) on $[t_1, \infty)$.

Necessity. Let *x*(*t*) be a bounded positive solution between two positive constants of Eq. (1). That is $0 < l \leq x(t) \leq L$. Set $z(t) = x(t) - p(t)x(t-\tau)$. When *n* is even, we have

$$\Delta_\tau^n z(t) = q(t)x(g(t)) \geq lq(t)$$

Summing the above inequality on i from 1 to *N* and set $N \to \infty$, Using Lemma 2.1 we get (2.2) holds. When *i* is odd, we have

$$\Delta_\tau^n z(t) = q(t)x(g(t)) \leq -lq(t)$$

Summing the above inequality in *i* from 1 to N and let $N \to \infty$, using Lemma 2.1 we get (2.2) holds.

Example 1. Consider the difference equation

$$\Delta_1^3\left(x(t) - \frac{(2t-1)^2}{2t(t-1)} x\left(t-\frac{1}{2}\right) \right) + \frac{3}{(t-1)(t+3)^{(3)}} x\left(\frac{t}{2}\right) = 0$$

In out notation, $n = 3$, $\tau = 1$, $\sigma = ½$, $q(t) = \frac{t}{2}$, $q(t) = \frac{3}{(t-1)(t+3)^{(3)}}$,

$1 < 1.5 \le p(t) = \frac{(2t-1)}{2t(t+1)} \le 2.$ It is easy to show that the conditions (2.1) (2.2) in Th. 2.1 are satisfied. Therefore, the equation has a bounded positive solution between two positive constants. In fact, $x(t) = 1 - \frac{1}{2t}$ is the solution satisfied the conditions.

Theorem 2.2. *Assume there positive constants $P_3 > 0$ such that*

$$0 \le p(t) \le p_3 < 1 \tag{2.4}$$

$p(t)$ and $q(t)$ will not become zero at the same time, and (2.2) holds. Then Eq. (1) has a bounded positive solution.

Proof. From (2.2), there exists t_2 such that

$$p_3 + \sum_{i=0}^{\infty} \frac{(n+i-2)^{(n-1)}}{(n-1)^{(n-1)}} q(t+ir) \le 1,\ t \ge t_2 \tag{2.5}$$

Let BC be the set as in the proof of Theorem 2.1. Set

$$\Omega = \{x \in BC : 0 \le x(t) \le 1\}$$

Define an operator $T : \Omega \to BC$ as follows :

$$(Tx)(t) = \begin{cases} p(t)x(t-\sigma) + \sum_{i=0}^{\infty} \frac{(n+i-2)^{(n-1)}}{(n-1)^{(n-1)}} q(t+ir)x(g(t+ir)) & t \ge t_2 \\ (Tx)(t_2) & t_0 \le t < t_2 \end{cases}$$

Set $x_0(t) = 1, x_k(t) = (Tx_{k-1})(t) \qquad k = 1,2,\ldots,n.t \ge t_0$

It is easy to show

$$x_1(t) \le x_0(t) = 1 \qquad t \in [t_0, \infty).$$

By induction we have $0 \le x_{k+1}(t) \le x_k(t) \le 1 \quad t \in [t_0, \infty), k = 1,2,\ldots,n$ In fact,

$$x_k(t) = (Tx_{k-1})(t) = p(t)x_{k-1}(t-\sigma) + \sum_{i=0}^{\infty} \frac{(n+i-2)^{(n-1)}}{(n-1)^{(n-1)}} q(t+i\tau)x_{k-1}(g(t+i\tau))$$

$$\geq p(t)x_k(t-\sigma)+\sum_{i=0}^{\infty}\frac{(n+i-2)^{(n-1)}}{(n-1)^{(n-1)}}q(t+i\tau)x_k(g(t+i\tau))$$

$$=x_{k+1}(t),$$

using (2.4) (2.5) we get

$$0\leq x_k(t)\leq p_3+\sum_{i=0}^{\infty}\frac{(n+i-2)^{(n-1)}}{(n-1)^{(n-1)}}q(t+i\tau)\leq 1.$$

By Lebesgue's convergence theorem we have

$$x(t)=\begin{cases} p(t)x(t-\sigma)+\sum_{i=0}^{\infty}\frac{(n+i-2)^{(n-1)}}{(n-1)^{(n-1)}}q(t+i\tau)x(g(t+i\tau)) & t\geq t_2 \\ x(t_2) & t_0\leq t<t_2 \end{cases}$$

We shall show $x(t) > 0$. If not, there exists $T > t_2$ such that $x(T) = 0$ and $x(t) > 0$ for $t_2 \leq t < T$. Under the condition of $p(t)$, $q(t)$, we get $x(T)\geq p(T)x(T-\sigma)$ or $x(T)\geq q(T)x(g(T))$. So $x(T-\sigma)=0$ or $x(g(T))=0$. Due to $T-\sigma<T$ and $g(T)<T$ we get a contradiction. Hence $x(t)$ is a bounded positive solution of Eq. (1).

Example 2. Consider the difference equation

$$\Delta_1^4\left(x(t)-\frac{(t-2)^2}{2t(t-1)}x(t-2)\right)=\frac{48}{(t+5)^{(5)}}x\left(\frac{t}{5}\right)$$

In our notation, $n=4$, $\tau=1$, $\sigma=2$, $g(t)=\frac{t}{5}$, $q(t)=\frac{48}{(t+5)^{(5)}}$, $0\leq p(t)=\frac{(t+2)(t-2)}{2t(t-1)}\leq \frac{3}{4}<1$, as $t\geq 4$. It is easy to show that the conditions (2.1) (2.2) in Th. 2.1 are satisfied. Therefore, the equation has a bounded positive solution between two positive constants. In fact, $x(t) = 1 + \frac{1}{t}$ is the solution satisfied the conditions.

Example 3. Consider the difference equation

$$\Delta_1^3\left(x(t)-\frac{1}{8}x(t-2)\right)+\frac{2^{-t+\sqrt{t}}}{16\left(2^{\sqrt{t}}+1\right)}x\left(\sqrt{t}\right)=0$$

In our notation, $n = 3$, $\tau = 1$, $\sigma = 2$, $p(t)=\frac{1}{8}$, $g(t)=\sqrt{t}$, $q(t)=\frac{2^{-t+\sqrt{t}}}{16\left(2^{\sqrt{t}}+1\right)}$ It is easy to show that the conditions (2.2) (2.4) in Th. 2.2 are satisfied. Therefore, the equation has a bounded positive solution. In fact, $x(t)=1+2^{-t}$ is the solution satisfied the conditions.

Theorem 2.3. *Assume* $0\le p(t)<1$ *and there exists* $\alpha>0$, *such that*

$$\limsup_{t\to\infty}\left\{p(t)\exp(\alpha\sigma)+\sum_{i=0}^{\infty}\frac{(n+i-2)^{(n-1)}}{(n-1)^{(n-1)}}q(t+i\tau)\exp\left[-\alpha\left(-t+g(t+i\tau)\right)\right]\right\}<1 \tag{2.6}$$

Then Eq. (1) has an eventually positive solution which tends to zero as $t\to\infty$.

Proof. We assume that there exists $t_3>t_0$ such that $t-\sigma\ge t_0$ and $g(t)\ge t_0$ for $t\ge t_3$ and

$$\beta(t)=p(t)\exp(\alpha\sigma)+\sum_{i=0}^{\infty}\frac{(n+i-2)^{(n-1)}}{(n-1)^{(n-1)}}q(t+i\tau)\exp\left[-\alpha\left(-t+g(t+i\tau)\right)\right]<1 \tag{2.7}$$

Let Y be the set of all continuous functions y defined on $[t_0,\infty)$ satisfying $0\le y(t)\le 1$ for $t\ge t_0$. Set Y be endowed with the usual point-wise ordering : $y_1\le y_2$ if $y_1(t)\le y_2(t)$ for all $t\ge t_0$. it is easy to see that for any subset A of Y, there exist *inf A* and *sup A*.

Define an operation on Y as follows :

$$(Sy)(t)=\begin{cases}p(t)\exp(\alpha\sigma)y(t-\sigma)+\\ \sum_{i=0}^{\infty}\frac{(n+i-2)^{(n-1)}}{(n-1)^{(n-1)}}q(t+i\tau)y\left(g(t+i\tau)\right)\exp\left[-\alpha\left(-t+g(t+i\tau)\right)\right] & t\ge t_3\\ (Sy)(t_3)+\exp\left[\varepsilon(t_3-t)\right]-1 & t_0\le t<t_3\end{cases}$$

where $\in=\dfrac{\ln\left(2-\beta\left(t_3\right)\right)}{t_3-t_0}$.

We will show SY $\in$ Y. In fact, for any $y\in\Omega$ and $t\geq t_3$, using (2.6) (2.7) we have

$$\begin{aligned}(Sy)(t)&=p(t)\exp(\alpha\sigma)y(t-\sigma)\\&\quad+\sum_{i=0}^{\infty}\frac{(n+i-2)^{(n-1)}}{(n-1)^{(n-1)}}q(t+i\tau)y(g(t+i\tau))\exp\left[-\alpha(-t+g(t+i\tau))\right]\\&\leq p(t)\exp(\alpha\sigma)+\sum_{i=0}^{\infty}\frac{(n+i-2)^{(n-1)}}{(n-1)^{(n-1)}}q(t+i\tau)\exp\left[-\alpha(-t+g(t+i\tau))\right]\\&<1,\ t\geq t_3,\end{aligned}$$

and

$$\begin{aligned}(Sy)(t)&=(Sy)(t_3)+\exp\left[\varepsilon(t_3-t)\right]-1\\&\leq\beta(t)+\exp\left[\varepsilon(t_3-t)\right]-1\\&\leq 1,\ t_0\leq t<t_3,\end{aligned}$$

and $(Sy)(t)>0$. So $(Sy)(t)\in Y$, i.e., $SY\in Y$. Moreover S is a nondecreasing mapping. By Knaster's fixed point theorem, there exists $y\in Y$ such that $Sy=y$. Since $y(t)>0$ for $t_0\leq t\leq t_3$, it follows that $y(t)>0$ for all $t\geq t_0$.

Let $x(t)=y(t)\exp(-\alpha t)$ we get

$$\Delta_\tau^n\left(x(t)-p(t)x(t-\sigma)\right)=(-1)^n q(t)x(g(t))\quad t\geq T$$

i.e., $x(t)$ is a positiove solution of Eq. (1) and $\lim_{t\to\infty}x(t)=0$.

Example 4. Consider the difference equatio

$$\Delta_1^3\left(x(t)-\frac{1}{18}x(t-1)+\frac{1}{9}.2^{-3t/4}x\left(\frac{t}{4}\right)\right)=0$$

In our notation, $n=3$, $\tau=1$, $\sigma=1$, $p(t)=\dfrac{1}{18}$, $g(t)=\dfrac{t}{4}$, $q(t)=\dfrac{1}{9}.2^{-3t/4}$. Clearly,

$0 \le p(t) < 1$. We will show (2.6) holds. In fact, let $\alpha = \frac{1}{2}$

$$p(t)\exp(\alpha\sigma) + \sum_{i=0}^{\infty} \frac{(n+i-2)^{(n-1)}}{(n-1)^{(n-1)}} q(t+i\tau)\exp\left[-\alpha\left(-t+g\left(t+i\tau\right)\right)\right]$$

$$= \frac{\sqrt{e}}{18} + \sum_{i=0}^{\infty} \frac{(i+1)^{(2)}}{18} 2^{\frac{-3(t+i)}{4}} e^{-\frac{1}{2}\left(-t+\frac{t+i}{4}\right)}$$

$$< \frac{1}{9} + \left(\frac{2}{\sqrt{e}}\right)^{-3t/4} . \sum_{i=0}^{\infty} \frac{(i+1)^{(2)}}{18} 2^{-\frac{7i}{8}} < 1$$

The above inequality holds for all large t, i.e. (2.6) holds. Therefore, the equation has a bounded positive solution positive solution which tends to zero as $t \to \infty$. In fact, $x(t) = 2^{-t}$ is the solution satisfied the conditions.

Theorem 2.4. *Assume* $p(t) \equiv 1$. *Then Eq.* (1) *has a bounded positive solution if and only if*

$$\sum_{k=0}^{\infty}\sum_{i=0}^{\infty} \frac{(n+i-2)^{(n-1)}}{(n-1)^{(n-1)}} q(t_0 + i\tau - k\sigma) < \infty \tag{2.8}$$

Proof. *Sufficiency.* Fro (2.8) there exists t_4 such that $t_4 > t_0$ and

$$\sum_{k=0}^{\infty}\sum_{i=0}^{\infty} \frac{(n+i-2)^{(n-1)}}{(n-1)^{(n-1)}} q(t + i\tau - k\sigma) \le \frac{1}{2} \tag{2.9}$$

Let BC the set as in the proof of Theoren 2.1. Set

$$\Omega = \left\{ x \in BC : \frac{1}{2} \le x(t) \le 1 \right\}$$

Define operator $T : \Omega \to BC$ as follows :

$$(Tx)(t) = \begin{cases} \frac{1}{2} + \sum_{k=0}^{\infty}\sum_{i=0}^{\infty} \frac{(n+i-2)^{(n-1)}}{(n-1)^{(n-1)}} q(t+i\tau-k\sigma)x\left(g\left(t+i\tau-k\sigma\right)\right) & t \ge t_4 \\ (Tx)(t_4) & t_0 \le t < t_4 \end{cases}$$

Set $x_0(t) = 1$

$x_k(t) = (Tx_{k-1})(t)$ $k = 1, 2, ...,$

It is easy to show $\frac{1}{2} \le x_1(t) \le x_0(t) = 1\, t \in (t_0 \infty)$. By induction we have

$$\frac{1}{2} \le x_{k+1}(t) \le x_k(t) \le 1, \quad t \in [t_0, \infty), \quad k = 1, 2, ...$$

By Lebergue's monotonic convergence theorem, T has a fixed point x, i.e. $Tx = x$. Hence $x(t)$ is a bounded positive solution of Eq. (1).

Necessity. Let $x(t)$ be a bounded positive solution of Eq. (1). So

$$\Delta_\tau^n \left(x(t_0 + i_1\tau + \cdots + i_n\tau - k\sigma) - x(t_0 - \sigma + i_1\tau + \cdots + i_n\tau - k\sigma) \right)$$

$$= \quad (-1)^n q(t_0 + i_1\tau + \cdots + i_n\tau - k\sigma) x\left(g(t_0 - i_1\tau + \cdots + i_n\tau - k\sigma)\right)$$

Summing above equality in $i_1, i_2, ..., i_n, k$ from 0 to N set $N \to \infty$, by Lemma 2.1 we have

$$x(t_0) = \sum_{k=0}^{\infty} \sum_{i_n = i_{n-1}}^{\infty} \cdots \sum_{i_2 = i_1}^{\infty} \sum_{i_1 = 0}^{\infty} q(t_0 + i_1\tau + i_2\tau + \cdots + i_n\tau - k\sigma) \times$$

$$x\left(g(t_0 + i_1\tau + i_2\tau + \cdots + i_n\tau - k\sigma)\right) \tag{2.10}$$

$$= \quad \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \frac{(n+i-2)^{(n-1)}}{(n-1)^{(n-1)}} q(t_0 + i\tau - k\sigma) x\left(g(t_0 + i\tau - k\sigma)\right).$$

Set $y(t) = x(t) - x(t-\sigma)$. We shall show that $y(t) > 0$. If not, there exists $T > t_0$, $l > 0$ such that $y(T) = -l < 0$. So $x(T) - x(T-\sigma) = -l < 0$, that is $\lim\limits_{k\to\infty} x(T + n\sigma) = \lim\limits_{k\to\infty} x(T) - nl = -\infty$, which contradicts the boundness of $x(t)$. So $y(t) > 0$ i.e. $x(t) > x(t-\sigma)$. So there exists $M>0$ and $T > t_0$ such that $x(g(t)) \ge M$ for $t \ge T$. From (2.10) we get

$$x(t_0) \ge \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \frac{(n+i-2)^{(n-1)}}{(n-1)^{(n-1)}} q(t_0 + i\tau - k\sigma) M$$

So (2.8) holds.

Theorem 2.5. *Assume there exists positive constants p_4 such that*

$$-1 < p_4 \le p(t) < 0 \tag{2.11}$$

and (2.2) *holds. Then Eq.* (1) *has a bounded positive solution.*

Proof. From (2.2), there exists $t_4 > t_0$ such that

$$\sum_{i=0}^{\infty}\frac{(n+i-2)^{(n-1)}}{(n-1)^{(n-1)}}q(t+i\tau)\le\frac{1+p_4}{2},\quad t>t_5 \tag{2.12}$$

Let BC be the set as in the proof of Theorem 2.1. Set

$$\Omega=\left\{x\in BC:\frac{p_4(1+p_4)}{2(p_4-1)}\le x(t)\le\frac{p_4}{p_4-1}\right\}$$

Define an operator $T:\Omega\to BC$ as follows :

$$(Tx)(t)=\begin{cases}-\frac{p_4}{2}+p(t)x(t-\sigma)+\sum_{i=0}^{\infty}\frac{(n+i-2)^{(n-1)}}{(n-1)^{(n-1)}}q(t+i\tau)x(g(t+i\tau)) & t\ge t_5\\ (Tx)(t_5) & t_0\le t<t_5\end{cases}$$

For every $x\in\Omega$, using (2.11) (2.12) we get

$$(Tx)(t)\le-\frac{p_4}{2}+\frac{1+p_4}{2}\times\frac{-p_4}{1-p_4}=\frac{p_4}{p_4-1}$$

and

$$(Tx)(t)\ge-\frac{p_4}{2}+p_4\left(\frac{-p_4}{1-p_4}\right)=\frac{p_4(1+p_4)}{2(p_4-1)}$$

So $T\Omega\in\Omega$.

Next we will show that T is a contraction on Ω. In fact, for any $x,y\in\Omega$ and $t\ge t_0$, using (2.11) (2.12) we have

$$\begin{aligned}\left|(Tx)(t)-(Ty)(t)\right| &\le\left|p(t)\right|\left|x(t-\sigma)-y(t-\sigma)\right|\\ &+\sum_{i=0}^{\infty}\frac{(n+i-2)^{(n-1)}}{(n-1)^{(n-1)}}q(t+i\tau)\left|x(g(t+i\tau))-y(g(t+i\tau))\right|\\ &\le\left(-p_4+\sum_{i=0}^{\infty}\frac{(n+i-2)^{(n-1)}}{(n-1)^{(n-1)}}q(t+i\tau)\right)\left\|x-y\right\|\end{aligned}$$

$$\le \left(-p_4 + \frac{1+p_4}{2} \right) \|x-y\|$$

$$= \frac{1-p_4}{2} \|x-y\|$$

where $\frac{1}{2} < \frac{1-p_4}{2} < 1$. Hence

$$\|Tx - Ty\| = \sup_{t \ge t_0} \left| (Tx)(t) - (Ty)(t) \right| \le \frac{1-p_4}{2} \|x-y\|$$

which shows that T is a contraction on Ω. Then by Banach contraction principle, T has a fixed point $x \in \Omega$, i.e. $Tx = x$. So $x(t)$ is a bounded positive solution of Eq. (1).

Example 5. Consider the difference equation

$$\Delta_1^3 \left(x(t) - \left(-\frac{1}{4} + \frac{1}{t} \right) x(t-4) \right) + \frac{33\sqrt{t}}{2(t+3)^{(4)} \left(\sqrt{t} + 3 \right)} x\left(\sqrt{t} \right) = 0$$

In our notation, $n = 3$, $\tau = 1, \sigma = 4, p(t) = -\frac{1}{4} + \frac{1}{t}, g(t) = \sqrt{t}, q(t) = \frac{33\sqrt{t}}{2(t+3)^{(4)} \left(\sqrt{t} + 3 \right)}$.

Clearly, conditions (2.2) (2.11) in Th. 2.5 hold. Therefore, the equation has a bounded positive solution. In fact, $x(t) = \frac{1}{3} + \frac{1}{t}$ is the solution satisfied the conditions.

Theorem 2.6. *Assume there exist negative constants* p_5, p_6 *such that*

$$-p_5^2 < p_6 \le p(t) \le p_5 < -1 \tag{2.13}$$

and (2.2) *holds. Then Eq.* (1) *has a bounded positive solution.*

Proof. There exists M such that $0 < M < -p_5 - \frac{p_6}{p_5}$ and. From (2.2), there exists $t_5 > t_0$ for $-\frac{1}{p_5} < l < 1$, such that

$$\frac{1}{|p_5|} \left(1 + \sum_{i=0}^{\infty} \frac{(n+i-2)^{(n-1)}}{(n-1)^{(n-1)}} q(t + i\tau + \sigma) \right) \le l < 1, \quad t > t_6 \tag{2.14}$$

and

$$\sum_{i=0}^{\infty}\frac{(n+i-2)^{(n-1)}}{(n-1)^{(n-1)}}q(t+i\tau+\sigma)<M \tag{2.15}$$

Let BC be the set as in the proof of Theorem 2.1. Set

$$\Omega=\left\{x\in BC:\frac{p_5M+p_5^2+p_6}{|p_5|(p_6M+p_5p_6-1)}\le x(t)\le\frac{p_6^2+p_5}{|p_6|(p_6M+p_5p_6-1)}\right\}$$

Define an operator $T:\Omega\to BC$ as follows :

$$(Tx)(t)=\begin{cases}\dfrac{-1}{p(t+\sigma)}\left(1-x(t+\sigma)+\displaystyle\sum_{i=0}^{\infty}\frac{(n+i-2)^{(n-1)}}{(n-1)^{(n-1)}}q(t+i\tau+\sigma)x\big(g(t+i\tau+\sigma)\big)\right) & t\ge t_6\\ (Tx)(t_6) & t_0\le t<t_6\end{cases}$$

For every $x\in\Omega$, using (2.13), (2.15) we get

$$(Tx)(t)\quad\le-\frac{1}{p_5}-\frac{p_5M+p_5^2+p_6}{p_5p_6(p_6M+p_5p_6-1)}+\frac{M\left(p_6^2+p_5\right)}{p_5p_6(p_6M+p_5p_6-1)}$$

$$=\frac{p_6^2+p_5}{|p_6|(p_6M+p_5p_6-1)}$$

and

$$(Tx)(t)\ge-\frac{1}{p_6}-\frac{p_6^2+p_5}{p_5p_6(p_6M+p_5p_6-1)}$$

$$=\frac{p_5M+p_5^2+p_6}{|p_5|(p_6M+p_5p_6-1)}$$

So $T\Omega\in\Omega$.

Now we will show that operator T is a contraction on Ω. In fact, for any x, y ∈ Ω and $t\ge t_6$, using (2.13), (2.14) we get

$$|(Tx)(t)-(Ty)(t)|$$

$$\le\frac{1}{|p(t+\sigma)|}\Big[|x(t+\sigma)-y(t+\sigma)|\Big]$$

$$+\sum_{i=0}^{\infty}\frac{(n+i-2)^{(n-1)}}{(n-1)^{(n-1)}}q(t+i\tau+\sigma)\left|x(g(t+i\tau+\sigma))\right|\Bigg]$$

$$\leq\frac{1}{|p_5|}\left[1+\sum_{i=0}^{\infty}\frac{(n+i-2)^{(n-1)}}{(n-1)^{(n-1)}}q(t+i\tau+\sigma)\right]\|x-y\|$$

$$\leq l\|x-y\|$$

where $0 < l < 1$. So T is a contraction. By Banach contraction principle, T has a fixed point T has a point x, i.e. $Tx = x$. Hence $x(t)$ is a bounded positive solution of Eq. (1).

Example 6. Consider the difference equation

$$\Delta_1^4\left(x(t)-\frac{4(t-1)}{t}x(t-1)\right)=\frac{24\ln t}{(t+4)^5(\ln t+1)}x(\ln t)$$

In our notation, $n = 4$, $\tau = 1$, $\sigma = 1$, $p(t)=\frac{4(1-t)}{t}$, $g(t)=\sqrt{t}$,

$$q(t)=\frac{24\ln t}{(t+4)^5(\ln t+1)}.$$

Clearly, the conditions (2.2) (2.13) in Th. 2.6 hold. Therefore, the equation has a bounded positive solution. In fact, $x(t)=1+\frac{1}{t}$ is the solution satisfied the conditions.

Theorem 2.7. *Assume $p(t)\equiv -1$, $\sigma = k\tau$, and (2.2) holds. Then Eq. (1) has a bounded nonoscillatory solution.*

Proof. From (2.2), there exists $t_7 > t_0$ such that

$$\sum_{i=0}^{\infty}\frac{(n+i-2)^{(n-1)}}{(n-1)^{(n-1)}}q(t+i\tau)=\sum_{i_{n-1}=0}^{\infty}\sum_{i_{n-2}=i_{n-1}}^{\infty}\cdots\sum_{i=i_1}^{\infty}q(t+i\tau)\leq\frac{1}{2},\ t>t_7 \tag{2.16}$$

Let BC be the set as in the proof of Theorem 2.1. Set

$$\Omega = \left\{ x \in BC : \frac{1}{2} \le x(t) \le 1 \right\}$$

Define an operator $T : \Omega \to BC$ as follows :

$$(Tx)(t) = \begin{cases} \frac{1}{2} + \sum_{j=1}^{\infty} \sum_{i_{n-1}=(2j-1)k}^{2jk-1} \sum_{i_{n-2}=i_{n-1}}^{\infty} \dots \sum_{i=i_1}^{\infty} q(t + i\tau + i_{n-1}\tau) x(g(t + i\tau + i_{n-1}\tau)) & t \ge t_7 \\ (Tx)(t_7) & t_0 \le t < t_7 \end{cases}$$

Using (2.16) it is easy to show $T\Omega \in \Omega$. Now we shall show that the operator T is a contraction on Ω. In fact, for every $x, y \in \Omega$, using (2.16) and $t \ge t_7$, we get

$$\begin{aligned} \left|(Tx)(t) - (Ty)(t)\right| &\le \sum_{j=1}^{\infty} \sum_{i_{n-1}=(2j-1)k}^{2jk-1} \sum_{i_{n-2}=i_{n-1}}^{\infty} \dots \sum_{i=i_1}^{\infty} q(t + i\tau + i_{n-1}\tau) \\ &\quad \left| x(g(t + i\tau + i_{n-1}\tau)) - y(g(t + i\tau + i_{n-1}\tau)) \right| \\ &\le \sum_{i_{n-1}=0}^{\infty} \sum_{i_{n-2}=i_{n-1}}^{\infty} \dots \sum_{i=i_1}^{\infty} q(t + i\tau + i_{n-1}\tau) \|x - y\| \\ &\le \frac{1}{2} \|x - y\| \end{aligned}$$

Hence

$$\|Tx - Ty\| = \sup_{t \to t_0} \left|(Tx)(t) - (Ty)(t)\right| \le \frac{1}{2} \|x - y\|$$

which shows that T is a contraction on Ω. Then by Banach contraction principle, T has a fixed point $x \in \Omega$,

$$x(t) = \frac{1}{2} + \sum_{j=1}^{\infty} \sum_{i_{n-1}=(2j-1)k}^{2jk-1} \sum_{i_{n-2}=i_{n-1}}^{\infty} \dots \sum_{i=i_1}^{\infty} q(t + i\tau + i_{n-1}\tau) x(g(t + i\tau + i_{n-1}\tau))$$

That is to say $x(t) + x(t + \sigma) = 1 + \sum_{i_{n-1}=0}^{\infty} \sum_{i_{n-2}=i_{n-1}}^{\infty} \dots \sum_{i=i_1}^{\infty} q(t + i\tau + i_{n-1}\tau) x(g(t + i\tau + i_{n-1}\tau))$

So $x(t)$ is bounded nonoscillatory solution of Eq. (1).

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