VIDYASAGAR UNIVERSITY JOURNAL OF PHYSICAL SCIENCES

(A yearly publication of Vidyasagar University)

Volume 8 **2002 - 2003**

Vidyasagar University
Midnapore - 721 102
West Bengal
India

Editorial

Fundamental researches in Physical Science can only explore the origin of Universe.

Mass to energy as well as energy to mass conversion including their joint conservation certainly brought a breakthrough in the continuous research train in the domain of Physical Sciences. Lot of many other break in the same domain were observed in century after centuries from the knowledge peeping brains of the great scientists. Still the above frontline rescarches could not explain the inherent essence, facts and reasons behind the creation of universe, mode of journey of it and its destination. This is because of non-explanation and non-exploration of fundamental natural parameters completely with strong scientific base. These parameters are mass, charge energy, force, space, time and some others.

To refer an examplanary issue we can take the case of charge. What is charge? Electron consumes some charge certainly, but what is it ? There is no strong and justified explanation behind it. Unless the proper composition and facts of these fundamental parameters are explained in a right avenue, the all-abouts of the universe cannot be explored at all in its full form. So this fundamental research can make only the whole thing transparent. We believed, the global research will be more molivated towards the proper finding of hidden truths behind the origin of universe. And this global research should be dominated more by the fundamental researches on basic natural parameters.

The morning will come when everything of our universe will be known.

Dr. Sourangshu Mukhopadhyay

Editor-in-Chief,
Vidyasagar University Journal of Physical Sciences

INTENSITY BASED ALL OPTICAL PARALLEL LOGIC OPERATIONS USING NON-LINEAR MATERIAL

Nirmalya Pahari
Archan Kumar Das
Parthe Pratime Das
and
Sourangshu Mukhopadhyay
Department of Physics and Technophysics
Vidyasagar University, Midnapore - 721102, W.B., India.

Abstract :

In the last few decades the computer scientists are dealing with a novel concept, which is computing with light. Using photons instead of electrons natural and artificial data can be handled parallelly at the maximum possible speed. It is already established that any all optical system can run with highest operational speed. To construct an all optical system non-linear material has an important role in developing super-fast switching device. In this paper an all optical system is proposed for developing the logic operations with optics in parallel possible for two inputs by proper exploitation of non-linear material character.

INTRODUCTION

In the development of optical computing and processing systems, the optical shadow casting technique is used where two inputs are coded in 2-D-formats for performing logical operations.[1-3] The input cells are coded by black and white rectangles, which are composed by some materials having response in external / optical signal.

Non-linear optical materials have wide uses in the optical controlled switching systems, and therefore, in all optical parallel computation, these materials show a potential prospect. [4-6]

In this paper, an all optical input encoding technique (a different concept) in proposed for performing some logic operations in parallel possible for two inputs. The operations are conducted as all optical in nature with proper use of non-linear material.

USE OF NON-LINEAR MATERIAL IN BEAM - GUIDANCE

The refractive index of an isotropic non-linear material is $n = n_0 + n_1 I$, where n_0 and n_1 are constant, I is the intensity of the used light passing through the material. In the above equation, the refractive index of the non-linear material increases with the increase of intensity of the guiding beam. But in case of a linear material, the refractive index does not depend on the intensity of the used light beam.

To develope the whole scheme, we take a system of having the combination of linear medium and non-linear medium as given in figure 1, where n is the refractive index of non-linear medium and n_L is the refractive index of linear medium. Here a light beam is made incident from the side of linear medium and is passed to non-linear medium with varying intensity. Let for a particular intensity, the beam approaches from linear medium to the non-linear medium in C-direction. Here θ in the incident angle of the beam.

The refractive index of the linear medium does not change with increasing intensity where the refractive index of the non-linear media increases following the above equation.

Hence the refractive index of non-linear media increases with increasing intensity of the input beam. This relative increase of refractive index of the non-linear medium is compensated by the decrease of the angle of refraction of that side. Hence with the increase of intensity of the non-linear medium the refracted beam will be closer to the normal of incidence for a particular incident angle.

OPTICAL IMPLEMENTATION OF LOGIC OPERATIONS HAVING TWO BINARY INPUTS

The optical implementation of the scheme is in figure 2. We need parallel beams for the logic operations. We have taken two 4-bit binary numbers A and B whose logic operations are to be performed. Let the input binary numbers A and B be encoded by pixels in two different planes so that any pixel is fully transparent when the bit is '1' and semi-transparent for the bit '0'. Here we have used four light beams through successive order encoded inputs. The input planes A and B are kept parallel to each other.

Thus a light beam passing through successive order pixels in input planes, may bear different intensities before reaching to NLM. Hence the beam having high intensity being passed through pixels for a particular order of both inputs, designated by '1' (both pixels transparent), comes closer to the normal when passed through NLM and falls on the plane (having the pixel form) giving the same order AND bit. Other beams will give their respective order AND bits. Thus we may have all AND bits.

A beam having less intensity being passed through pixels designated by '1' and '0' of a particular order for two inputs A and B, comes less closer to the normal when passed through NLM and falls on the output plane giving the same order XOR bit.

Thus we may have two logic operations from this concept. In the output we consider a pixel indicating '1' having light and '0' having no light.

CONCLUSION

This system is completely all optical in nature so its rate of operation is very high. The input light beam should be a polarised light beam (preferably a laser beam) for activating the non-linear material.

FIGURE CAPTION

Figure 1 : A composite slab of & linear and non-linear material.
Figure 2 : AND and XOR logic system by non-linear material.

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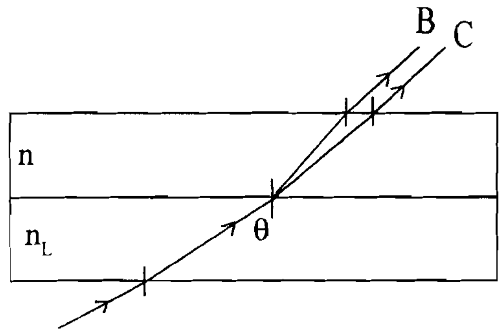

Fig. 1

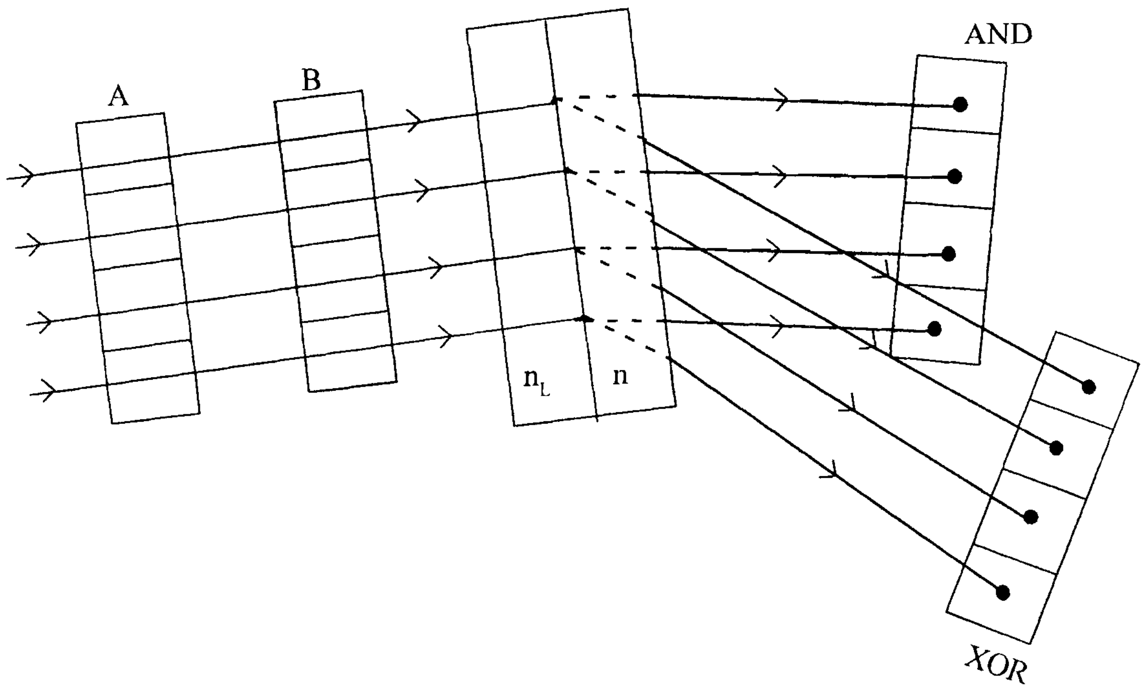

Fig.- 2

SOLITON-SOLITON INTERACTION IN LONG DISTANCE REMOTE CABLE SWITCHING

Abhijit Sinha
Harihar Bhowmik
Tushar Kanti Dalal
Ajitesh Biswas
Sugata Ghosh
Piyasi Mukherjee
Pradip Dhak
Subhamoy Singha Ray
Tushar Kanti Tapadar
and
Sourangshu Mukhopadhyay
Department of Physics and Technophysics
Vidyasagar University, Midnapore - 721102, West Bengal India.

Abstract :
Optical Soliton propagation in optical fibre has been established with its own tremendous advantages in communication. Many scientists and technologists around the world are working in this high by potential field. In this paper the authors propose a new concept of using a Soliton-Soliton interaction in remote switching through the optical fibre cable.

INTRODUCTION

An optical pulse propagation through a linear dispersive medium undergoes temporal broadening as well as chirping. For operation at a wave length greater than zero (0) dispersion wave length, the instantaneous frequency decreases with increasing time.

A pulse propagating through a non-linear non-dispersive medium undergoes no temporal broadening but undergoes only chirping.

A soliton is a pulse propagating through a non-linear dispersive medium that broadens in neither the temporal domain nor the spectral domain i.e. soliton is the abbreviation of solitary photon.(1-4)

The optical fibre as a linear medium, that is the intensities associated with the propagating optical pulse were assumed to so small that there was no significant effect on the propagation characteristics of the wave guide.(5) In actual practice all media exibit non-linear effects. In the case of Silica optical fibres, one of the manifestations of the nonlinearity as the intensity-dependent refractive index according to the following equation -

$$n = n_0 + n_2 I.$$

where n_0 is the liner refractive index of silica (for low intensity levels),

n_2 is the non-liner refractive index co-efficient and

$I =$ is the effective intensity in the medium with P being the power carried by the mode and *Aeff* the effective area of the fiber mode.

Thus, when an optical pulse travels through the fibre, the higher intensity portions of the pulse encounter a higher refractive index of the fibre compared with the low intensity regions. This intensity dependent refractive index leads to the phenomenon known as self-phase modulation (S.P.M).

Self-phase modulation leads to a chirping with lower frequencies in the leading edge and higher frequencies in the trailing edge, which is just opposite the chirping caused by linear dispersion in the wave length region above the zero (0) dispersion wave length; thus by a proper choice of pulse shape (a hyperbolic secant shape) and the power carried by the pulse, we can indeed compensate one effect with the other. In such a case the pulse would propagate undistorted by a mutual compensation of dispersion and S.P.M.

PROPERTIES AND APPLICATION OF OPTICAL SOLITON

(a) The pulse propagation power of soliton is very high. So the capacity to overcome the barriers is high enough.

(b)The attenuation through an optical fibre of soliton is very low. The chirping effect due to dispersion and nonlinearity cancel each other where soliton propagates through a optical fibre, therefore we would have a pulse that remains unaltered both in time and in frequency domains.

(c) Utilising these above effects, we can obtain undistorted-digital-long distance communication fibre by soliton.

We can divide our work in three parts -

(a) Firstly we study the refractive indices of optical fibre's material when an optical soliton pulse propagates through an optical fibre and when two optical soliton pulses of different frequencies propagates through the same optical fibre.

(b) We study the dependence of refractive index of optical fibre's material on soliton pulses frequency when two optical soliton pulses of different frequencies propagates in same direction through an optical fibre.

(c) We study the refractive index of optical fibre material on soliton pulse's frequency when two optical soliton pulses of same frequency and same phase but in opposite direction.

2. OPTICAL SOLITON GENERATION IN OPTICAL FIBRE

In a linear medium the electric polarization is assumed to be a linear function of the electric field

$$P = \epsilon_0 \chi E \quad \text{.............................} (1)$$

where for simplicity, a scalar relation has been written. The quantity χ is termed the linear dielectric susceptibility. At high optical intensities (or equivalently at high optical fields) all media behave non-linearly that is, the relation expressed in equation - (1) is approximate and one has

$$P = \epsilon_0 \chi E + \epsilon_0 \chi^{(2)} E^2 + \epsilon_0 \chi^{(3)} E^3 + \text{———} \quad . \quad \text{...................} (2)$$

For non-crystalline media such as the optical fibre $\chi^{(2)} = 0$ and the lowest order non-linearity is due to $\chi^{(3)}$.

If we consider a plane optical wave with an electric field variation of the form

$$E = E_0 \cos(\omega t - kz). \quad \text{......................} (3)$$

Then,

$$P = \epsilon_0 \chi E_0 \cos(\omega t - kz) + \epsilon_0 \chi^{(3)} E_0^3 \cos^3(\omega t - kz) \quad \text{...................} (4)$$

Now,

$$\cos^3\theta = {}^1/_4 (\cos 3\theta + 3\cos\theta). \quad \text{............................} (5)$$

Hence,

$$P = \epsilon_0 (\chi + {}^3/_4\, \chi^{(3)} E_0^2)\, E_0 \cos(\omega t - kz) + \epsilon_0 \frac{\chi^{(3)}}{4} E_0^3 \cos 3(\omega t - kz) \quad \text{......} (6)$$

The second term on the RHS corresponds to third harmonic generation, which is negligible in optical fibres due to phase mismatch between frequency ω and 3ω. The polarization at frequency ω is

$$P = \epsilon_0 (\chi + {}^3/_4\, \chi^{(3)} E_0^2)\, E_0 \cos(\omega t - kz). \quad \text{...................} (7)$$

For a plane wave given by equation - (3), the intensity is given by

$$I = {}^1/_2\, C \epsilon_0 n_0 E_0^2. \quad \text{.....................} (8)$$

Where, n_0 is the refractive index of the medium at low fields, Hence

$$P = \epsilon_0 (\chi + \frac{3}{2} \frac{\chi^{(3)}}{C \epsilon_0 n_0} I)\, E_0 \cos(\omega t - kz). \quad \text{....................} (9)$$

The general relationship between polarization and refractive index is given by

$$P = \epsilon_0 (n^2 - 1)\, E_0 \cos(\omega t - kz). \quad \text{.........................} (10)$$

Comparing equations (9) and (10), we see that the non-linear term containing $\chi^{(3)}$ leads to an intensity dependent refractive index

$$n^2 = 1 + \chi + \frac{3}{2}\,\frac{\chi^{(3)}}{C \epsilon_0 n_0}\, I. \quad \ldots\ldots\ldots\ldots\ldots (11)$$

Since the last term in the above equation is usually very small even for very intense light beams, we may approximate by a Taylor series expansion

$$n \approx n_0 + \tfrac{3}{4}\,\frac{\chi^{(3)}}{C \epsilon_0 n_0^2}\, I$$

$$\approx n_0 + n_2 I. \quad \ldots\ldots\ldots\ldots\ldots (12)$$

Where, $n_0^2 = 1 + \chi$. .... (13)

and $n_2 = \tfrac{3}{4}\,\frac{\chi^{(3)}}{C \epsilon_0 n_0^2}$ is the non-linear co-efficient.

For fused silica fibres,

$n_0 \approx 1.46$; $n_2 \approx 3.2 \times 10^{-20}\, m^2/W$.

If we consider the propagation of a mode carrying 100 mW of power in a single mode fibre with an effective mode area $\approx 50\ \mu m^2$, then the resultant intensity is $2 \times 10^9\ W/m^2$ and the change in refractive index due to non-linear effects is

$\Delta n = n_2 I \approx 6.4 \times 10^{-11}$. .... (14)

Although this change in refractive index seems too small, due to very long interaction length (10 - 10,000 km.) in an optical fibre, the accumulated effects become significant. In fact, it is thus small non-linear term that is responsible for the formation of solitons.

3. OPTICAL SOLITON PULSES PROPAGATION IN OPTICAL FIBRE WHEN FREQUENCIES ARE DIFFERENT AND IN SAME PHASE (S.P.M.)

In a linear medium the electric polarization is assumed to be a linear function of the electric field,

$P = \epsilon_0 \chi E$. .... (1)

where χ is the linear dielectric susceptibility, E is the electric field ϵ_0 is the permittivity of free space and P is the polarization.

At high optical intensities or optical fields, all media behave non-linearly and polarization is expressed as

$P = \epsilon_0 \chi E + \epsilon_0 \chi^{(2)} E^2 + \epsilon_0 \chi^{(3)} E^3 +$ ——— . ... (2)

for non-crystalline media, such as optical fibre the term $x^{(2)}$ is equals to zero. Therefore for optical fibre the lowest order non-linearity arises due to the term $\chi^{(3)}$.

Now, we consider two plane optical waves of frequencies ω_1 and ω_2 propagates in z-direction with same wave vector k and their electric field variation of the form

$$E_1 = E_0\cos(\omega_1 t\text{-}kz). \quad ....(3)$$

$$E_2 = E_0\cos(\omega_2 t\text{-}kz). \quad ....(4)$$

The resultant of these two electric fields should be

$$E = E_1 + E_2 = E_0(\omega_1 t\text{-}kz) + E_0(\omega_2 t\text{-}kz). \quad ....(5)$$

Therefore, through optical fibre the polarization should be

$$P = \in_0\chi E + \in_0\chi^{(3)}E^3 ....(6)$$

$$P = \in_0\chi E_0[\cos(\omega_1 t\text{-}kz) + \cos_0(\omega_2 t\text{-}kz)] + \in_0\chi^{(3)}E_0^3[\cos(\omega_1 t\text{-}kz) + \cos_0(\omega_2 t\text{-}kz)]^3$$

$$= \in_0\chi E_0[\cos(\omega_1 t\text{-}kz) + \in_0\cos(\omega_2 t\text{-}kz)] + \in_0\chi^{(3)}E_0^3[\cos^3(\omega_1 t\text{-}kz) + 3\cos^2(\omega_1 t\text{-}kz)\cos(\omega_2 t\text{-}kz) + 3\cos(\omega_1 t\text{-}kz)\cos^2(\omega_2 t\text{-}kz) + \cos^3(\omega_2 t\text{-}kz)]$$

$$\therefore \quad P = \in_0\chi E_0\cos(\omega_1 t\text{-}kz) + \in_0\chi E_0\cos(\omega_2 t\text{-}kz) + \in_0\chi^{(3)}E_0^3[{}^1/_4\cos3(\omega_1 t\text{-}kz) + {}^3/_4\cos(\omega_1 t\text{-}kz) + \{{}^3/_2 + {}^3/_2\cos2(\omega_1 t\text{-}kz)\}\cos(\omega_2 t\text{-}kz) + \{{}^3/_2 + {}^3/_2\cos2(\omega_2 t\text{-}kz)\}\cos(\omega_1 t\text{-}kz) + {}^1/_4\cos3(\omega_1 t\text{-}kz) + {}^3/_4\cos(\omega_2 t\text{-}kz)]$$

Neglecting the mismatched term i.e. the other frequencies rather than ω_1 andω_2. we can write down the porization expression as

$$P = [\in_0\chi E_0\cos(\omega_1 t\text{-}kz) + \in_0\chi E_0\cos(\omega_2 t\text{-}kz) + {}^3/_4\in_0\chi^{(3)}E_0^3\cos(\omega_1 t\text{-}kz) + {}^3/_2\in_0\chi^{(3)}E_0^3\cos(\omega_2 t\text{-}kz) + {}^3/_2\in_0\chi^{(3)}E_0^3\cos(\omega_1 t\text{-}kz) + {}^3/_4\in_0\chi^{(3)}E_0^3\cos(\omega_2 t\text{-}kz)]$$

$$= \in_0\chi E_0\cos(\omega_1 t\text{-}kz) + \in_0\chi E_0\cos(\omega_2 t\text{-}kz) + {}^9/_4\in_0\chi^{(3)}E_0^3\cos(\omega_1 t\text{-}kz) + {}^9/_4\in_0\chi^{(3)}E_0^3\cos(\omega_2 t\text{-}kz)$$

$$= \in_0(\chi + {}^9/_4\chi^{(3)}E_0^2)E_0\cos(\omega_1 t\text{-}kz) + \in_0(\chi + {}^9/_4\chi^{(3)}E_0^2)E_0\cos(\omega_2 t\text{-}kz)$$

$$\therefore \quad P = \in_0 E_0(\chi + {}^9/_4\chi^{(3)}E_0^2)\{\cos(\omega_1 t\text{-}kz) + \cos(\omega_2 t\text{-}kz)\}. \quad ....(7)$$

For a plane wave the intensity is given by

$$I = {}^1/_2 C\in_0 n_0 E_0^2$$

$$\text{and} \quad E_0^2 = \frac{2I}{C\in_0 n_0}. \quad ....(8)$$

where, C is the velocity of high at vaccum and n_0 is the refractive index at low frequencies. So in this case we can write down the intensity as-

$$I = I_1 + I_2 = {}^1/_2 C\in_0 n_0 E_0^2 + {}^1/_2 C\in_0 n_0 E_0^2$$

$$\text{and} \qquad E_0^2 = \frac{(I_1 + I_2)}{C\in_0 n_0}$$

$$\therefore \quad P = \in_0[\chi + {}^9/_4 \chi^{(3)} \frac{(I_1 + I_2)}{C\in_0 n_0}] E_0\{\cos(\omega_1 t\text{-}kz) + \cos(\omega_2 t\text{-}kz)\}$$

$$\therefore \quad P = \in_0[\chi + {}^9/_4 \chi^{(3)} \frac{(I_1 + I_2)}{C\in_0 n_0}] E_0\cos(\omega_1 t\text{-}kz) + \in_0[\chi + {}^9/_4 \chi^{(3)} \frac{(I_1 + I_2)}{C\in_0 n_0}] E_0\cos(\omega_2 t\text{-}kz)$$

Let, $P = P_1 + P_2$

where, $P_1 = \in_0[\chi + {}^9/_4 \chi^{(3)} \frac{(I_1 + I_2)}{C\in_0 n_0}] E_0\cos(\omega_1 t\text{-}kz)$ ....(9)

$P_2 = \in_0[\chi + {}^9/_4 \chi^{(3)} \frac{(I_1 + I_2)}{C\in_0 n_0}] E_0\cos(\omega_2 t\text{-}kz)$................(10)

Compairing general relationship between polarization and refractive inden

$P = \in_0[x_i^2 - 1]E_0\cos(\omega t\text{-}kz)$..................(11)

$$\therefore x_i^2 = 1 + x + {}^9/_4 \, x^{(3)} \frac{(I_1 + I_2)}{C\in_0 n_0}$$

(Compairing p with p_1 and compairing p with p_2)

Now, we may approximate by taylor serise expention,

$x_i = (1+x)^{1/2} + 9/8 \, x^{(3)} \frac{(I_1 + I_2)}{C\in_0 n_0^2}$ ....(12)

$\therefore$ Therefore, total x, i

$$x = 2x_i$$

$= 2(1+x)^{1/2} + 9/4 \, x^{(3)} \frac{(I_1 + I_2)}{C\in_0 n_0^2}$ ....(13)

where we considering x_i from p_1 and x_i from p_1 and x_i from p_2 and non linear $x_2 = 9/4 \frac{x^{(3)}}{C\in_0 n_0^2}$

From equation - (13) we conclude that we get the r.i. is frequency independent when $\omega_1 = \omega_2$.

For fused silicon fibre,

$$n_0 = 1.46 \; ; \; n_2 = 3.2 \times 10^{-20} \ m^2 / W.$$

If we consider the propagating of a mode carrying 100 mW of power in a single mode fibre with an effective mode area 50 μm^2, then the resultant intensity 2×10^9 W/m^2.

$\therefore$ Change in r.i. is due to the non-linear effects

$$\Delta n = n_2 I \approx 6.4 \times 10^{-11}.$$

If P is the power carried by a mode in an optical fibre

$$I_1 + I_2 = I = \frac{P}{A_{eff}}$$

Thus for a fibre we may write from equation - (13) we get,

$$n = n_0 + n_2(I_1 + I_2) = n_0 + n_2 \frac{P}{A_{eff}}$$

Now, β_0 is the propagation constant in a linear case then new propagation constant can be written as

$$\beta = \beta_0 + \frac{k_0 n_2 P}{A_{eff}}$$

Hence, an incident wave of the form

$$Ae^{i(\omega_1 t - \beta z)} + Ae^{i(\omega_2 t - \beta z)}$$

$$= A \left[\exp\{i(\omega_1 t - \beta_0 z - \frac{k_0 n_2 P}{A_{eff}} z)\} + \exp\{i(\omega_2 t - \beta_0 z - \frac{k_0 n_2 P}{A_{eff}} z)\}\right]$$

If the input wave is pulse with a power variation given by P(t), then the output phase dependence would be

$$\exp[i(\omega_1 t - \beta_0 z - \frac{k_0 n_2 P(t)}{A_{eff}} z)] + \exp[i(\omega_2 t - \beta_0 z - \frac{k_0 n_2 P(t)}{A_{eff}} z)]$$

If we consider an input Gaussian pulse is given by

$$E(z = 0,t) = E_0 e^{-t^2/\tau_0^2}(e^{i\omega_1 t} + e^{i\omega_2 t})$$

After propagating the length L of an optical fibre the pulse become

$$E(z = L,t) = E_0 e^{-(t - \frac{L}{v_g})^2/\tau_0^2}$$

$$\times \left[\exp\{i(\omega_1 t - \beta_0 L - \frac{k_0 n_2 P(t)}{A_{eff}} L)\} + \exp\{i(\omega_2 t - \beta_0 L - \frac{k_0 n_2 P(t)}{A_{eff}} L)\}\right]$$

The quantity P(t) represents the temporal variation of the pulse, which is given by

$$E(z = L,t) = E_0 \exp\{-(t - \frac{L}{v_g})^2/\tau_0^2\}$$

$$\times [\exp\{i(\omega_1 t - \beta_0 L - \frac{k_0 n_2 P(t)}{A_{eff}} L)\} + \exp\{i(\omega_2 t - \beta_0 L - \frac{k_0 n_2 P(t)}{A_{eff}} L)\}]$$

$$E^*(z = L,t) = E_0 e^{-(t - \frac{L}{v_g})^2/\tau_0^2}$$

$$\times [\exp\{-i(\omega_1 t - \beta_0 L + \frac{k_0 n_2 P(t)}{A_{eff}} L)\} - \exp\{-i(\omega_2 t - \beta_0 L - \frac{k_0 n_2 P(t)}{A_{eff}} L)\}]$$

The quantity P(t) represents the temporal variation of power in the pulse, which is given by,

$$P(z,t) = EE^* = E_0^2 \exp[-2(t - \frac{L}{v_g})^2/\tau_0^2]$$

$$\times \exp[\{i(\omega_1 t - \beta_0 L - \frac{k_0 n_2 P(t)}{A_{eff}} L)\} + \exp\{i(\omega_2 t - \beta_0 L - \frac{k_0 n_2 P(t)}{A_{eff}} L)\}]$$

$$\times \exp[\{-i(\omega_2 t - \beta_0 L - \frac{k_0 n_2 P(t)}{A_{eff}} L)\} + \exp\{-i(\omega_1 t - \beta_0 L - \frac{k_0 n_2 P(t)}{A_{eff}} L)\}]$$

$$P(z,t) = E_0^2 \exp[-2(t - \frac{L}{v_g})^2/\tau_0^2] [\exp\{i(\omega_1 t - \omega_2 t) + \exp\{i(\omega_2 t - \omega_1 t)\}+2]$$

$$= E_0^2 \exp[-2(t - \frac{L}{v_g})^2/\tau_0^2] [\exp\{it(\omega_1 - \omega_2)\} + \exp\{it(\omega_2 - \omega_1)\}+2] \quad \ldots\ldots\ldots\ldots\ldots\ldots (14)$$

Putting $\omega_1 = \omega_2$ we get

$$P(z,t) = 4E_0^2 \exp[-2(t - \frac{L}{v_g})^2/\tau_0^2]$$

$$= P_0 \exp[-2(t - \frac{L}{v_g})^2/\tau_0^2]$$

where $4E_0^2 = P_0$

CONCLUTION

From the above treatment we can conclude -

The refractive index of the fiber medium does not depend at all with the frequency variation of solition pulses lat it depends on the intensities of the individual pulses. It is also clear from the equation of P(z,t) with w_1 & w_2 that propagating intereated power has a dependence on the frequency variation of the soliton sigrals. The power variation can lead to the remote switching cases by organising proper interaction.

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Design, Analysis and Simulation of 140 GHz p^+-n-n^+ $Si_{1-x}Ge$/ Si Heterostructure IMPATTs.

S. Karmakar and J.P Banerjee
Institute of Radio Physics and Electronics
University of Calcutta
92, Acharya Prafulla Chandra Road,
Kolkata - 700 009, West Bengal, India

Abstract :

Design and optimization of p^+-n-n^+ heterostructure **IMPATTs** based on a material system , i.e., unstrained $Si_{1-x}Ge_x$ /Si with the molefraction x=10% have been carried out at the atmospheiric window frequency of 140 GHz by using draft-diffusion model and computer simulation technique. The results of AC analysis of the device show that the peak negative conductance is obtained at the optimum frequency of 135 GHz and 145 GHz for the bias current density of 1.2 x 10^8 Am^{-2} and 2 x 10^8 Am^{-2} respectively. The device negative resistance per unit area increases from -1.5 x $10^{-9}\Omega$ to -2.3 x $10^{-9}\Omega$ when the bais current increases from 1.2 x 10^8 Am^{-2} to 2 x 10^8 Am^{-2} . The results indicate that $si_{1-x}Ge_x^{/si}$ IMPATT devices would be highly suitable for power genaration at high millimeter wave frequencies.

INTRODUCTION

IMPACT Avalanche Transit Time (IMPATT) diodes have come to the fore as very important solid state device capable of generation high RF power at microwave, milimeter wave and submillimeter wave frequency bands,when mounted on a heat sink and put inside a suitable microwave cavity.

Around 94 GHz and above Impatt diodes are virtually without any serious competition with other microwave solid state devices such as Metal Semiconductor field Effect Transistors (MESFETs), High Electron Mobility Transistors (HEMTs) and Gunn diodes in terms of output power and DC to RF conversion efficiency. In view of the simple arrangement, low cost, tuning capability, frequency coverage, output power and reliability Impatt devices are finding useful applications as solid state sources of high frequency power. These device are now being used as active elements in missile guidence, tracking radars and microwave and millimeter wave communication systems.

Till date Impatt oscillators have been developed using Si, Ge, GaAs and InP as base materials. In addition, some heterojunction Impatt diodes and microstrip Impatts are also being developed in recent years .

In last two decades, a new material system based on epitaxial $Si_{1-x}Ge_x$ layers on Si has been devaloped for the fabrication of Heterojunction Bipolar Transistors (HBTs), Modulation Doped Field Effect Transistors (MODFETs), and Infrared Photodectctors that operate into the millimeter wave frequency spectrum which make $Si_{1-x}Ge_x$ a viable material for microwave and millimeter wave source and related circuit applications.

NASA has undertaken the research on $Si_{1-x}Ge_x$ based devices for the development of RF integrated circuits for communication application with an emphasis on the X, Ku and Ka frequency bands to cover the NASA Deep Space Network (DSN), the new Low Earth Orbit(LEO) commercial satellite communication networks, the terrestrial Local Multiple Distributed Service (LMDS) .

Research works are going on to design and fabricate an Microwave Monolithic Integrated Circuit (MMIC) using $Si_{1-x}Ge_x$ Heterojunction Bipolar Transistor (HBT) as active devices and excellent results are reported in published literature. Metal Oxide Field Effect Transistors (MOSFET), Modulation Doped Field Effect Transistors (MODFET). p-i-n and Avalanche photodetector have also been designed and fabricated using $Si_{1-x}Ge_x$ material. The performance of these davices is reported to be excellent.

It is worthwhile to mention that no work has yet been reported in the literatures on the design and development of Impatt diode using $Si_{1-x}Ge_x$ / Si material system. The potential application of $Si_{1-x}Ge_x$ based devices in different areas have aroused great interest to study Impatt diodes based on this new naterials system. An effort has been made in this work to investigate the possibility of genaration of nagative resistance in the $Si_{1-x}Ge_x$ /Si heterostructure system, so that Impatt diodes could be developed using this heterostructure.

In this paper dc and small signal analysis have been carried out by computer simulation technique for single drift p^+nn^+ structures of Impatt diodes that uses $Si_{1-x}Ge_x$ material as the drift layer for x = 0. (i.e., mole fraction of Ge) from 90 GHz to 200 GHz.

DESIGN AND ANALYSIS

Computer design and analysis has been made in this paper to design a p^+nn^+ Impatt diode using $Si_{1-x}Ge_x$ in the window frequency of 140 GHz where the atmospheric attenuation of the millimeter wave signal is minimum. A drift-diffusion model has been employed to study the effect of space charge and carrier diffusion and realistic exponential and error function doping profiles have been used for DC analysis of $Si_{1-x}Ge_x$ IMPATT devices. The material parameters of $Si_{1-x}Ge_x$ with x = 10% used in the simulation are shown in Table 1. A double iterative field maximum computer simulation method [7] has been used for this purpose.

In modeling the device, different material parameters of $Si_{1-x}Ge_x$ as reported in the literature have been used. The electric field profiles in the depletion layer of the diode have been obtained from DC analysis. The conductance susceptance characteristics have also been obtained from small signal computer analysis.

Design of Doping Profile : The frequency of operation of an Impatt diode essencially depends on the transit time of charge carriers to cross the depletion layer of the diode.The depletion layer width of a single p^+nn^+ diode is initially estimated from a simple transit time considaration in which the transit angle for maximum device nagative resistance is 0.74π radian.

Thus,

$$2\pi f_d(w/v_{sn}) = 0.74$$

$$\text{Or,} \quad w = 0.37\, v_{sn}/\pi f_d$$

where, w is the depletion layer width, f_d is the design frequency and v_{sn} is the saturated drift velocity of electrons in a p^+nn^+ diode.

The schematic diagram of one dimensional flat profil p^+nn^+ SDR Impatt diode structure and the corresponding doping profile, $N(x)=|N_D-N_A|$ are shown in Fig (1b). The direction of the electric field (E) and the carrier current densities(Jn, Jp) are indicated in Fig.(1a).Taking the origin at the plane of the metallurgical junction, the doping profile near the p^+n junction is taken to be an exponential function. The doping profile at the interface of epitaxy and substrate, i.e, nn^+ surface is also an exponential funtion which resembles very well with the complementary error function. The equations of doping profile N(x) in different regions are given by

$$N(x) = N_D[1-\exp(x/s)] \qquad \text{for } -w_n<x<0$$

$$N(x) = -N_h \exp[-1.08\lambda - 0.78\lambda^2] \qquad \text{for } x>0$$

$$N(x) = N_h \exp[-1.08\lambda - 0.78\lambda^2] \qquad \text{for } x<-w_n$$

Material parameters : The field variation of ionization rates of electrons and holes in $Si_{1-x}Ge_x$ / Si are given by

$$\alpha_n(E) = A_n\exp(-B_n/E)$$

$$\beta_n(E) = A_p\exp(-B_p/E)$$

The values of the constants of ionization rate (A_n, B_n, A_p, B_n) at T=300K are obtained from [13]. The field dependence of hole and electron drift velocifies are given by

$$v_p = v_{sp}[1-\exp\{-\mu_p E(x)/v_{sp}\}]$$

$$v_n = v_{sn}[1-\exp\{-\mu_n E(x)/v_{sn}\}]$$

where v_{sp} and v_{sn} are the satured drift velocities for holes and electrons, respectively.

The material parameters of $Si_{1-x}Ge_x$ used here correspond to 300K which are listed given in Table 1.

Table 1: Material parameters of $Si_{1-x}Ge_x$ at 300K for x =10%

Carrier	Electric Field (x 10^8 Vm $^{-1}$)	Constants of ionization rates An,p (x 10^8 m^{-1})	Bn,p (x 10^8 m^{-1})	Mobility (m^{2}V^{-1}s^{-1})	Saturated drift velocity (x 10^5 ms^{-1})
Electron	0.01—1.00	5.78	1.2	0.05478	0.72
Hole	0.01—1.00	0.1078	1.569	0.05138	0.70

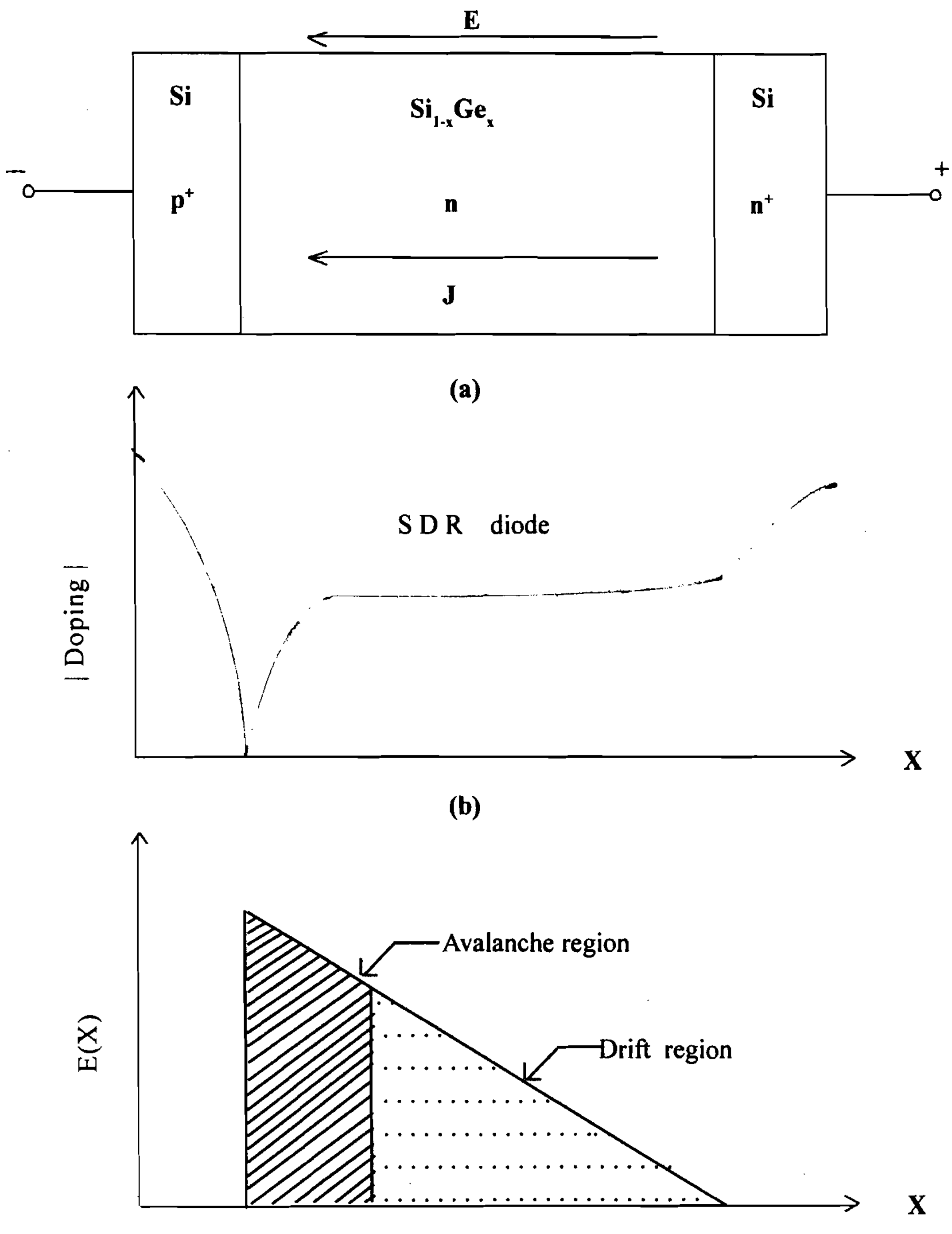


Fig.1 (a) Structure and (b) the doping profile and (c) field profile of flat profile SDR Impatts

Table 2: Other parameters of $Si_{1-x}Ge_x$ Impatt diode at 300K

X	Permittivity of $Si_{1-x}Ge_x$ layer (x 10^{-10} Fm^{-1})	Current dencity (x 10^8 Am^2)	Concentration of p^+ and n^+ regions (x 10^{27} m^{-3})	Concentration of drift $Si_{1-x}Ge_x$ layer
10%	1.08993	1.0	1.0	1.0

Computer Simulation Technique

The DC electric field profiles E(x) of p^+nn^+ $Si_{1-x}Ge_x$ Impatt have been obtained by using a double iterative computer method which involves simultaneous numerical solution of possion's equation continuity equation and space charge equation subject to the appropriate boundary conditions for E(x) at the depletion layer edges.

The avalanche zone width (x_A) is defined as the high field region near the p^+n junction where 95% growth of current density takes place due to impact ionization and avalanche multiplication.The voltage drop across the avalanche and drift zones (V_A,V_D) and breakdown voltage (V_B) are also calculated from the DC characteristics. A semequantative estimate of DC to RF conversion efficiency (η) is obtained by using Schafetter-Gummel formula [7]

$$\eta = (2m/\pi)/[V_D/(V_A+V_D)]$$

where the RF voltage modulation(m) is taken to be 50%.

Small signal analysis of SDR $Si_{1-x}Ge_x$ Impatts is carried out by using a generalized computer simulation method based on Gummel - Blue Technique [8]. The simulation program developed for this purpose provides admittance(conductance versus susceptance) of the device for differnt bia current density. These plots are shown in Fig.(3a) and Fig.(3b).

RESULTS AND DISCUSSION

The doping concentration of different regions and the corresponding depletion layer width of p^+nn^+ $Si_{1-x}Ge_x$ Impatt diodes are simultaneously designed for device operation at 140 GHz window frequency. The doping profile of the device is also optimized with respect to high DC to RF conversion efficiency and high output power from the device at the above mentianed frequency.

The design parameters are listed in Table 1 and 2.

The electric field profiles for x =10% $Si_{1-x}Ge_x/s_i$, Impatt diodes at current densities 1.2 x 10^8 Am^{-2} and 2 x 10^8 Am^{-2} are shown in Fig.(2a) and Fig.(2b)

The conductance-susceptance plots of the diodes for x=10% at current densities

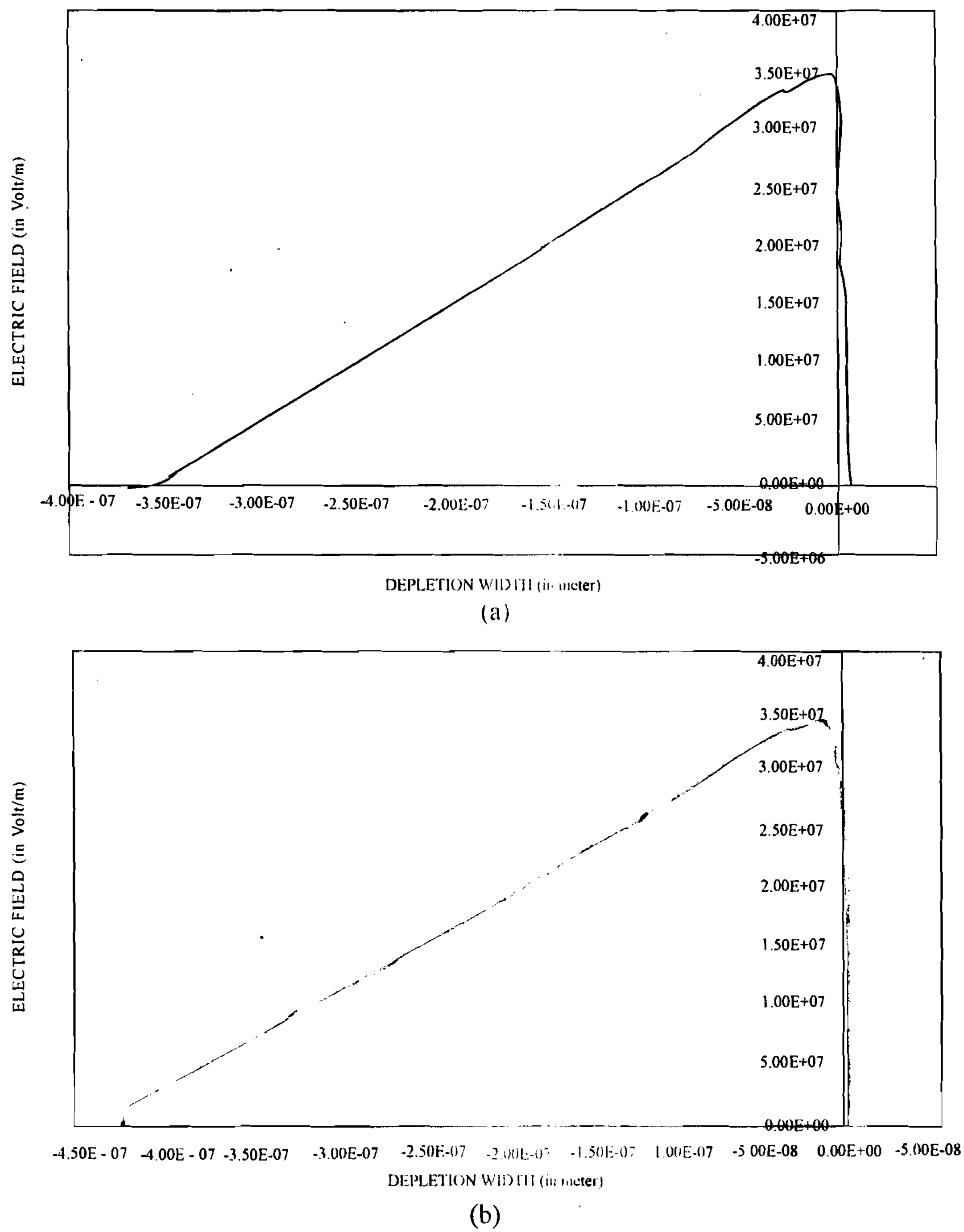
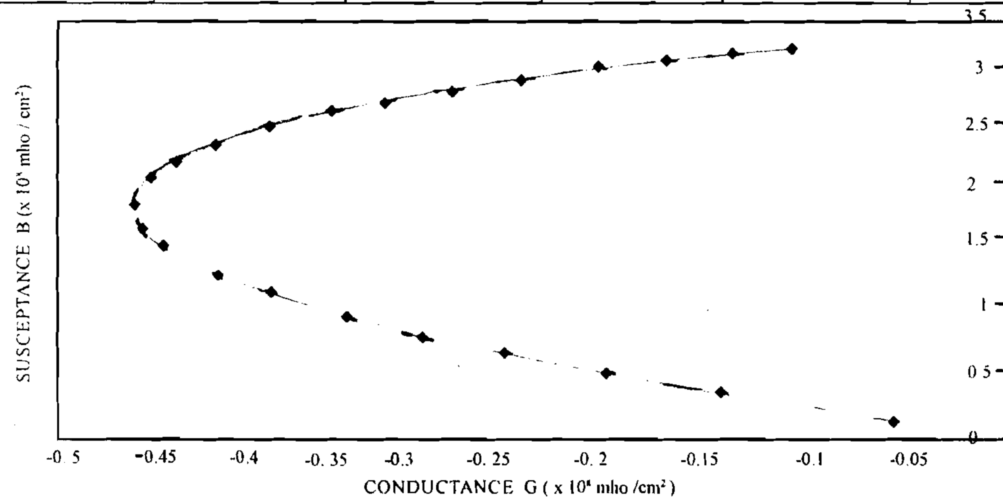


Fig.2 Diode electric field versus depletion width plot for current densties (a) 1.2×10^8 Am^{-2} and (b) 2.0×10^8 Am^{-2}

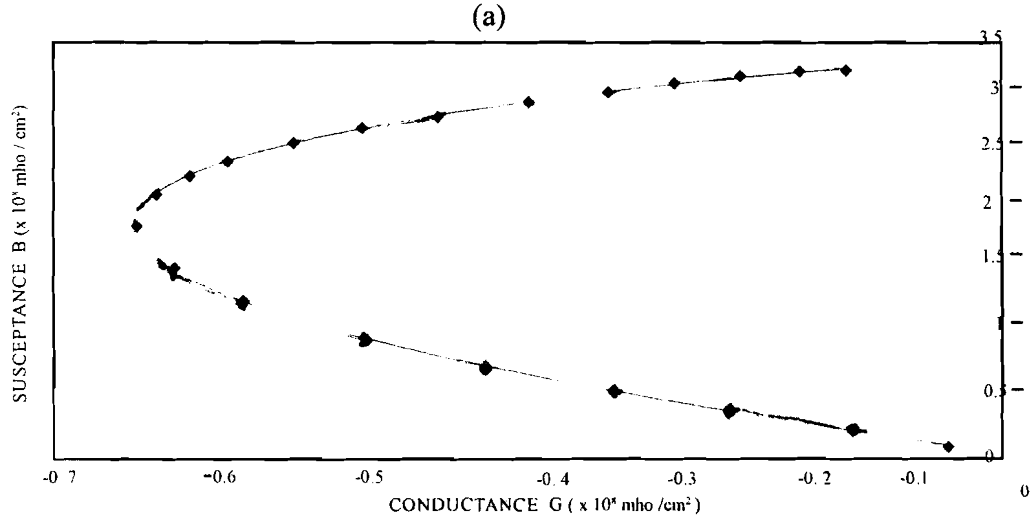
The following table summarizes the DC properties of the p^+nn^+ heterostructure Impatt diodes at 140 GHz.

Table 3: DC properties of heterostructure $Si_{1-x}Ge_x/s_i$ Impatt diode for x= 10%

J_0 ($x10^{-8}$ Am^{-2})	E_m (x 10^8 Vm^{-1})	X_0 (x 10^{-9} m)	V_A (V)	V_B (V)	η (%)
1.2	0.345	0.865	4.356	6.308	9.84
2	0.344	1.39	4.626	6.782	10.12



(a)



(b)

Fig.3 (a) Diode admittance (susceptance versus conductance) as a function of frequency and current dencity (a) 1.2 x 10^8 Amp/m^2 and (b) 2 x 10^8 Amp/m^2

1.2 x 10^8 Am^{-2} and 2 x 10^8 Am^{-2} are shown in Fig. 3(a) and Fig. 3(b)

Table 4: High frequency properties at 140 GHz heterostructure $Si_{1-x}Ge_x$ Impatt diode for x= 10%.

J_0 (10^8 Am^{-2})	Z_R (x 10^{-8} m^2)	Z_X (x 10^{-8} m^2)	G_p (x 10^{-8} mho m^{-2})	B_p (x 10^{-8} mho m^{-2})	$-Q = B_p / G_p$
1.2	-.15080	-.55354	-.45816	1.6817	-3.67055
2	-.23096	-.56228	-.62505	1.5217	-2.43452

CONCLUSION AND DISCUSSION

The results show that it is possible to generate negative resistance in the millimeter wave frequencies in the unstrained $Si_{1-x}Ge_x/s_i$ Impatt diodes for the values of mole fraction x=105% so, Impatt diodes can be designed and fabricated using $Si_{1-x}Ge_x$ / Si hterostructure.

The result of small signal analysis show that the peak of G-B plot is obtained at 135 GHz for p^+nn^+ $Si_{1-x}Ge_x/s_i$ diode at current densities 1.2 x 10^8 AM^{-2} and at 145 GHz at current densities 2 x $10^8 AM^{-2}$ which fall well within the W-band frequencies. But the value of nagative quality factor (-Q) at these frequencies are not good. So the band of frequencies of operation of $Si_{1-x}Ge_x$ Impatt will be narrow for amplifier circuits.

It can be concluded that the design and development of Impatt diodes using a totally new material viz., unstrained $Si_{1-x}Ge_x/s_i$ are presented in this work and the results are encouraging and suggest that these $Si_{1-x}Ge_x/s_i$ Impatt diodes can be fabricated and used in Si-based MMIC to integrate the low-cost, high reliability, advanced and easy technology to have low-power high-speed wireless RF circuits.

There x fore $Si_{1-x}Ge_x$ based Impatt devices will bring a revolution in the millimeter wave communication systems and rader system in the present century.

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Invited paper

ON THE CARRIER CONTRIBUTION TO THE ELASTIC CONSTANTS IN NANOSTRUCTURED NONLINEAR OPTICAL AND OPTOELECTRONIC MATERIALS : THEORY AND SUGGESTION FOR EXPERIMENTAL DETERMINATION

Professor Kamakhya Prasad Ghatak
Department of Electronic Science
University of Calcutta
92, Acharya Prafulla Chandra Road, Kolkata - 700 009.

Abstract :
In this paper an attempt is made to study the carrier contribution to the second and third order elastic constant in quantum wells (QWs), quantum well wires (QWWs) and quantum dots (QDs) of nonlinear optical materials on the basis of a newly formulated electron dispersion law allowing various type of anisotropies of the energy band spectrum within the framework of **k.p** formalism. We have also studied the said contribution in nanostructured II-VI and IV-VI optoelectronic compounds in accordance with the appropriate band models. The result of quantum confined III-V materials form a special case of our generalized formalism. It is found, taking QWs, QWWs and QDs of $CdGeAs_2$ InAs, CdS and PbTe as examples that the said contribution changes with nanothickness in various oscillatory manners and the rates of variation are totally band structure dependent. We have also suggested an experiment method of determining the carrier contribution to the elastic constant in nanostructured materials have arbitrary dispersion laws and the theoretical formulation is in agreement with the suggested experimental method.

With the advent of fine lithographical method, molecular beam epitaxy and other experimental techniques, low dimensional structures having quantum confinement in one, two and three dimensions such as QWs, QWWs and QDs have in the last few years attracted much attraction not only for their potential in uncovering new phenomena in nanoscience and technology but also for their interesting device applications [1]. Heterostructures find extensive application in quantum confined laser, high speed digital network, quantum wire transistors, optical modulator and also in other nanostructured device [2]. In QWs, the restriction of the motion of the carriers leading to the quantum size effect allowing the 2D carrier transport parallel to the surface of the film. In QWWs the carriers are quantized in two transverse directions, and only one dimensional motion is allowed. Besides in QDs the all three directions of motion are quantize. In this paper, we shall study the electronic contribution to the elastic constant in nanostructured in nonlinear optical, III-IV, II-VI and VI-VI optoelectronic compounds respectively.

In this context we wish to note that Sreedhar and Gupta [3] have developed a theory for determining the electronic contribution to the elastic in bulk specimens of small-gap materials. It was shown that the carrier contribution to the elastic constant depends on the density-of-states function. Therefore it would of much interest to study the said contribution in nanostructured nonlinear optical materials by considering the various anisotropies of the energy band spectrum. This class of compound is being increasingly used in nonlinear optics and light emitting diodes [4]. Out study is based on a newly formulated electron energy spectrum within the framework of **k.p** formalism considering the crystal field splitting together with the anisotropies of the effective electron mass and the spin orbit splitting constant since these are the physical featurs of such compounds. The electronic contribution to the elastic constants in nanostructured III-V materials can be obtained as special cases of our generalized analysis. The said study for nanostructured II-VI and IV-VI compounds have subsequently been studied. The importance of III-V, II-VI and IV-VI optoelectronic materials is already well-known in the literature [5]. We shall suggest an experimental method of determining the carrier contribution to the elastic constants in nanostructured nonlinear optical and optoelectronic compound having arbitrary dispersion laws. The influence of nanothickness on the said contribution has been studied taking quantum confined $CdGeAs_2$, InAs CdS and PdTe as examples for numerical computations.

THEORETICAL BACKGROUND

The generalized electron energy spectrum in bulk specimens of nonlinear optical materials can be expressed as [6].

$$\gamma(E) = f_1(E)k_s^2 + f_2(E)k_z^2 \qquad (1)$$

where the notations are defined in [5]. The carrier statistics in QWs, QWWs and QDs can, respectively, be written as $n_0 = (2\pi d_x)^{-1} C_1[L_1(E_F) + L_2(E_F)]$, (2)

$$n_0 = (d_x d_y)^{-1} (2/\pi) C_2[L_3(E_F) + L_4(E_F)], \qquad (3)$$

and

$$n_0 = (2/d_x d_y d_z)\, C_4[F_{-1}(\eta)], \qquad (4)$$

where *di* (i = x, y, z) is the nanothickness,

$C_1 = \sum_{n_z=1}^{n_{zmax}}$, n_i is the size quantum number for the i^{th} axis, $L_1(E_F) = [[\gamma(E_F) - f_2(E_F)(n_z\pi/dz)^2]/f_1(E_F)$, $L_2(E_F) = C_3k_1$, $C_3 = \sum_{r=1}^{s}$,

$K_t = (K_B T)^{2r}(1-2^{1-2r})\, U(2r)\,(d^{2r}/dE_F^{\,2r})\,[L_t(E_F)]$, k_B is the Boltzmann constant, T is temperature, U(2r) is the Zeta function of order 2r, $L_3(E_F) = [(\gamma(E_F) - f_1(E_F)(n_y\pi/d_y)^2 - f_2(E_F)(n_z\pi/d_z)^2]/ f_1(E_F)]^{1/2}$, $L_4(E_F) = C_3K_3$,

$$C_2 = \sum_{n_y=1}^{n_{ymax}} \sum_{n_z=1}^{n_{zmax}} \quad , \quad C_2 = \sum_{n_x=1}^{n_{xmax}} \sum_{n_y=1}^{n_{ymax}} \sum_{n_z=1}^{n_{zmax}} ,$$

$F_j(n)$ is the one parameter Fermi – Dirac integral of order j [7], $\eta = (E_F - E')/k_BT$, E_F is the corresponding Fermi energy and E′ is the root. The carrier contribution to the second and third order elastic constants can, in general, be expressed as [3]

$$\Delta C_{44} = (-G_0^{\,2}/9)(\partial n_0/\partial E_F) \qquad (5)$$

$$\Delta C_{456} = (-G_0/3)(\partial/\partial E_F)[\Delta C_{44}] \qquad (6)$$

where G_0 is the deformation potential constant. Thus combining Eqs. (2) to (6) we can study ΔC_{44} and ΔC_{456} in QWs, QWWs and QDs of nonlinear optical materials. Under the conditions $\Delta_{11} = \Delta_1 = \Delta$ (the isotropic spin-orbit splitting constant), $\delta = 0$ and $m^*_{11} = m^*_1 = m^*$ (the isotropic effective electron mass at the edge of the conduction band). Eq (1) assumes the form

$$(h^2k^2/2m^*) = [\{E(E + Eg)(E + Eg + \Delta)(Eg + 2/3\Delta)\} / \{(E(E + \Delta)(E + Eg + 2/3\Delta)\}] \qquad (7)$$

The eq.(7) is the well-known three-band model of Kane [8] which is a valid model for studying the electronic properties of III-V materials. Thus we study the ΔC_{44} and ΔC_{456} for nanostructured III-V optoelectronic materials under the aforementioned simplifications.

The carriers statistics in quantum confined II-VI compounds can, respectively, be expressed as

$$n_0 = (16\pi A^2 dx)^{-1} [C_1\{L_5(E_F) + L_6(E_F)\}] \qquad (8)$$

$$n_0 = (\pi d_x d_y)^{-1} [C_2\{L_7(E_F) + L_8(E_F)\}] \qquad (9)$$

and

$$n_0 = (d_x d_y d_z)^{-1}\, C_4[F_{-1}(\eta)] \qquad (10)$$

Where $L_5(E_F) = [2C^2 - 4AB(n_z\pi/d_z)^2 + 4AE_F \pm 2C\{C^2 - 4AB(n_z/d_z)^2 + 4AE_F\}^{1/2}]$, $L_6(E_F) = C_3K_5$, $L_7(E_F) = (4A^2)^{-1}\{L_5(E_F) - (n_y\pi/d_y)^2\}^{1/2}$ and $L_8(E_F) = C_3K_7$

Thus using eqs(5), (6), (8), (9), (10)we can study ΔC_{44} and ΔC_{456} in nanostructured II-VI compounds.

Similarly, the carrier statistics in quantum confined IV-VI optoelectronic materials can be written as

$$n_o = (2\pi g_v \sqrt{(m1m3)}/d_x h^2) \sum_{n_y=1}^{n_{ymax}} [L_9(E_F)+L_{10}(E_F)\}] \qquad (11)$$

$n_o = (2g_v\sqrt{(2m1)}/\pi\hbar d_x d_y)C_2[L_{11}(E_F) + L_{12}(E_F)\}]$ (12)

$n_o = (2g_v/d_x d_y d_z)C_4[F_{-1}(\eta)]$ (13)

where $L_9(E_F) = [E_F-(\hbar^2/2m_2)(n_y\pi/d_y)^2][1+pE_F+(p\hbar^2/2m_2')(\pi n_y/d_y)^2], L_{10}(E_F) = C_5K_9$, $p=1/Eg$, $L_{11}(E_F) = [L_9(E_F) - (\hbar^2/2m_3)(n_z\pi/d_z)^2]^{1/2}$, $L_{12}(E_F) = C_3K_{11}$, m_1 and m_3 are the band edge carrier effective mass in the longitudinal directions, m_2 is the same mass in the longitudinal direction and m_2' is the longitudinal band edge effective mass of holes for electrons. Thus using eqs (5), (6), (11), (12), and (13) we can study the ΔC_{44} and ΔC_{456} for nanostructured IV-VI materials.

The magnitude of the thermoelectric power in the present case be expressed as [9]

$L_0 = (eTn_0)^{-1} \int_{-\infty}^{\infty} R(E)(-\partial f_0/\partial E)dE$ (14)

Where R(E) is the total number of states and f_0 is the Fermi-Dirac function. Thus using (5), (6) and (14) we get

$\Delta C_{44} = (-n_0G_0^2eL_0/3\pi^2k_B^2T)$ and $\Delta C_{456} = (n_0eG^3_0L_o^2/3\pi^4k_B^3T)(1 + n_0/L_0)(\partial L_0/\partial n_0))$ (15)

Thus we can experimentally determine ΔC_{44} and ΔC_{456} for all types of nanostructured materials as discussed in this paper knowing L_0 which is an experimentally measurable quantity.

RESULTS AND DISCUSSION

Using the appropriate equations together will the parameters as given in [5, 10] we have plotted the normalized ΔC_{44} and ΔC_{456} in Fig 1 for QWs of $CdGeAs_2$, InAs, CdS and PbTe as a function of d_z as shown by the curves a, b, c and d and a′, b′, c′ and d′ respectively. The circular plots have been obtained by using eq. (15) and the experimental results of L_0 versus n_0 curves for QWs of n - InAs as given in [10]. In Figs 2 and 3 all curves of Fig 1 have been plotted for QWWs and QDs of the said nanostructured compounds respectively.

The influence of quantum confinement is immediately apparent from Fig 1 since ΔC_{44} and ΔC_{456} depend strongly on the thickness of the nanostructures in contrast with the bulk specimens of the said materials. The appearance of the humps in the figures is due to the redistribution of the electrons amount the quantized energy level when the size quantum number corresponding to the one fixed set changes from one fixed value to the other. The ΔC_{44} and ΔC_{456} in nanostructured materials can become several order of magnitude larger than that in the bulk specimens of the same compounds which is a direct consequence of nanostructures.

Our experimental suggestion for the determination of ΔC_{44} and ΔC_{456} are valid for materials having arbitrary dispersion laws. Only the experimental values of L_0 as a function of n_0 for any model will give the experimental values of ΔC_{44} and ΔC_{456} for that model for that ranges of n_0. Since the experimental values of L_0 in the present case is not available in the literature excluding QWs of n - InAs, to the best of the knowledge of the author the theoretical formulation can not be compared with the proposed experiment for the other types of nanostructures as considered here.

It is worth remarking that the influence of energy band models on the ΔC_{44} and ΔC_{456} in nanostructured materials can be assessed from this work and this simplified analysis also covers various quantum confined compounds having different dispersion relations. We have not considered other types of nanostructured nonlinear optical and optoelectronic materials or other physical variables for plotting the ΔC_{44} and ΔC_{456} for the purpose of condensed presentation. Finally it may be noted the experimental values of L_0 for various other cases will provide an experimental check of the present work and also a technique for probing the band structured of nanostructured materials.

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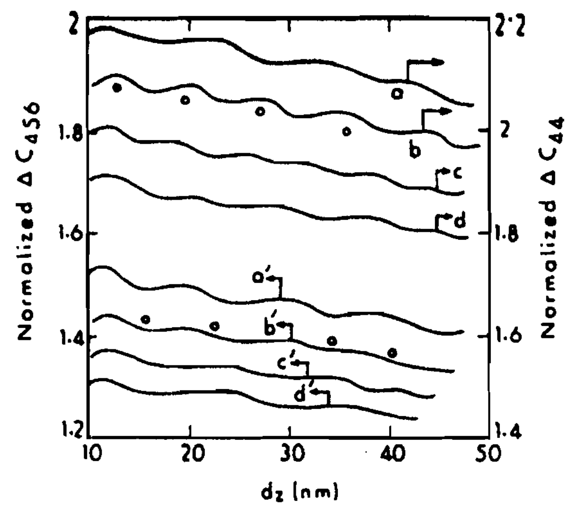

Fig. 1 Plots of the normalized ΔC_{44} and ΔC_{456} versus d_z for quantum wells of all types of materials as discussed in the paper.

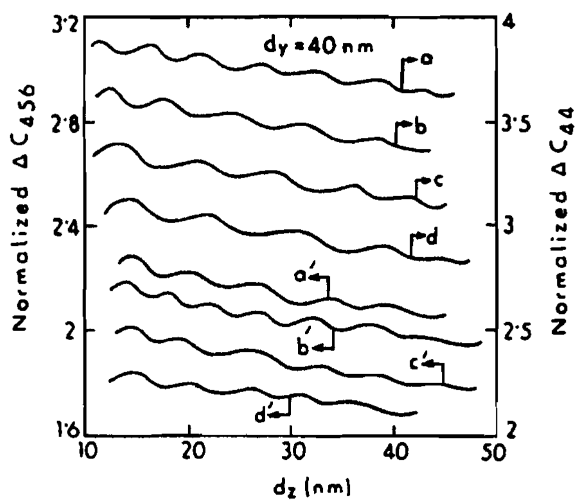

Fig. 2 Plots of the normalized ΔC_{44} and ΔC_{456} versus d_z for quantum well wires of all types of materials as discussed in the paper.

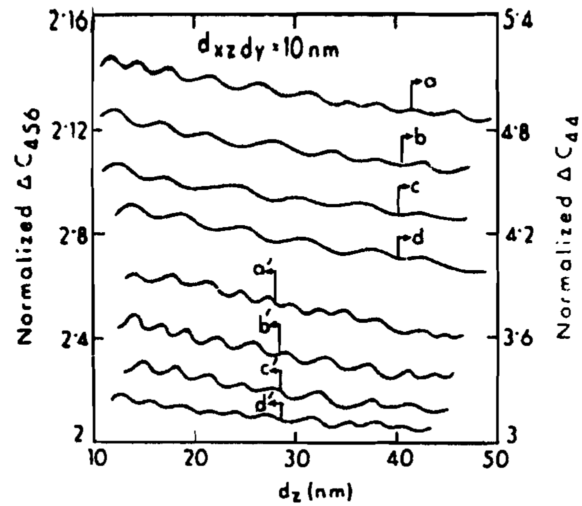

Fig. 3 Plots of the normalized ΔC_{44} and ΔC_{456} versus d_z for quantum dots of all types of materials as discussed in the paper.

GRAFT COPOLYMERIZATION BASED ON POLYSACCHARIDES

[1]Tridib Tripathy & [2]B. R. De
[1]Department of Chemistry
Narajole Raj College
Narajole, Paschim Midnapore
[2]Department of Chemistry & Chem. Tech.
Vidyasagar University
Midnapore

INTRODUCTION

The method of graft copolymerization remains one of the most widely used among the recent methods for synthesizing new forms of polymeric materials and also modifying the properties of natural and synthetic polymers. Molecular combinations of both natural and synthetic derivatives attract special attention because graft copolymers, as a rule, combine the properties of the individual components. This is an innovative area of research. The present article focuses on literature survey of graft copolymerization based on polysaccharides. A graft copolymer is a polymer where one or more species of block connected to the main chain as side chains, having constitutional or configurational features different from in the main chain, exclusive of branch points[1]. A simple structure of graft copolymer can be represented by the following structure (Fig. 1)

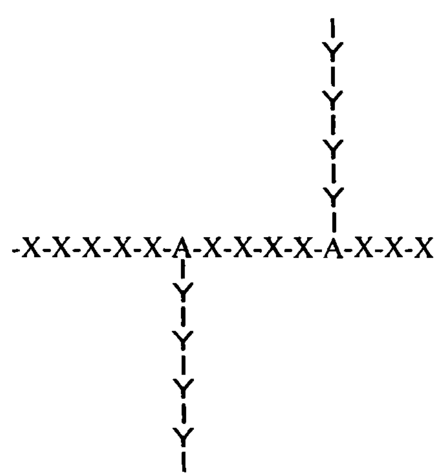

Fig. 1 Structure of graft copolymer

Where a sequential arrangement of the monomer units X is referred to as the main chain or backbone. The sequence of Y units is the side chain or graft and A is the unit in the backbone to which the graft is attached. In graft copolymers, the backbone and side chain both may be copolymeric or homopolymeric or one is copolymeric and another is homopolymeric. Because the main chain and the branch chain are usually thermodynamically incompatible, most graft copolymers can be classified as multiphase polymers in the solid state, analogous to polymer blends[2].

Early research of Duke *et al*[3,4] had provided the evidence, both kinetic and spectroscopic, of the oxidation of alcohols and glycols by ceric ion via disproportionation of intermediate coordination complexes, Mino and kaizerman[5] showed that the free radicals formed during disproportionation of these complexes can initiate vinyl polymerization. A vast quantity of research effort has been directed towards the ceric ion initiated graft copolymerization of vinyl monomers onto suitable substrates containing 1, 2- diol groups: the polysaccharides being the majority of them. As a result, the study of the preparation and application of graft copolymers of vinyl monomers onto polysaccharide backbones has grown into a separate field of its own. A survey of literature based on grafted polysaccharides can be divided into three sections ; (A) preparation, (B) characterization and (C) application.

1. PREPARATION OF GRAFT COPOLYMERS :

Graft copolymers are prepared by generation of free radical sites on polysaccharide backbone and allowing these free radicals to serve as macroinitiators for the vinyl or acrylic monomers. Mainly following methods have been reported for creating free radicals on the polysaccharide backbone.

a) Photo Initiation, b) Radiation Initiation, c) Initiation by Mastication d) Chemical Initiation.

a) PHOTO INITIATION

The method involves the dissociation of excited molecules to give free radicals mostly on the polymer backbone which then initiate graft copolymerization. Some photosensitizers have been used in the photoinitiation method, for example, anthraquinone[6] has been used for grafting onto cellulose, uranyl nitrate and ceric ammonium nitrate[7] have been used for U.V. radiation induced graft copolymerisation of styrene and acrylonitrile onto cotton cellulose. Merlin *et al.*[8] made a photo chemical investigations on cellulosic materials. Fouassier[9] has reported U.V. light initiated grafting onto wood cellulose using several vinyl monomers. Fe^{+3} being highly photo sensitive, the high rate of polymerization and grafting transformation were observed in the photo activated system. Due to the generation of radical in the photo system, the formation of homopolymer may take place. In the field of photo induced grafting onto cellulose, significant contribution has made by Stannett *et al.*[10,11]

Recently photo initiated graft copolymerisation of polyacrylamide occurred on cellulosic and ligno cellulosic substrates using a novel photo catalytic system. The samples were photo exposed with oxalic acid solutions which functionalized the backbone materials to different extents with carboxyl groups, and at the same time rendered them photo active. Afterwards, the functionalized samples were again photo exposed with monomer solution and photo grafting was found to take place without any initiator from outside.

b) RADIATION INITIATION

UV radiation has been used to initiate grafting. Merlin and Fouassier[13] showed by ESR study that irradiation of starch both in the presence or absence of photosensitizers results in chain scission and the formation of free radicals that can subsequently react with monomer. Several monomers were graft copolymerized onto dilute water suspension of starch (0.67%) by this technique[14]. Trimnell and Stout[15] prepared starch-g poly acrylic acid by irradiating 10-20% suspension of unmodified and modified starches with acrylic acid in the absence of sensitizer. The amount of free radicals generated on wool was estimated by M. Burke[16] by ESR technique. Gilbert and Stannet[17] studied the simultaneous grafting of a monomer onto wool fibers by radiation method. Photo induced graft copolymerisation of vinyl monomers onto deoxy (thio-sulfato) chitin (S_2O_3 - Chitin) has recently been studied by Morita et al[13]. Chitin was first tosylated and subsequently transformed into S_2O_3 - Chitin. They carried out graft copolymerisation of S_2O_3 - Chitin by UV irradiation without catalyst and have got grafting efficiency of 95.67% with acrylonitrile and 42.24% with methyl methacrylate. Here grafting does not take place at the backbone rather it is at the side chain (Scheme-I, Fig. 2)

c) NITIATION BY MASTICATION

Free radicals on starch have been created by mechanochemical means such as mastication, ball-milling, freezing of starch dispersions. When starch is subjected to shear force, the starch molecules are broken apart producing free radical sites at the break points. If such application of shear is done in the presence of monomers, copolymerization is initiated and block copolymers are produced, attached to the starch at the site of free radial formation. Mixtures of starch on mastication with methyl methacrylate, styrene, vinyl acetate and acrylonitrile result in grafting[19,20].

d) CHEMICAL INITIATION

(i) INITIATION BY Ce^{+4}

The most widely used method of chemical initiation for graft copolymerization onto polysaccharides has been with ceric such as ceric ammonium nitrate (CAN) or ceric ammonium salt (CAS). Although Smith and Duke[21] examined the oxidation of several organic compounds, including cellulose, by ceric ions, Mino and Kaizerman[22] for the first time investigated the initiation of vinyl polymerization by the transient free radicals produced from an oxidation-reduction reaction of ceric salts with 1,2-glycols. Since then the ceric ion initiation method has been used to graft a number of monomers onto polysaccharides.

The mechanism by which Ce (IV) ion generates free radicals is believed to involve the formation of a coordination complex between the oxidant and the hydroxyl groups of polysaccharides. The complex formation between cerium (IV) and cellulose has been studied by several investigators[23-28]. Others have proposed complex formation on the basis that the initial absorption of Ce(IV) was rapid and the amount absorbed was much greater than the carbonyl content of the cellulose[26,27,28].

Model compound studies of Ce (IV) oxidations of monohydric alcohols and 1,2 glycols support the postulated mechanism and suggest that the C_2-C_3 glycol and the C_5 hydroxyl of D-anhydroglucose unit of starch may be the preferred sites for free radical generation[29-30]. Muhammad and Rao[32] have suggested a mechanism for direct oxidation is sulfuric acid. However, Hinz and Johnson[31] pointed out that the oxidation proceeds through an intermediate complex where the equilibrium constant for complex formation is small. The complex formed between Ce (IV) and a 1,2-glycol may be either a chelated or an acyclic species. Considering the enhanced stability of chelate complexes over acyclic ones, it is reasonable to assume that they will be formed whenever the reaction condition is favourable. Oxidation of monohydric alcohols[31] which cannot form chelate intermediates and 1,2-glycols[31] by Ce (IV) in 1.0 M perchloric acid shows the equilibrium constant for complex formation is significantly larger for the 1,2-glycols. This increase is attributed to chelate stabilization of the Ce(IV)-1,2-glycol complex. This not only indicates the complex formed to be a chelated one, but also minimizes the possibility of involvement of the hydroxyl group at 6-position in complex formation.

On the basis of the available evidences, the following mechanism[22,33] has been proposed for the initiation of graft copolymerization by ceric ion action on the anhydroglucose units as depicted in Scheme-II (Fig. 3). The hydroxyl groups at C_2 and C_3 of a D-anhydroglucose unit form a chelate complex with Ce(IV). The complex disproportionates forming a free radical which is rapidly oxidized by a second mole of Ce(IV). In the presence of monomer, the intermediate radical could generate a stable graft.

Grafting of acrylamide onto various polysaccharide backbones using ceric ammonium nitrate was reported by Singh et al[34-38]. During graft copolymerisation of acrylamide onto CMC, guargum, starch, (amylopectin and amylose), sodium alginate and xanthangum it was reported that the length of grafted chains depends upon the catalyst as well as monomer concentration. Increment of catalyst concentration by keeping all other parameter constant the length of grafted chains become shorter. But the length gradually increases with increasing monomer concentration keeping all other factors constant.

(ii) INITIATION BY TRIVALENT MANGANESE

Duke[39] reported that 1,2-glycols can be oxidized by Mn^{3+}. The oxidation proceed by way of electron transfer reaction via free radical mechanism. Singh et al[40] have applied the same reaction, using a manganic sulfate-sulfuric acid system, for initiation of grafting onto polysaccharides. But the use of the sulfate complex of Mn^{3+} introduces a problem in the sense that these ions in sulfuric acid media of low acidity are unstable and disproportionate according to the reaction.

Disproportionation can be prevented at high acid medium but this leads to acidic hydrolysis and consequent degradation of the carbohydrate chains during the grafting reaction. This problem was overcome by Mehrotra and Ranby[41-44] by the use of pyrophosphate complex of Mn^{3+} ions, which is stable to disproportionation and exists as $[Mn(H_2P_2O_7)_3]^{3+}$. In a series of four research communications, Mehrotra and Ranby[41-44] have studied the grafting of monomers like methyl methacrylate (MMA), acrylonitrile (AN) and acrylamide (AM) onto starch by use of pyrophosphate complex on Mn^{3+} as an initiator. They have observed high conversion in case of both AN and MMA but very low conversion in case of AM. However, running the same experiment under identical conditions but in the absence of starch substrate, produces negligible amount of homopolymers in case of AN or MMA, but a considerably large amount of the same in case of AM. This shows that Mn^{3+} ion is unreactive to monomers of AN and MMA in absence of starch substrate but reacts to initiate polymerization in case of AM. They have proposed a similar mechanism as in case of Ce^{4+} ion for the initiation of grafting by Mn^{3+} ion as shown in Scheme III (Fig. 4). The mechanism primarily involves the cleavage of glycol groups of D-anhydroglucose units (AGU). The rate-determining step in the glycol-cleaving reaction of manganic pyrophosphate appears to be dissociation of glycol-Mn^{3+} complex. The overall glycolcleaving rate is governed by several factors, such as Mn^{3+} concentration, "free" pyrophosphate concentration, the acidity and glycol concentration.

The initiation of homopolymerization in case of acrylamide in absence of starch substrate has been proposed to be proceeding through the oxidation of the enol of β-hydroxy propionamide formed by vinylogous addition of water molecule to the monomer catalyzed by H^+ ions as shown in Scheme IV (Fig. 5).

(iii) INITIATION BY HYDROGEN PEROXIDE/Fe^{2+} SYSTEM (FENTON'S REAGENT)

Brockway and Moser[45,46] have studied the initiation of grafting of methyl methacrylate (MMA) onto starch by use of hydrogen peroxide and ferrous ammonium sulfate. As in case of other systems where grafting is accomplished via free radical reactions, the present process is presumed to involve the formation of radical sites on the backbone polymer. This may occur either by reaction of a radical from initiator or transfer from a growing chain with the starch molecules. In either event, with hydrogen peroxide and ferrous ion, the well known Fenton's reagent, the first step is undoubtedly the formation of hydroxyl radicals. The possible steps in the grafting process are illustrated in the following reactions :

$$H_2O_2 + Fe^{2+} \rightarrow HO^{\cdot} + OH^- + Fe^{3+} \quad (1)$$

$$HO^{\cdot} \rightarrow M^{\cdot} \quad (2)$$

$$M^{\cdot} + (n\text{-}1)M \rightarrow M_n^{\cdot} \quad (3)$$

$$HO^{\cdot} + \text{Starch-H} \rightarrow \text{Starch}^{\cdot} + H_2O \quad (4)$$

$$M_a^{\cdot} + \text{Starch-H} \rightarrow \text{Starch} + M_aH^{\cdot} \quad (5)$$

$$\text{Starch}^{\cdot} + M \rightarrow \text{Starch-}M^{\cdot} \quad (6)$$

$$\text{Starch-}M^{\cdot} + (n-1)M \rightarrow \text{Starch-}M_a^{\cdot} \quad (7)$$

It is evident from the above scheme of reaction mechanism that either reaction (4) or (5) would produce the necessary radical site on the starch chain. But use of azobisisobutyronitrile (AIBN) produces little grafting. This indicates that transfer of PMMA radicals with starch occurs only to a minor extent, if at all. As far as the positions of the grafted chains on starch backbone are concerned, those will be more or less randomly distributed at all possible positions in the anhydroglucose ring system in view of the ability of hydroxyl radicals to abstract hydrogen from any C-H link[47].

In a variation of the above method, hydrogen peroxide has been replaced by organic hydroperoxides or inorganic persulfate salts and such reducing agents as sodium bisulfite has been substituted for ferrous ion[47]. Mixtures of ferrous ammonium sulfate and ascorbic acid have also been used as the reducing reagent of the redox system[48]. Bajpai et al.[49] have grafted poly (acrylonitrile) onto guar gum using potassium persulfate/ascorbic acid redox initiating system. The mechanism in all these cases is similar to that with the Fenton's reagent. Free radicals are generated by the reaction between ascorbic acid and persulfate. Reaction between persulfate and ascorbic acid involves a chain mechanism[50] due to the formation of sulfate ion radicals which are well known chain carriers. The overall reaction mechanism may be represented as follows :

a) Formation of free radicals

$S_2O_8^{2-}$	$\rightarrow$	$2SO_4^{\cdot-}$	(1)
$SO_4^{\cdot-} + H_2O$	$\rightarrow$	$SO_4^{-} + OH + H^{-}$	(2)
$\cdot OH + AH^{-}$	$\rightarrow$	$\cdot AH + OH^{-}$	(3)
$\cdot AH + S_2O_8^{2-}$	$\rightarrow$	$A + SO_4^{2-} + SO_4^{-} + H^{+}$	(4)
$\cdot AH + \cdot OH$	$\rightarrow$	$A + OH^{-} + H^{+}$	(5)
$\cdot AH + SO_4^{-}$	$\rightarrow$	$SO_4^{2-} + H^{+} + A$	(6)

b) Mechanism of graft copolymerization

$R\cdot + M$	$\rightarrow$	$RM\cdot$	(7)
$RM\cdot + nM$	$\rightarrow$	RM_{n+1}	(8)
$GOH + RM_n\cdot$	$\rightarrow$	$GO\cdot + RM_nH$(Homopolymer)	(9)
$GOH + R\cdot$	$\rightarrow$	$GO\cdot + RH$	(10)
$GO\cdot + M$ (Guar gum radical)	$\rightarrow$	$GOM\cdot$	(11)
$GOM\cdot + nM$	$\rightarrow$	$GOM_n\cdot$	(12)
$2GOM_n\cdot$	$\rightarrow$	Graft Copolymer	(13)

(iv) INITIATION BY THIOLATION OF STARCH

Among the various methods suggested for the grafting of vinyl monomers onto polysaccharides substrates, the xanthate (dithiocarbamates of alcohols are known as xanthates) method has a number of distinct advantages. First, the formation of initiating species can be carried out directly on the backbone chain. This, unlike in many other methods, minimizes the formation of homopolymer and this increases the grafting efficiency. Secondly, the process neither requires special conditions like maintaining inert atmosphere nor does it necessitate the use of expensive reagents. Moreover, grafting frequencies are notably increased when part of the redox system involved in free-radical grafting is incorporated in the polysaccharide backbone. For example, grafting frequency of acrylonitrile on starch[51] increases in the following order : initiation with ceric ion[52] gives products having 600 - 4000 AGU/graft and graft molecular weight of 75,000 - 8,000,000; grafting by ferrous ion-hydrogen peroxide system[53] gives higher grafting frequencies (300-400) and lower molecular weight of the grafts (4,000-90,000). On the other hand, redox grafting of acrylonitrile onto cellulose xanthate[54] gives 80 - 100 AGU/graft and graft molecular weight of 13,000 - 15,000.

Various aspects of xanthate process have been tackled by several researchers. Notable among them is the study of Dimov and Pavlov[55]. Cellulose xanthogenate and xanthogenates in general have reducing properties due to the presence of the hydrosulfide (-SH) group. This in the presence of an oxidizing agent (e.g. hydrogen peroxide) forms xanthogenate-oxidant redox system which could be used to obtain graft copolymers of cellulose. They have successfully grafted acrylonitrile onto cellulose using this method and the results indicated low homopolymer content, especially when substrates with high xanthate content were used. The essential reaction is given in Scheme V (Fig. 6).

The mechanism of disintegration of the initially formed radical on the xanthogenate group resembles the disintegration of the similar radical formed on the carboxyl group under similar circumstances[56]. As can be seen, cellulose macroradicals are obtained as a result of both the oxidation of the xanthogenate group and as consequence of radical or chain transfer. The OH radicals can recombine among themselves or with some of the other created radicals or they can split off the hydrogen atom from either the hydroxyl groups of cellulose or from the -SH group of the xanthogentate. They can result in homopolymerization or can stop transfer or propagation of growing chains. However, the xanthate system of graft copolymerization has a disadvantage in that the graft are attached to the polysaccharide in part through thioester linkages that are cleaved by hydrolysis[57]. Trimnell et at.[58] have introduced thiol groups onto cellulose via starch tosylate route by the sequence:

Starch-→ Starch-OTs → Starch-SC(=S)OEt→ Starch-SH

(where OTs is tosyloxy) so that the grafts would become attached through stable thioester linkages. In addition to acrylonitrile, they have grafted monomers like styrene, acrylamide, acrylic acid and dimethylaminomethyl methacrylate (DMAEMA) onto thiol starch in presence of hydrogen peroxide. The peroxide causes both grafting of monomer at the thiol site and coupling of the thiol groups to disulfide. Treating of graft copolymers with sodium borohydride cleaves the disulfide links to regenerate the thiol groups so that the grafting sequence could be repeated. In this way, add-on and grafting frequently could be greatly increased over what is normal in a single grafting step. The process may be illustrated as in

Scheme VI (Fig.7).

Thiol groups have also been introduced by Axen et at.[59]. In this method starch is reacted with epichlorohydrin under acid rather than alkaline catalysis to from the chlorohydrin instead of the epoxide. The echlorohydrin is then reacted with sodium thiosulfate and the resulting
thiosulfate is reduced to the thiol.

Starch-OH$\rightarrow$ Starch-$OCH_2CH(OH)CH_2Cl$$\rightarrow$

Starch-$OCH_2CH(OH)CH_2S_2O_3Na$$\rightarrow$Starch-$OCH_2CH(OH)CH_2SH$(Thiol)

Bayazeed et al.[60] have studied the graft copolymerization of acrylamide onto starch using ferrous-starch thiosulfate-persulfate redox system with persulfate as the oxidant under different conditions. In this study the graft yield was shown to be significantly favoured in acidic as well as with increasing persulfate concentration.

(v) INITIATION BY Cu(II) ION

In 1962 Kimura, Takitani and Imoto[61] found that an aqueous solution of starch with Cu(II) ions could easily polymerize methyl methacrylate (MMA) and about half of the polymerized MMA was grafted onto starch. This novel polymerization method called "uncatalyzed polymerization", was studied in detail later on by Imoto el al.[62]. They have concluded that the polymerization proceed in the water phase, Particularly in hydrophobic areas (or micelles) formed by starch. They have proposed[63] the initiation mechanism to be proceeding via complex of CU(II) ion with starch, water and MMA as represented in Scheme VII (FIg.2.8). The formation of the complex of Cu(II) ion with starch and water have been confirmed by the measurements of electric conductance and difference spectrum of the starch solution in the presence of $CuCl_2$.

Graft copolymerization has also been initiated by pretreatment of starch with ozone-oxygen mixtures. Such pretreatment forms hydroperoxy groups on starch which, when decomposed by heat or reducing agents in the presence of monomer, form free radical sites to initiate graft copolymerization. The mechanism of ozone attack on carbohydrates has been extensively studied by Katai and Schuerch[64].

(vi) INITIATION BY AZO-BIS-ISOBUTYRO NITRILE (AIBN)

The use of azobisisobutyronitrile (AIBN) as initiator for vinyl graft polymerization onto cellulose[65-67] and modified celluloses[68] as well as wool was reported. It was thought that the grafting reaction proceeds through production of macroradicals resulting from a chain transfer reaction between a growing vinyl homopolymer chain and the substrate present in the polymerization reaction. AIBN decomposes to give rise a cyanoisopropyl radical which can initiate polymerisation. These studies have also revealed that the magnitude of grafting varies significantly with the type and amount of solvent used.

(vii) CHROMIUM (VI) INITIATION

Much work was done on Cr (VI) initiated graft copolymerization of vinyl monomers onto polysaccharide backbone. Chromium (VI) has been used by Nayak et al. as an initiator for graft copolymerization[69,70] as well as homopolymerization[71]. Mishra et al.[72,73] reported the feasibility of chromium (VI) to induce graft copolymerization of methylmethacrylate onto cellulose. In a system of potassium dichromate, perchloric acid, MMA and cellulose, Cr (VI) reacts with cellulose to form cellulose macroradicals which react with vinyl monomers resulting in the formation of graft copolymer on the backbone of the fibre. It was also reported that for grafting of MMA on cellulose, increasing the Cr^{+6} concentration up to 24.9×20^{-11} M, the graft yield increases ; thereafter it decreases.

(viii) INITIATION BY V^{+5}

Pentavalent V^{+5} can also be used for graft copolymerization of vinyl monomers into the polysaccharide backbone. Lenka et al.[74] carried out graft copolymerization of methyl methacrylate on cellulosic materials with the use of quinquelevalent vanadium as an initiator was studied. Increase of V^{+5} ion concentration upto 0.0025 mole/litre increases graft yield, and with further increase of the initiator the graft yield decreases. The graft yield increases with increase of monomer concentration. The graft yield is medium and substrate dependent.

Grafting of acrylonitrile into acetylated chemically modified jute was carried out in the temperature range 40-60^{0}C using V^{+5}-cyclohexanone redox initiator system and grafting of MMA onto jute fibres using redox systems consisting of V^{+5} and cyclohexanone or cyclohexanone was reported by Mohanty et al.[75] V^{+5} with a series of organic species gives free radicals which can initiate the polymerisation of vinyl monomers.

(ix) INITIATION BY METAL CHELATES

Recently attention have been paid to find new initiating systems of radical reactions based particularly on chelate complexes of transition metals. The initiating species are believed to be the ligand radicals formed, i.e. under homolytic scission of the metal-oxygen bond of metal acetylacetonates. At the same time the formal valence of the metal is reduced by one which is confirmed by spectral and ESR measurements. The efficiency of initiation of metal chelates could be enhanced by addition of various compounds, mainly halogen containing compounds and compounds of electron donating substances such as dioxane and dimethyl sulfoxide etc. In the majority of the cases reported so far, the polymerization proceeds via a typical free radical process. Kasting et al.[76] reported the polymerization of MMA by Mn $(acac)_3$ halogen compounds. Tripathy et al.[77] reported grafting of MMA onto cellulose by using acetylacetonato manganese (III) complex.

2. PROOF OF GRAFTING

(i) Extraction of Homopolymer and IR Spectra of Graft Copolymers

IR has served as one of the important tools in the hands of polymer chemists for the study of proof of grafting. The IR spectra of grafted branches (isolated after depolymerization of starch) of PAN[78] and polyacrylamide[78], all showed bands due to carbohydrate end groups. The carbohydrate end groups on polyacrylonitrile branches have also been benzoylated and the resulting benzoyl groups have been detected by both IR and UV spectroscopy[79]. In a typical experiment Goni et al.[82,83] have carried out the graft copolymerization of methyl acrylate (MA), ethyl acrylate (EA), ethyl methylacrylate (EMA) and n-butyl methacrylate (n-BMA) on amylose, initiated by ceric ammonium nitrate. The reaction products were extracted with dilute alkali (0.5 M NaOH) to remove any ungrafted carbohydrate. Afterwards, the homopolymers produced were removed by extraction with acetone for poly (methyl acrylate) (PMA), THF for poly (ethyl acrylate) (PEA) and poly (ethyl methacrylate) (PEMA), toluene for poly (butyl methacrylate) (PBMA), 1,2-dichloro ethane, THF and toluene for PBA. Measurement of FT IR spectra of all the purified graft copolymers showed absorption bands at 3450 and 1100-1000 cm^{-1} characteristic of amylose and additional absorption bands at 2960, 1730, 1450 and 1250 cm^{-1} characteristics of acrylic polymers.

Under conditions that lead to poor grafting efficiency, selective solvent extraction of crude reaction products with a good solvent easily removes the ungrafted homopolymers of acrylonitrile[79,80] and methyl methacrylate[81] while ungrafted starch could be removed by water extraction. The inability to achieve such separations with other polymers thus provides evidence for chemical bonding between starch and the acrylic content.

(ii) Molecular Weight of Grafted Chains

It has been possible to calculate the molecular weight of the grafted chains after selective hydrolysis of the carbohydrate backbone. Two methods of hydrolysis have been used : acid hydrolysis and enzyme hydrolysis. The acids used for the hydrolysis of carbohydrate backbone are perchloric acid $(HClO_4)$[85,86], hydrochloric acid (HCl)[81], sulfuric acid (H_2SO_4)[84] and $NaIO_4/NH_4OH$ system, etc. The enzyme method of hydrolysis is particularly suitable when the graft copolymer has polyacrylamide branches (PAM forms imide links which get crosslinked in presence of acids). α -Amylase has been used for hydrolysis of starch based graft copolymers. McCormick et al.[85,86] have used diazyme for the hydrolysis of dextran based graft copolymers with acrylamide. Molecular weight of grafted chains have been calculated from intrinsic viscosity, $[\eta]$, using Mark-Houwink relationship : the 'K' and 'a' values depending on the nature of the polymer involved. For instance :

For polyacrylamide[87,88]

$$[\eta] = 6.8 \times 10^{-4} (\overline{M}_n)^{0.66}$$

$$[\eta] = 6.31 \times 10^{-5} (\overline{M}_w)^{0.80}$$

for polyacrylonitrile[79] $$[\eta] = 3.92 \times 10^{-4} (\overline{M}_n)^{0.75}$$

for poly(vinyl alcohol)[89] $$[\eta] = 0.96 \times 10^{-4} (\overline{M}_n)^{0.69}$$

Where, $\overline{M}_n$ = number average molecular weight
$\overline{M}_w$ = weight average molecular weight.

(iii) Use of ^{13}C NMR Spectroscopy

Since early 1960s, the ^{13}C nuclear magnetic resonance (^{13}C NMR) has provided stereochemical information on a large number of chemical compounds. More recently, this technique has also been applied to the investigation of stereochemical investigation of polymers. Goni et al[90,91] in a series of some interesting research communications have for the first time reported the identification of graft copolymers by independent use of ^{13}C NMR spectroscopy. They have reported the proof of grafting of acrylic monomers onto starch directly from ^{13}C NMR result without prior degradation of the graft copolymer as has been done earlier. One of the characteristics of NMR spectroscopy is the proportionality between the signal intensity and the concentration of each carbon type. So depending upon the carbon type (primary, secondary, etc.) and its chemical environment, i.e. depending on the carbon surrounding or lattice, the time (the relaxation time) needed to achieve equilibrium in the magnetic field will be different. Hence, to establish any comparison among the different peaks, the relaxation time must be taken into account. Goni et.at have calculated the percent grafting(%G= ratio between the weight of the acrylic grafted polymers and the grafted carbohydrate) directly without the previous degradation of the products from a measurement of the relaxation time of the graft copolymers.

(iv) Proof of Grafting by Enzyme Hydrolysis of the Grafted products

Rath and Singh[92] reported the proof of grafting by a combined use of viscometry and enzyme hydrolysis of the polyacrylamide grafted amylopectin. A three fold approaches were made for this purpose. First, a series of graft copolymers were synthesized with varying catalyst and monomer concentrations to obtain a variation in the number and length of polyacrylamide chains in the series as evidenced from their intrinsic viscosity measurements. Second, α -amylase enzyme was used to hydrolyse the amylopectin backbone so that most of the polyacrylamide chains are released. Measurement of intrinsic viscosity both before and after the cleavage of polysaccharide backbone resulted into a great change, because of the drastic change in molecular weight as well as molecular structure. The change was in accordance with the trend in original viscosity of graft copolymers in the series. The change was observed as the products are graft copolymers and not the physical mixtures of the polysaccharide and polyacrylamide. The physical mixture was ruled out by measuring the intrinsic viscosity of the physical mixture of amylopectin and polyacrylamide after the treatment by the enzyme α -amylase. The intrinsic viscosity before and after hydrolysis of this physical mixture remain the same.

3. Calculation of Grafting Parameters

Various grafting parameters can be calculated by using the following equations.

I. % Graft Level[93] (P_g) = (Weight of grafted polymer / Weight of polysaccharide)× 100.

II. Rate of Graft Copolymerization[93] (R_g) = [(Weight of grafted polymer) / {(Molecular weight of grafted polymer) x (Reaction time) x (Reaction volume)}] × 1000.

III. Frequency of Grafting[93] (F_g) = [{(Weight of grafted polymer) x (Molecular weight of AGU)} / (molecular weight of grafted polymer)] × 10^4.

IV. % Conversion[94] = [(Weight of product - Weight of polysaccharide) / (Weight of monomer)× 100.

V. % Add-on[94] = (Weight of pure copolymer - Weight of polysaccharide) / Weight of pure copolymer.

VI. % Polymer grafted[94] = [(Weight grafted polymer) / (Weight grafted polymer + Weight ungrafted polymer)] × 100.

VII. Grafting frequency (AGU/graft)[95] = [{(100-9% Add-on) / Weight of AGU}] / (%Add-on / Molecular weight graft).

VIII. Rates of homopolymerization[95] (R_n) = [(Weight of homopolymer) / (M_w of Monomer x Reaction time (see) x Reaction volume (ml)] × 1000.

IX. Rate of total polymerization[95] (R_p) = R_g + R_n.

Okieimen et al[96] calculated the copolymer composition during grafting of mixture of ethyl acrylate and acrylonitrile on starch by using the following relationship.

Mole fraction of acrylonitrile in the copolymer =

$$[(N/14) \times W] / [N/14 \times W + \{W - (N/14) \times (53.06 / W)\}] / 100 \times 12$$

Where N being the weight of nitrogen per gram of copolymer sample and W the weight of the sample.

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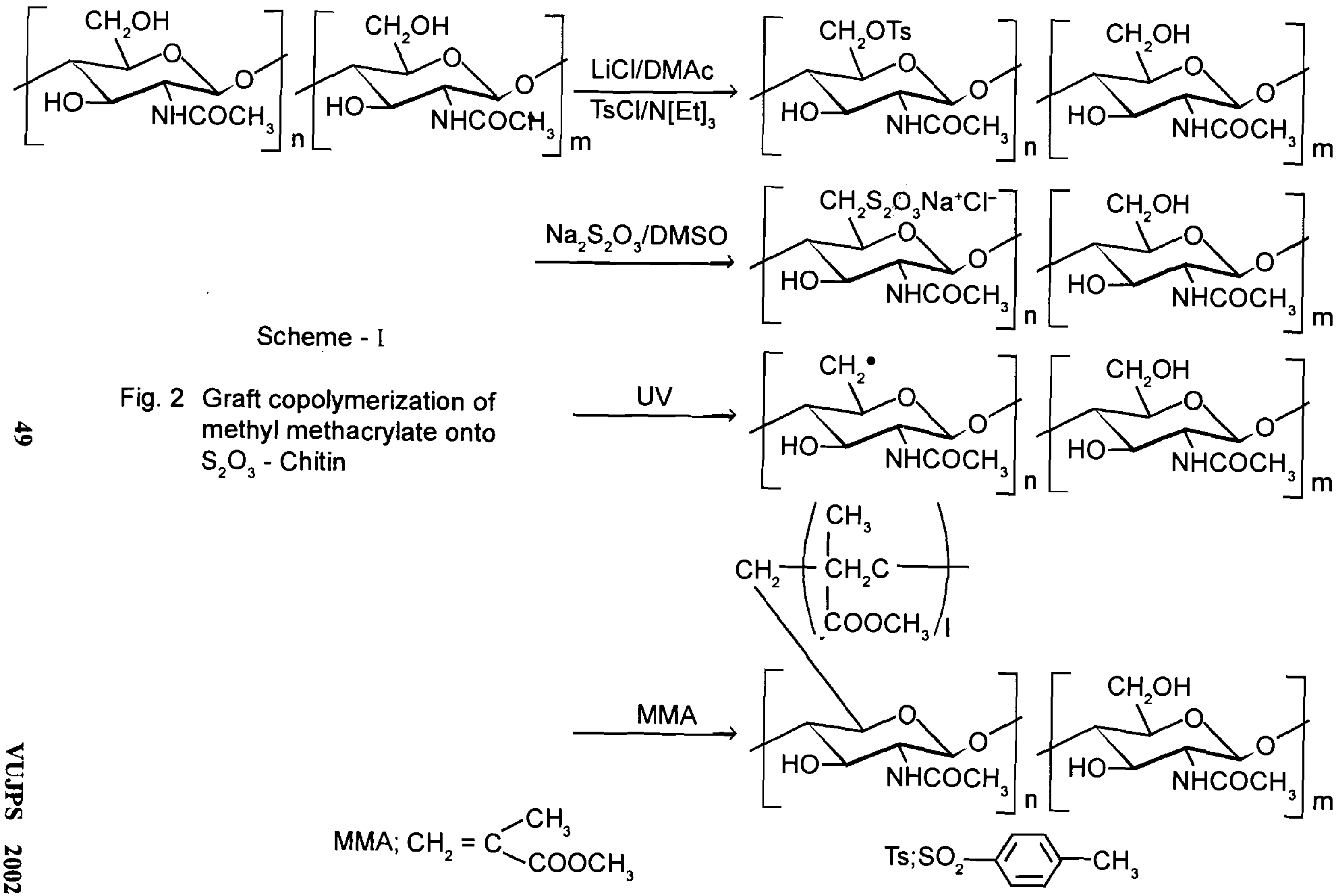
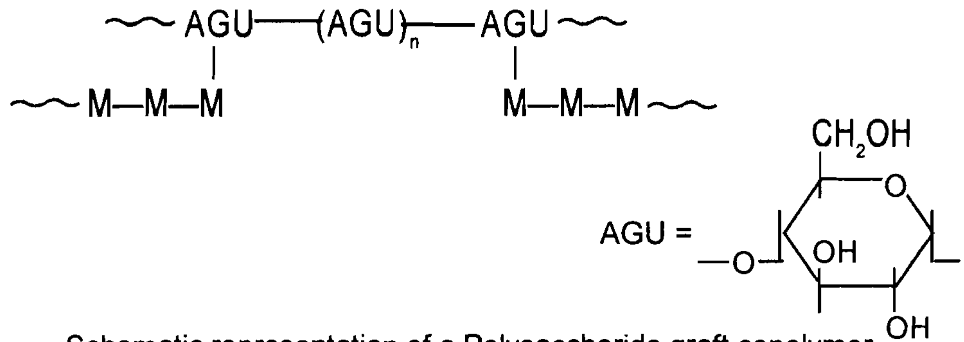
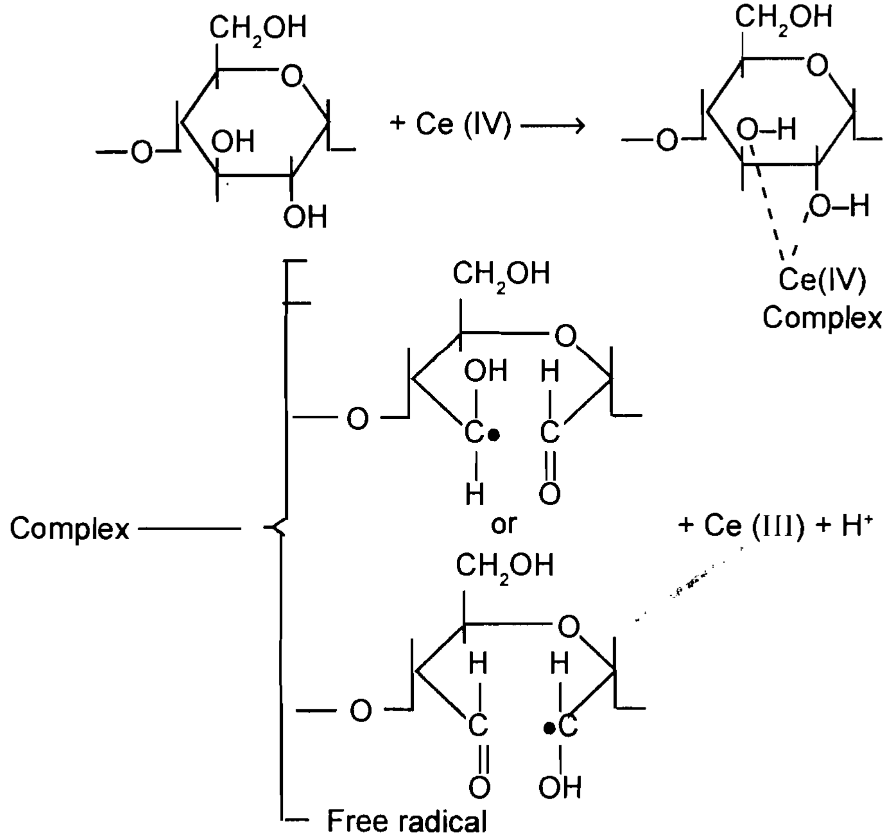


Scheme - I

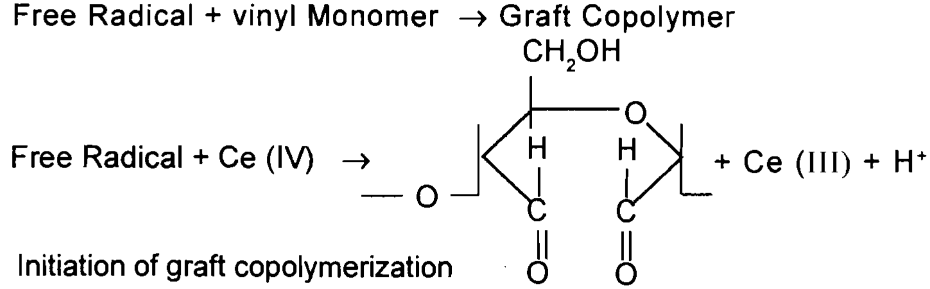
Fig. 2 Graft copolymerization of methyl methacrylate onto S_2O_3 - Chitin



Schematic representation of a Polysaccharide graft copolymer



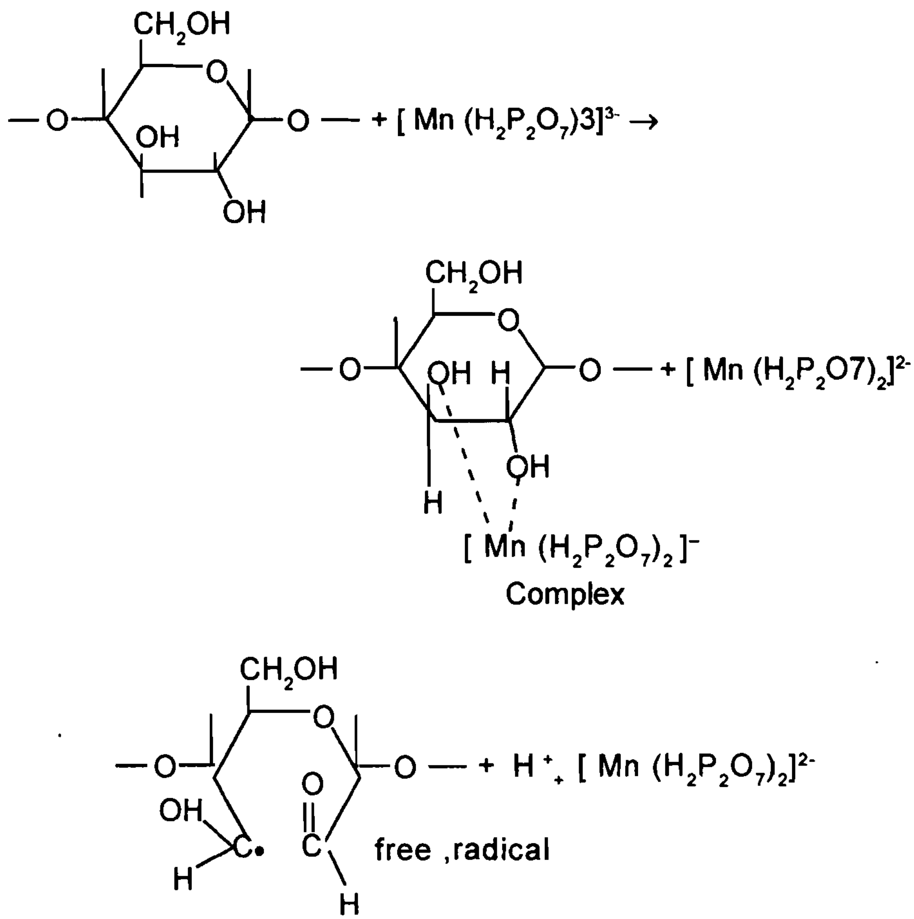
Free Radical + vinyl Monomer → Graft Copolymer



SCHEME II - Initiation of graft copolymerization by ceric ion

Fig. 3

INITATION :



PROPAGATION :

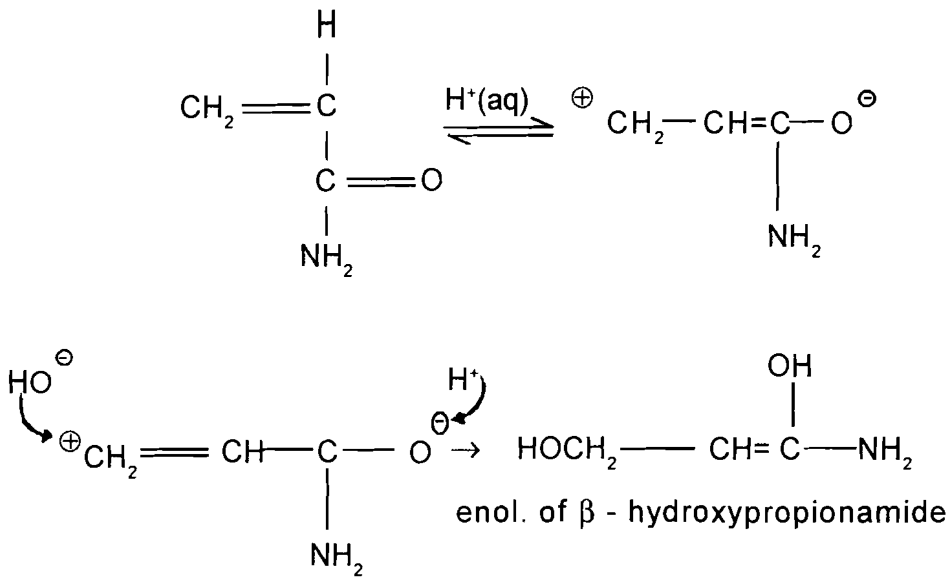
Free radical + monomer → graft copolymerzation

TERMINATION :

$(M)_X M^{\bullet}$ + Mn^3 → Mn^{2+} + H^+ + Ploymer

SCHEME III

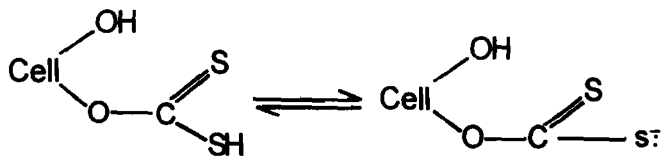
Fig. 4 Mechanism of glycol Cleavage and intiation of graft copolymerization with Mn^{+3} ions



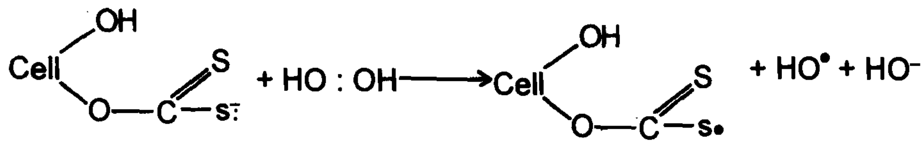
$$\text{enol} + Mn^{3+} \rightarrow Mn^{2+} + H^{+} + \text{free radical}$$
$$\text{free radical} + \text{monomer} \longrightarrow \text{homopolymerization}$$

SCHEME IV

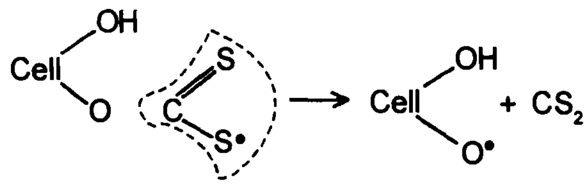
Fig. 5 Iinitation of homopolymerization of acrylamide by Mn^{3+} initation



(1)



(2)



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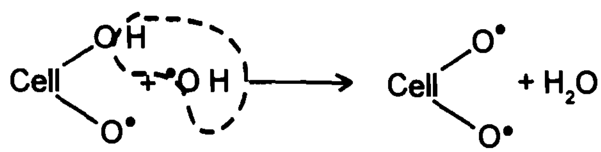
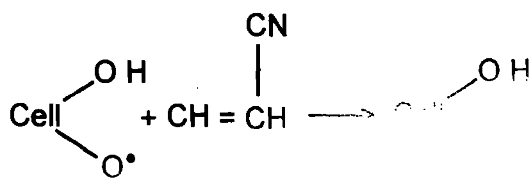
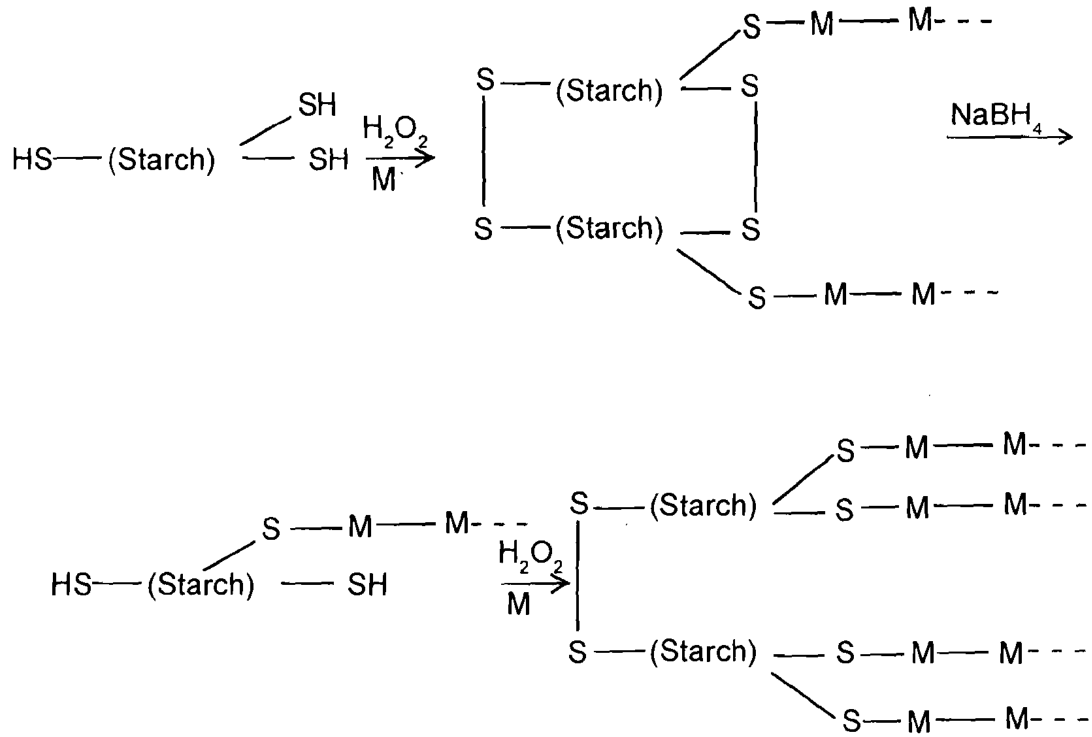


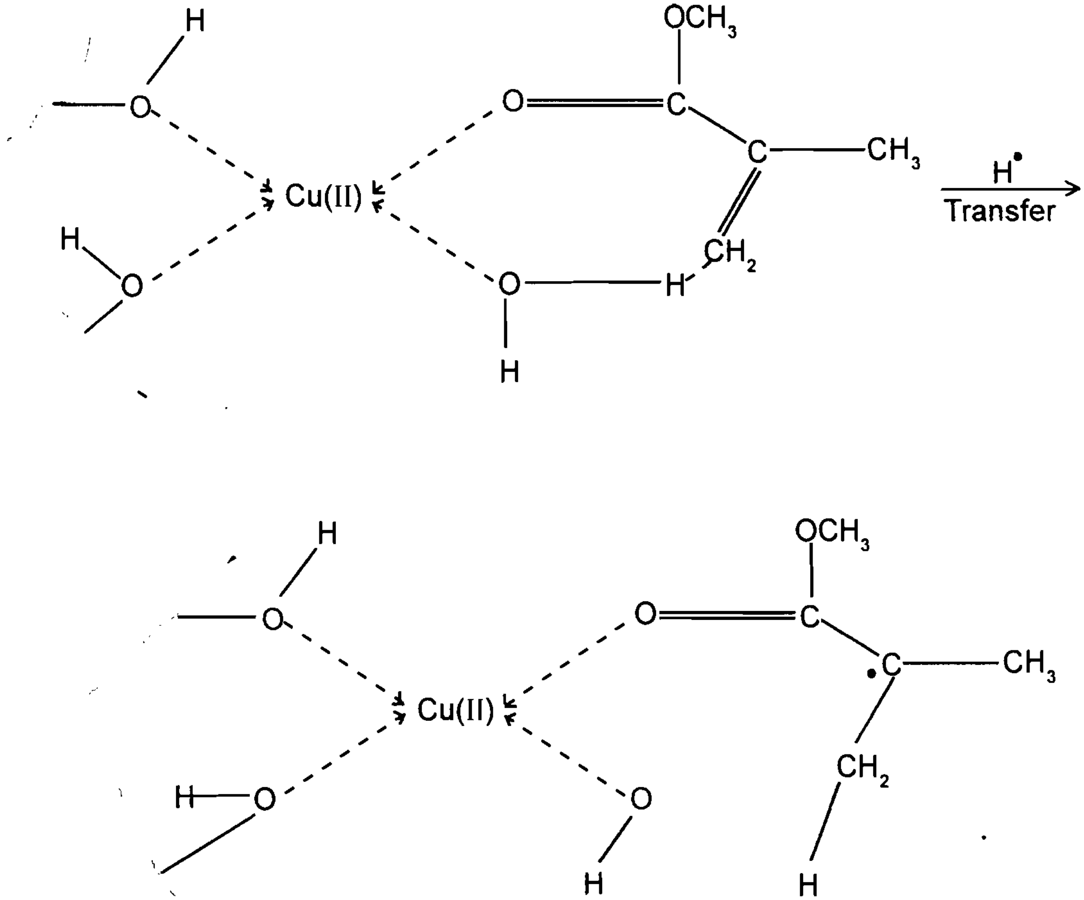
(4)





SCHEME VI

Fig. 7 Initation by thiolation of starch



SCHEME VII

Fig. 8 Schematic representation of the complex of Cu (II)—H_2O—starch—MMA

A CRITICAL APPRAISAL OF KITIS APPROXIMATION TO THE TEMPERATURE INTEGRAL

Manabesh Bhattacharya[a] and W. G. Devi[b]
[a]Department of Physics
Krishnath College
Berhampore - 742 101, West Bengal, India
[b]Department of Chemistry
T. S. Paul Manipur Women's College
Monsangei, Imphal - 795 008, Manipur, India

Abstract :
In the present paper we critically assess the Kitis approximation to the temperature integral by using it to analyse numerically computed and experimental DTA curves.

INTRODUCTION

The analysis of Differential Thermal Analysis (DTA) curves requires the calculation of well known temperature integral

$$I(T) = \int_0^T \exp(-E/RT')\, dT' \qquad (1)$$

Where E is the activation energy, R is the universal gas constant and T is the temperature in K. Kitis et al [1] proposed an approximation to the temperature integral. In the present paper we analyse the suitability of the Kitis approximation to the temperature integral by using it to evaluate the kinetic parameters namely E (activation energy), n (order of kinetics) and A (pre-exponential factor) from numerically computed as well as experimental DTA curves.

THEORY

We start from equation (1) by putting $x' = E/RT'$ and obtain

$$I(x) = (E/R) \int_x^\infty \exp(-x')/(x')^2\, dx'$$

Now by using the integral transform of complementary incomplete Gamma function $\Gamma(u,x)$ [2] given by

$$\Gamma(u,x) = \int_x^\infty \exp(-t)\, t^{u-1}\, dt \tag{3}$$

I(x) can be expressed as

$$I(x) = (E/R)\, \Gamma(-1,x) \tag{4}$$

Now the asymptotic series expansion of the complementary incomplete Gamma function [2] is given by

$$\Gamma(-1,x) = \exp(-x)\, [1 - 2/x + 6/x^2 - \ldots\ldots\ldots]/x^2 \tag{5}$$

From equations (4) and (5) one gets

$$I(x) = (E/R) \exp(-x)\, [1 - 2/x + 6/x^2 - \ldots\ldots]/x^2 \tag{6}$$

so that equation (1) reduces to

$$I(T) = [RT^2/E]\exp(-E/RT)\, [1 - 2RT/E + 6R^2T^2/E^2 - \ldots\ldots\ldots] \tag{7}$$

Now retaining the first two terms of equation (7) we obtain

$$I_K(T) = [RT^2/E]\exp(-E/RT)\, [1 - 2RT/E] \tag{8}$$

Here $I_K(T)$ is the approximation to the temperature integral proposed by Kitis et al [1]. Now following Devi et al [3] one can write for first order (n=1) and non-first order ($n \neq 1$) DTA curves

$$\Delta T/(\Delta T)_m = \exp[x_m - x + F(x, x_m)] \qquad \text{for } n = 1 \tag{9}$$

$$\Delta T/(\Delta T)_m = \exp(x_m - x)\, [1 - (n-1)F(x, x_m)/n]^{n/(1-n)} \qquad \text{for } n \neq 1 \tag{10}$$

with

$$F(x, x_m) = x_m^2 \exp(x_m)\, [\Gamma(-1, x_m) - \Gamma(-1, x)] \tag{11}$$

where (ΔT) is the temperature deviation from the base line at temperature T, $(\Delta T)_m$ is the maximum value of ΔT and $x_m = E/RT_m$, T_m being the peak temperature.

Now from equation (5) after retaining first two terms, equation (11) reduces to

$$F(x, x_m) = (1 - \Delta_m) - (T/T_m)^2\, (1 - \Delta)\exp[E(T - T_m)/(RT)] \tag{12}$$

with $A_m = 2RT_m/E$ and $A = 2RT/E$

Substituting equation (12) into equations (9) and (10) one gets

$$\Delta T/(\Delta T)_m = \exp[1 + E((T - T_m)/T_m)/RT - (T^2/T_m^2)\exp(E((T - T_m)/T_m)/RT)\, (1 - \Delta) - \Delta_m] \qquad \text{for } n = 1 \tag{13}$$

$$\Delta T/(\Delta T)_m = (n)^{n/(n-1)} \exp[E((T - T_m)/T_m)/RT] - [(n-1)\, (T^2/T_m^2) \times \exp(E((T - T_m)/T_m)/RT)\, (1 - \Delta) + Z_m^2] \qquad \text{for } n \neq 1 \tag{14}$$

with $Z_m = 1 + (n-1)\, \Delta_m$

RESULTS AND DISCUSSIONS

We apply equations (13) and (14) to fit some numerically computed DTA peaks by using the rigorous curve fitting code developed by Devi [4] and the corresponding results are shown in Table 1. From this Table it is evident that there is very good agreement between the input values E_{in}, n_{in} and A_{in} and the values E_K, n_K and A_K obtained by using the above mentioned curve fitting procedure. As a test of goodness of fit, in Table 1, the values of chi squared (χ^2) have been incorporated. The values of χ^2 have been evaluated by using the relation [5]

$$\chi^2 = [\Sigma_{i=1}^{M} |f(X_i) - Y_i|^2]/(M - m_f) \quad (15)$$

where $f(X_i)$ are the values of the function used for fitting, Y_i are the values of DTA curve fitted at the points X_i, M is the total number of points used for the determination of χ^2 and m_f is the number of parameters that are fitted. But contrary to the observations of Chakravarty et al [6] the most reliable test of goodness of fit is figure of merit (FOM) [7-10] defined as [11,12]

$$FOM = 100 \times ([\Sigma_{i=1}^{M} |f(X_i) - Y_i|]/A_{fit}) \quad (16)$$

where A_{fit} is the area of the fitted peak. The contention of Chakravarty et al [6] is not correct because FOM is a more convenient expression in comparison with χ^2 because standard deviations of the quantity $\Delta T/(\Delta T)_m$ [equations (9), (10), (13) and (14)] for DTA curves are normally not known accurately in the case of experimental DTA curves. A FOM value of about 5% usually indicates a good fit. The fitting program developed by Devi [4] has been upgraded by incorporating the provision for the calculation of χ^2 and FOM. The values of χ^2 and FOM depicted in Table 1 indicate a good quality fitting.

We apply the present method of curve fitting to a number of experimental DTA curves of (i) $Ca(DMP)_2(MCA)_2$ [13] and (ii) Calcitic limestone [14]. From Table 2 it is observed that as in the case of numerically computed DTA peaks (Table 1) here also the values of the kinetic parameters E_K, n_K and A_K obtained by using a curve fitting scheme in which the approximation to the temperature integral proposed by Kitis et al [1] is used are in fair agreement with the values of parameters E_{cf}, n_{cf} and A_{cf} obtained by Devi by employing a rigorous code of curve fitting [4].

CONCLUSION

In the paper it is demonstrated that Kitis approximation to the temperature integral may be successfully used to analyse both numerically computed and experimental DTA curves.

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Table 1 : Calculation of trapping parameters of some numerically computed DTA peaks. Notation A(B) stands for A x 10^B. All the peaks correspond A_{in} = 10 x 10^{12} s^{-1}, E_{in} = 96.44 KJ/mol.

n_{in}	n_K	E_K (KJ/mol)	A_K (sec^{-1})	χ^2	FOM (%)
0.7	0.7	96.40	9.97(12)	6.62(-5)	0.145
0.8	0.8	96.45	9.92(12)	7.18(-5)	0.356
0.9	0.9	96.41	9.97(12)	1.05(-5)	0.105
1.0	1.0	96.42	9.95(12)	3.47(-5)	0.238
1.1	1.1	96.44	1.00(12)	1.38(-5)	0.462
1.2	1.2	96.42	1.02(12)	1.73(-5)	0.185
1.3	1.3	96.41	1.04(12)	3.48(-5)	0.234
1.4	1.4	96.44	9.98(12)	7.57(-5)	0.321
1.5	1.5	96.42	9.99(12)	2.08(-5)	0.367
1.6	1.6	96.40	1.01(12)	6.18(-5)	0.284
1.7	1.7	96.45	9.98(12)	3.45(-5)	0.154
1.8	1.8	96.43	9.94(12)	2.37(-5)	0.372
1.9	1.9	96.44	9.98(12)	1.54(-5)	0.189
2.0	2.0	96.45	1.03(12)	5.38(-5)	0.457
2.1	2.1	96.43	1.00(12)	3.72(-5)	0.198
2.2	2.2	96.45	9.95(12)	6.25(-5)	0.542
2.3	2.3	96.42	1.00(12)	8.31(-5)	0.785
2.4	2.4	96.44	9.97(12)	2.45(-5)	0.310
2.5	2.5	96.45	9.96(12)	4.85(-5)	0.395

Table 2 : Kinetic parameters of some experimental DTA peaks. Notation A(B) stands for A x 10^B.

System	E_K (KJ/mol)	E_{cf} (KJ/mol)	n_K	n_{cf}	A_K (sec^{-1})	A_{cf} (sec^{-1})	χ^2	FOM (%)
$Ca(DMP)_2(MCA)_2$[13]	34.87	35.00	1.0	1.0	80.51	82.35	1.95(-6)	0.0985
Calcitic limestone[14]	142.52	143.90	1.5	1.5	7.51(16)	7.83(16)	2.54(-6)	0.138

ONE - TIME INVENTORY MODEL WITH VARIABLE REPLENISHMENT AND PRICE - DEPENDENT DEMAND VIA GENETIC ALGORITHM

S.Mondal and M.Maiti*
Department of Applied Mathematics with oceanology and Computer Programming
Vidyasagar University
Midnapur - 721 102, India
E-mail : vidya@dte.vsnl.net.in

Abstract :
A one-time inventory model for a deteriorating seasonable product with variable replenishment is formulated. Here, price of the item depends on the availability of the product in the market and demand is inversely proportional to price. A genetic algorithm (GA) is developed and applied for the solution of this one-time business problem. For the present GA, binary coded representation for the chromosomes is considered and a heuristic principle based on the present problem has been derived to increase the number of initial feasible individuals. The basic operation - (i) selection / reproduction (ii) cross-over and (iii) mutation are iteratively applied on a set of coded solutions and optimum results for the said inventory problem are obtained. The model is illustrated with some numerical data.

Key words : Inventory, Genetic Algorithms, Variable Replenishment, Price-dependent demand.

INTRODUCTION

Since the development of Harris's [1915] EOQ model, many researchers worked on the inventory models with different types of demand and replenishment. In deterministic inventory, normally there are four types of demands - constant and time, price and on-hand/initial stock-level dependent demands. Earlier inventory models (cf. Handley and Whitin [1963], Naddor [1969] etc) were developed with constant demand. But it is usually observed in the market the sales of the fashionable goods, electronic gadgets, food-grains, etc increases with time. For this reason, the models with time dependent demand have been studied by Hariga and Benkherouf [1994] , Hong, Cavalier and Hayya [1993] and others. Again, according to market research, glamorous display of items with the help of modern electronic systems in the super markets motivates the customers to buy more. That is why Mandal and Phaujdar [1989], Mandal and Maiti [1997] and others have developed models with stock - dependent demand. It is an universal fact that whenever the selling price of an item increases, the demand for that decreases and vice versa. Several authors like Urban [1992], Das, Bhunia and Maiti [1999] and others have investigated the inventory models with price-dependent demand. In the real world, deterioration is a natural phenomenon. Volatile liquids like gasolin, alcohol, etc., radioactive substances, blood, food stuffs and electronic components looses their utilities while in stock. A lot of research papers have been published by several authors like Cheng and Ting [1994], Wee [1997] and others taking the deterioration of items into account.

In classical inventory system, replenishment is either infinite or finite. In reality, it may be crisp but vary with stock, time and / or demand. Usually replenishment increases with demand but it decreases as the stock in go-down increases. Balkhi and Benkherouf [1989], Bhunia and Maiti [1997], Mandal and Maiti [1997] and others have solved the inventory models with variable replenishment.

Normally, it is assumed that if the price of a comodity is slashed down, demand of that item goes up, consumption of the item in the society will be more and to meet the extra demand, replenishment of the item is stepped up. But, in the literature, none has considered the situation when the above phenomena occur in the reverse way. A question may be asked : can the increased replenishment bring down the price and hence increase the demand?

It is very often observed that over production or availability of a comodity in large quantities in the market brings down its unit price and as a result, the demand of the comodity goes up along with the production / availability. It is assumed here that market has the capacity to absorb the increased production. This phenomenon is very often observed in the developing countries for the seasonal products like potato, tometo, mango, etc. The production of these consumable products is not controlled by the pre-determined future price but depends on the rain, favourable weather, etc. in the third world countries like India, Nepal, etc., if there is a bumper crop for a product, say, tometo, the market is flooded with that as preserving / storing facilities in cold storages are very limited in those countries. As a result, the price of the comodity, i.e., tometo reaches rock-bottom level and the lower-income group of people who normally could not afford to buy tometo due to its high price, may like to have a taste of it now and the middle class people who were buying tometo in small quantity, may prefer to eat more. Naturally, the demand of the comodity goes up. Hence, over-production of a comodity automatically creates the demand in the market. In this case, both price and demand are replenishment dependent, though their dependencies are opposite in nature.

These seasonal products are available in abundant only for a fixed period of a year though the materials from cold storages are available and sold throughout the year at a much higher price. Normally, at the beginning of the season, the particular product of that season starts to arrive at the market and its availability increases with time as the season progresses. After sometime, it stabilizes i.e. flow of the comodity to the market becomes constant for a short period. Towards the end of the season, the supply of the product recedes as the products from the late producers only are available in the market. When the season is over, the market comes back to its position earlier to the commencement of the season and only the cold storage stuffs are sold in the market at a higher price. The price and demand of a seasonal product fluctuates with the above mentioned variations in the flow of the comodity in the market. Till now, none has examined this natural phenomenon mathematically i.e. one-time business of a seasonal product. This paper gives as answer to this problem.

In inventory problems, decision is made to find out the quantity to be purchased / produced and the time-period in order to have maximum profit or minimum cost. In solving decision - making problems like inventory problems, etc., normally gradient based optimization methods are used for computation. These methods have limitations. One limitation is that these methods very often stuck to the local optimum. Recently, to examine the decision making problems, Genetic Algorithm (GA), one of the soft computing methods are very popular. It works synergetically and possesses flexible information processing capabilities for handling real - life ambiguous situations. Normally, it stucks to global optimum. It mimicks some of the processes observed in the natural evolution and is based on the Darwin's "survival of the fittest principle". The evolution starts from a set of individuals and proceeds from generation to generation through genetic operations. Replacement of an old population with a new one is known as generation. It is executed iteratively on a set of coded solutions called population with three basic operations - (i) Selection / reproduction (ii) cross-over and (iii) mutation.

Selection / Reproduction : The selection/ reproduction process copies individual coded solutions i.e., strings (called parent chromosomes) into a tentative new population (called offspring) for genetic operations. Number of copies reproduced for the next generation by an individual is expected to directly proportional to its fitness value, there by mimicking the natural selection procedure to some extent.

Crossover : The main purpose of crossover is to exchange information between randomly selected parent chromosomes by recombining parts of their corresponding strings. Actually, it recombines genetic material of two parent chromosomes to produce offspring for the next generation.

Mutation : The main aim of mutation is to introduce genetic diversity into the population. Sometimes, it helps to regain the information lost in earlier generation.

Due to several advantages of GA, it has been successfully applied to various fields like neural network (cf. Pal, De and Ghosh [1997], travelling salesman (cf. Forrest [1993]), scheduling (cf. Davis [1991]), numerical optimization (cf. Michalewicz and Janikow [1991]), pattern recognition (cf. Gelsema [1995]), etc. But, till now a very few researchers (cf. Hung, Shih and Chen [1997], khouja, michalewicz and Wilnot [1998] and others) has applied GA to the production and lot-size scheduling problems.

In this paper, GA has been developed and applied for the solution of a highly non-linear crisp inventory problem in the following way. First, the coded representation of the chromosomes representing the potential solutions of the given problem is selected. Here, binary representation of 20 bits has been used for this purpose. Next, the chromosomes are randomly initialized. In some cases, a heuristic principle based on the present problem has been used to increase the number of feasible initial individuals. Each solution is evaluated to give some measure of its fitness. Here fitness of a solution has been measured by the corresponding value of the objective function. Then, a new population is formed by selecting the more fit individuals. Some members of this new population undergo alterations by means of crossover and mutation to form new solutions. Here, crossover has acted with predefined probabilities, 0.2 and 0.42. A cutting point has been randomly selected and genetic materials have been exchanged to create two offsprings by gluing parts of the parent chromosomes. Now, mutation has been applied to alter one gene randomly selected chromosome with a mutation probability equal to 0.05. Actually, mutation gives some extra variability into the population. All these processes are iteratively repeated till the total improvement of the last 10 best solutions is less than 0.01.

The above GA has been applied to solve a one-time crisp inventory problem with price directly and demand indirectly being dependent on replenishment. The said inventory model has been developed for a deteriorating seasonal item for which the business period is only the season time. Here, replenishment varies with time. There is a constant supply of the item throughout the year. At the beginning of the season, the supply increases with time, t for an initial period; after that, becomes constant for another period and then reduces with the increase of time and ultimately comes back to the value of constant supply which prevailed before the season. Here, price of the item varies inversely with the amount of replenishment whereas the demand in the market is inversely related to the price. The model has been formulated to maximize the total average profit for one-time business. There will be seven scenarios for the problem depending on the time upto which replenishment is made. For each scenario, expression for the total average profit has been derived and optimized using GA developed for this purpose. The above model is illustrated with an numerical example and the results for different GA parameters are presented.

MODEL FORMULATION

Assumptions :

Lead time is zero
Shortages are not allowed
This is an one-time business for a seasonal product.
The time horizon of one-time inventory system is finite and crisp but unknown.
Replenishment rate is deterministic but a function of time, t. At time t = 0, it is constant and then increases with time, t upto t = t_1. During the time period (t_1, t_2), it is again constant and after t = t_2, it decreases upto t = T. At this point, it assumes its earlier value at t = 0 and retains this value upto t = H, scheduling period. Here, t_1, t_2 and T are known.
Purchasing price is continuous and a function of replenishment.
Demand is continuous and dependent on price.
Inventory is built up upto a certain time, t = t_s (unknown) and after that, demand is met from the stock.
Two types of deterioration are assumed, one during the inventory building period and another after that.

Notations :

C_1 = inventory holding cost per unit quantity per unit time,
C_3 = set-up cost for the business period,
m = mark-up,
a_0 = deterioration per unit quantity per unit time during the inventory building period,
a_1 = deterioration per unit quantity per unit time during which no replenishment is made
k(t) = replenishment at time, t = $k_0 + k_1 t$, k_0, k_1 being the shape parameters of the replenishment function and k_0 is the constant replenishment available before and after the season.
Mathematically, it may be represented as (Fig - 1)

$$k(t) = \begin{cases} k_0 + k_1 t & , 0 \le t \le t_1 \\ k_0 + k_1 t_1 (= k_{12}, say) & , t_1 \le t \le t_2 \\ k_{12} - k_2 (t - t_2) & , t_2 \le t \le T \\ k_0 & , T \le t \le H \end{cases} \qquad (1)$$

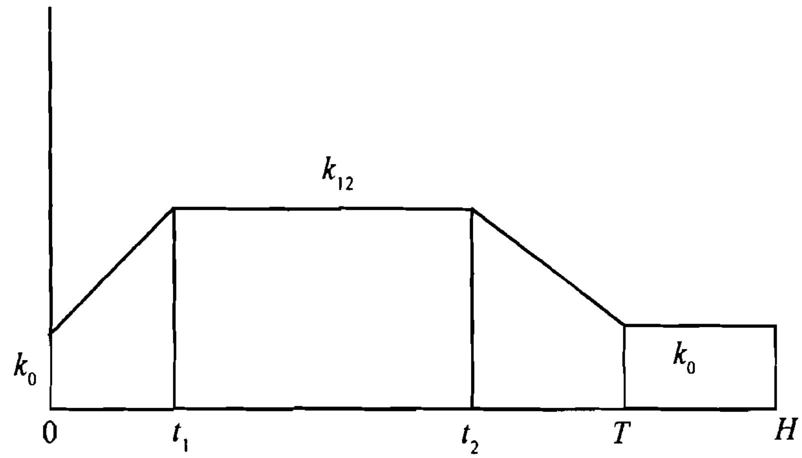

Fig. - 1

$p\{k(t)\}$ = purchasing price per unit quantity at time, t
$= p' - p''k(t)$, p', p'' being the shape parameters of the price function.
This can be expressed as (Fig - 2)

$$p\{k(t)\}\ (= p(t)) = \begin{cases} p_0 - p_1 t & , 0 \le t \le t_1 \\ p_0 - p_1 t_1 (= p_{12}, say) & , t_1 \le t \le t_2 \\ p_{12} + p_2 (t - t_2) & , t_2 \le t \le T \\ p_0 & , T \le t \le H \end{cases} \quad (2)$$

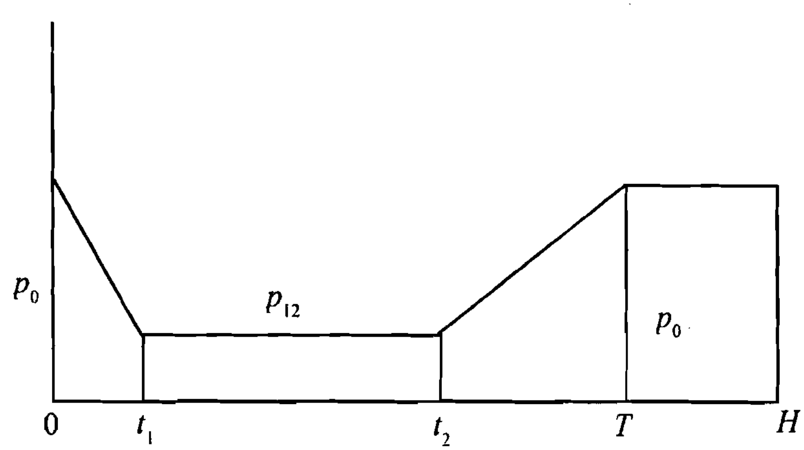

Fig. - 2

p_0 being the constant price per unit quantity before and after the season.

$d\{p(t)\}$ = demand rate at time, t = $d' - d''$ p(t), d' and d'' being the shape parameters of the demand function.
This can be expressed as (Fig. - 3)

$$d\{p(t)\}\ (= d(t)) = \begin{cases} d_0 + d_1 t & , 0 \le t \le t_1 \\ d_0 + d_1 t_1 (= d_{12}, say) & , t_1 \le t \le t_2 \\ d_{12} - d_2 (t - t_2) & , t_2 \le t \le T \\ d_0 & , T \le t \le H \end{cases} \qquad (3)$$

d_0 being the constant demand per unit quantity before and after the season

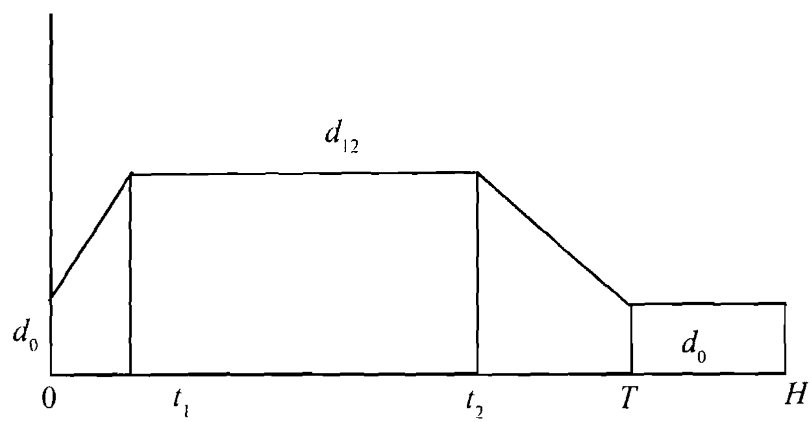

Fig. - 3

H = time period of business i.e., scheduling period
q(t) = inventory level at time, t
t_1 = the time upto which the replenishment increases.
t_2 = the time upto which the replenishment remains constant after t = t_1
T = seasonal time period i.e., the time upto which the replenishment decreases after t = t_2. At this point, replenishment attains the value which prevailed before the commencement of the season.
t_s = the time upto which inventory is built up (cf. Fig - 4)
Q_s, Q_1, Q_2, Q_T = inventory levels at time t = t_s, t_1, t_2, T respectively.

MATHEMATICAL ANALYSIS

Under the above assumptions, inventory is built up upto t = t_s after meeting the demands and wastage. After this period, there is no more replenishment and demand is met from the stock allowing the wastage due to stocking. This is continued till the inventory reduces to zero. Here t_s may have any value in between (0, H). Therefore, depending upon the position of t_s, there will be seven scenarios - 1, 2, 3, 4, 5, 6 and 7 due to (i) $0 < t_s < t_1$, (ii) $t_s = t_1$, (iii) $t_1 < t_s < t_2$, (iv) $t_s = t_2$, (v) $t_2 < t_s < T$, (iv) $t_s = T$ and (viii) $T < t_s < H$ respectively. The mathematical formulation of the model for the scenario - 1 is given in detail and for other scenarios, only differential equations and average profit expressions are presented.

Scenario - 1 : $0 < t_s < t_1$ (cf. Fig - 4) :

In this case, q(t) satisfies

$$\frac{dq}{dt} = \begin{cases} k_0 + k_1 t - \{(d_0 + d_1 t) + a_0 q\} & , 0 \le t \le t_s \\ -\{(d_0 + d_1 t) + a_1 q\} & , t_s \le t \le t_1 \\ - d_{12} - a_1 q & , t_1 \le t \le t_2 \\ - d_{12} - d_2 (t - t_2) - a_1 q & , t_2 \le t \le T \\ - d_0 - a_1 q & , T \le t \le H \end{cases} \tag{4}$$

with terminal conditions

$$q = \begin{cases} 0 , t = 0 \\ 0 , t = H \end{cases} \tag{5}$$

where t_s and H are dècision variables

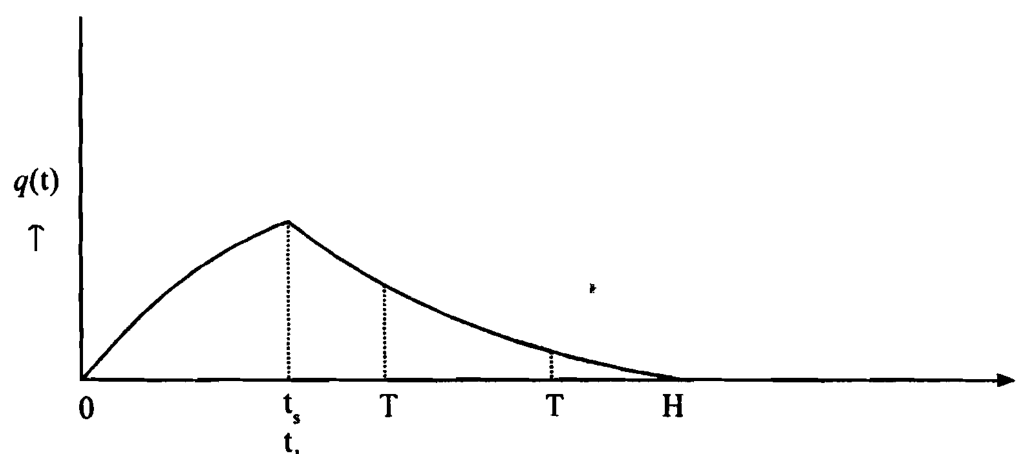

Fig. - 4

Due to the continuity of the replenishment, demand and price functions at t = T,

$$d_1 t_1 = d_2(T - t_2) \tag{6}$$

$$p_1 t_1 = p_2(T - t_2) \tag{7}$$

$$\text{and } k_1 t_1 = k_2(T - t_2) \tag{8}$$

Solving the above differential equation, we have,

$$q(t) = \frac{1}{a_0}[k_0 - d_0 + (k_1 - d_1)(t - \frac{1}{a_0})] + R_1 e^{-a_0 t}, \; 0 \le t \le t_s \tag{9}$$

$$= \frac{-1}{a_1}[d_0 + d_1(t - \frac{1}{a_1})] + R_2 e^{-a_1 t}, \; t_s \le t \le t_1 \tag{10}$$

$$= -\frac{d_{12}}{a_1} + R_3 e^{-a_1 t}, \; t_1 \le t \le t_2 \tag{11}$$

$$= -\frac{d_{12} + d_2 t_2}{a_1} + \frac{d_2}{a_1}(t - \frac{1}{a_1}) + R_4 e^{-a_1 t}, \; t_2 \le t \le T \tag{12}$$

$$= -\frac{d_0}{a_1} + R_5 e^{-a_1 t}, \; T \le t \le H \tag{13}$$

where

$$R_1 = \frac{1}{a_0}[\frac{k_1 - d_1}{a_0} - (k_0 - d_0)] \tag{14}$$

$$R_2 = [q_s + \frac{1}{a_1}(d_0 + d_1 t_s - \frac{d_1}{a_1})]e^{a_1 t_s} \tag{15}$$

$$R_3 = (q_1 + \frac{d_{12}}{a_1})\, e^{a_1 t_1} \tag{16}$$

$$R_4 = [q_2 + \frac{1}{a_1}(d_{12} + \frac{d_2}{a_1})]e^{a_1 t_2} \tag{17}$$

$$R_5 = (q_T + \frac{d_0}{a_1})e^{a_1 T} \tag{18}$$

q_s, q_1, q_2 and q_T being the levels of inventory at $t = t_s, t_1, t_2$ and T respectively and are obtained from (9), (10), (11) and (12) respectively. The terminal condition i.e., inventory at t = H vanishes gives an expression for H as

$$\frac{d_0}{a_1} e^{a_1 H} = R_5 \tag{19}$$

Therefore, total inventory,

$$TI\,(say) = \int_0^H q\,dt = TI_1 + TI_2 + TI_3 + TI_4 + TI_5 \tag{20}$$

where

$$TI_1 = \int_0^{t_s} q\,dt = \int_0^{t_s} \{\frac{1}{a_0}[(k_0 - d_0) + (k_1 - d_1)(t - \frac{1}{a_0})] + R_1 e^{-a_0 t}\}dt$$

$$= \frac{t_s}{a_0}[(k_0 - d_0) + (k_1 - d_1)(\frac{t_s}{2} - \frac{1}{a_0})] - \frac{R_1}{a_0}(e^{-a_0 t_s} - 1) \tag{21}$$

$$TI_2 = \int_{t_s}^{t_1} q\,dt = \frac{t_1 - t_s}{a_1}[-d_0 - \frac{d_1}{2}(t_1 + t_s) + \frac{d_1}{a_1}] - \frac{R_2}{a_1}[e^{-a_1 t_1} - e^{-a_1 t_s}] \tag{22}$$

$$TI_3 = \int_{t_1}^{t_2} q\,dt = -\frac{1}{a_1}[d_{12}(t_2 - t_1) + R_3(e^{-a_1 t_2} - e^{-a_1 t_1})] \tag{23}$$

$$TI_4 = \int_{t_2}^{T} q\,dt = \frac{T - t_2}{a_1}[-(d_{12} + d_2 t_2) + \frac{d_2}{2}(T + t_2) - \frac{d_2}{a_1}]$$

$$- \frac{R_4}{a_1}[e^{-a_1 T} - e^{-a_1 t_2}] \tag{24}$$

$$TI_5 = \int_T^H q\,dt = -\frac{1}{a_1}[d_0(H - T) + R_5\{e^{-a_1 H} - e^{-a_1 T}\}] \tag{25}$$

The purchase cost of the units is

$$PC = \int_0^{t_s} (k_0 + k_1 t)(p_0 - p_1 t)dt$$

$$= t_s[k_0 p_0 - (p_1 k_0 - k_1 p_0)\frac{t_s}{2} - p_1 k_1 \frac{t_s^2}{3}] \tag{26}$$

Revenue earned during the period (0, H) is

$$SP = S_1 + S_2 + S_3 + S_4 + S_5 \tag{27}$$

where, with the help of (2) and (3) ,

$$S_1 = \int_0^{t_s} F(t)dt$$

$$= m[d_0 p_0 t_s + (d_1 p_0 - d_0 p_1) \frac{t_s^2}{2} - d_1 p_1 \frac{t_s^3}{3}] \tag{28}$$

$$S_2 = \int_{t_s}^{t_1} F(t)dt = m[d_0 p_0(t_1 - t_s) + (d_1 p_0 - d_0 p_1) \frac{t_1^2 - t_s^2}{2} - d_1 p_1 \frac{t_1^3 - t_s^3}{2}] \tag{29}$$

$$S_3 = \int_{t_1}^{t_2} F(t)dt = md_{12}\, p_{12}(t_2 - t_1) \tag{30}$$

$$S_4 = \int_{t_2}^{T} F(t)dt$$

$$= m[(d_{12} + d_2 t_2)(p_{12} - p_2 t_2)(T - t_2)$$

$$+ \{p_2(d_{12} + d_2 t_2) - d_2(p_{12} - p_2 t_2)\} \frac{T^2 - t_2^2}{2} - d_2 p_2 \frac{T^3 - t_2^3}{3}] \tag{31}$$

$$S_5 = \int_T^H F(t)dt = md_0 p_0 (H - T) \tag{32}$$

where $F(t) = md(t)p(t)$.

$$\textit{Therefore, total average profit} = (SP - PC - C_3 - TI \times C_1) / H$$
$$= TAP(t_s, H) \tag{33}$$

Now, the problem is to maximize TAP with respect to t_s and H and to find the corresponding optimum values of the inventory parameters alongwith the optimum order quantity.

Scenario - 2 : $t_s = t_1$:
In this case, differential equation for q(t) will be the one replacing t_s by t_1 in scenario - 1. For all other expression of scenario - 2, q_s and t_s are changed by q_1 and t_1 respectively in the corresponding expression of scenario - 1. The corresponding TAP function in which only H is unknown, is maximized with respect to H.

Scenario - 3 : $t_1 < t_s < t_2$

The differential equation for this case is

$$\frac{dq}{dt} = \begin{cases} (k_0 + k_1 t) - \{(d_0 + d_1 t) + a_0 q\} & , 0 \leq t \leq t_1 \\ k_{12} - \{d_{12} + a_0 q\} & , t_1 \leq t \leq t_s \\ - d_{12} - a_1 q & , t_s \leq t \leq t_2 \\ - [d_{12} - d_2 (t - t_2) + a_1 q] & , t_2 \leq t \leq T \\ - (d_0 + a_1 q) & , T \leq t \leq H \end{cases}$$

with the boundary and continuity conditions, (5) - (8). Proceeding as before, the inventory q(t) are given by (9) - (13) as in scenario - 1 with the following changes :
In (9), t_s is replaced by t_1, the equation (10) does not exist and in its place,

$$q(t) = \frac{1}{a_0}[(k_0 - d_0) + (k_1 - d_1)t_1] + R_2 e^{-a_0 t}, \; t_1 \leq t \leq t_s \tag{34}$$

and in (11), the time range is $t_s \leq t \leq t_2$.
The changes in constants are :
in (15), R_2 is expressed as $R_2 = [q_1 - \frac{1}{a_0}\{(k_0 - d_0) + (k_1 - d_1)t_1\}]e^{a_0 t_1}$;
in R_3 (i.e., 16), q_1 and t_1 are replaced by q_s and t_s respectively.
The corresponding changes in the total inventories given by (20) to (25) are :
in TI_1(i.e., 21), t_1 replaces t_s ,
in (22), TI_2 is replaced by,

$$TI_2 = \frac{1}{a_0} \{(k_0 - d_0) + (k_1 - d_1)t_1\}(t_s - t_1) - \frac{R_2}{a_0}\{e^{-a_0 t_s} - e^{-a_0 t_1}\}, \tag{35}$$

in TI_3 (i.e., 23), t_1 is replaced by t_s.
Total purchased cost is

$$PC = t_1\{k_0 p_0 - (p_1 k_0 - k_1 p_0)\frac{t_1}{2} - p_1 k_1 \frac{t_1^3}{3}\} + k_{12} p_{12} (t_s - t_1) \tag{36}$$

The selling price given by (27) - (32) remain the same with the following changes :
in (28), t_s is replaced by t_1 , in (29),

$$S_2 = m d_{12} p_{12} (t_s - t_1) \tag{37}$$

and in S_3 (i.e., 30), t_1 is replaced by t_s.
Now, with these changed expressions, TAP function is evaluated and maximized with respect to t_s and H.

Scenario - 4 : $t_s = t_2$

The differential equations and all other expressions for this case are obtained from scenario - 3 by replacing t_s by t_2 and q_s by q_2. The corresponding TAP function is evaluated and is maximized with respect to H only.

Scenario - 5 : $t_2 < t_s < T$

The differential equations for this scenario is

$$\frac{dq}{dt} = \begin{cases} (k_0 + k_1 t) - \{(d_0 + d_1 t) + a_0 q\} & , 0 \le t \le t_1 \\ k_{12} - \{d_{12} + a_0 q\} & , t_1 \le t \le t_2 \\ [k_{12} - k_2(t - t_2)] - [\{d_{12} - d_2(t - t_2)\} + a_0 q] & , t_2 \le t \le t_s \\ - [\{d_{12} - d_2 (t - t_2)\} + a_1 q] & , t_s \le t \le T \\ - (d_0 + a_1 q) & , T \le t \le H \end{cases}$$

with the conditions (5) - (8). Proceeding as before, the expression (9) - (18), (20) - (25) and (27) - (32) are evaluated and these are the same as in scenario - 3 with the following changes :

In (34), time range is $t_1 \le t \le t_2$; (11) is replaced by

$$q(t) = \frac{1}{a_0} \{(k_0 - d_0) + (k_1 - d_1)t_1 + (k_2 - d_2)t_2\} - \frac{k_2 - d_2}{a_0}(t - \frac{1}{a_0}) + R_3 e^{-a_0 t}, \; t_2 \le t \le t_s \tag{38}$$

and in (12), time duration is $t_s \le t \le T$; in (16), R_3 is replaced by,

$$R_3 = [q_2 - \frac{1}{a_0} \{(k_0 - d_0) + (k_1 - d_1)t_1 + (k_2 - d_2)t_2\} + \frac{k_2 - d_2}{a_0}(t_2 - \frac{1}{a_0})]e^{a_0 t_2} \tag{39}$$

and in (17), R_4 is expressed as,

$$R_4 = [q_s + \frac{1}{a_1}(d_{12} + d_2(t_2 - t_s + \frac{1}{a_1}))]e^{a_1 t_s} , \tag{40}$$

in TI_2(i.e., 35) and S_2(i.e., 37), t_2 takes the place of t_s.
in (23), TI_3 is replaced by,

$$TI_3 = \frac{1}{a_0}\{(k_0 - d_0) + (k_1 - d_1)t_1 + (k_2 - d_2)t_2\}(t_s - t_2)$$
$$- \frac{k_2 - d_2}{2a_0}(t_s^2 - t_2^2) + \frac{k_2 - d_2}{a_0^2}(t_s - t_2) - \frac{R_3}{a_0}(e^{-a_0 t_s} - e^{a_0 t_2}) \qquad (41)$$

in (30), S_3 is replaced by,

$$S_3 = m[(d_{12} + d_2 t_2)(p_{12} - p_2 t_2)(t_s - t_2)$$
$$+\{p_2(d_{12} + d_2 t_2) - d_2(p_{12} - p_1 t_2)\}\frac{t_s^2 - t_2^2}{2} - d_2 p_2 \frac{t_s^3 - t_2^3}{3}] \qquad (42)$$

and in TI_4 (i.e., 24) and S_4 (i.e., 31), t_2 is changed to t_s;
The total purchased cost

$$PC = k_0 p_0 t_1 - (p_1 k_0 - k_1 p_0)\frac{t_1^2}{2} - p_1 k_1 \frac{t_1^3}{3} + k_{12} p_{12}(t_2 - t_1) + (k_{12} + k_2 t_2)(p_{12} - p_2 t_2)(t_s - t_2) + \{p_2(k_{12} + k_2 t_2) - k_2(p_{12} - p_2 t_2)\}\frac{t_s^2 - t_2^2}{2} - k_1 p_2 \frac{t_s^3 - t_2^3}{3}$$

With these expressions, TAP is calculated and as before, is optimized with regards to t_s and H.

Scenario - 6 : $t_s = T$:
For this scenario, q_s and t_s in the equations and expressions of scenario - 5 are replaced by q_T and T respectively. TAP is now obtained and maximized for H only.

Scenario - 7 : $T < t_s < H$:
The differential equations for this scenario is :

$$\frac{dq}{dt} = \begin{cases} k_{12} - \{(d_0 + d_1 t) + a_0 q\} & , 0 \le t \le t_1 \\ k_{12} - \{d_{12} + a_0 q\} & , t_1 \le t \le t_2 \\ [k_{12} - k_2(t - t_2)] - [\{d_{12} - d_2(t - t_2)\} + a_0 q] & , t_2 \le t \le T \\ k_0 - (d_0 + a_0 q) & , T \le t \le t_s \\ -(d_0 + a_1 q) & , t_s \le t \le H \end{cases}$$

with the same continuity and boundary conditions (5) - (8). Proceeding as in other scenarios, the expressions and equations of this scenario are the same as in scenario - 5, except the following changes :
in (38), time range is $t_2 \leq t \leq T$,

$$q(t) = \frac{k_0 - d_0}{a_0} + R_4 e^{-a_0 t}, \quad T \leq t \leq t_s \tag{43}$$

and in (13), time duration is $t_s \leq t \leq H$; in (39), R_4 is replaced by,

$$R_4 = [q_T - \frac{k_0 - d_0}{a_0}] e^{a_0 T} \tag{44}$$

and in R_5 (i.e., 18), q_T and T are replaced by q_s and t_s respectively ; in TI_3 (i.e., 40) and S_3 (i.e., 41), T takes the place of t_s and in TI_5 (i.e., 25) and S_5 (i.e., 32), T is changed to t_s , whereas

$$TI_4 = (k_0 - d_0)\frac{t_s - T}{a_0} - \frac{R_4}{a_0}(e^{-a_0 t_s} - e^{-a_0 T}) \text{ and } S_4 = md_0 p_0(t_s - T).$$

The total purchase cost is

$$PC = \{k_0 p_0 t_1 - (p_1 k_0 - k_1 p_0)\frac{t_1^2}{2} - p_1 k_1 \frac{t_1^3}{3}\} + k_{12} p_{12}(t_2 - t_1) + (k_{12} + k_2 t_2)(p_{12} - p_2 t_2)(T - t_2) + \{p_2(k_{12} + k_2 t_2) - k_2(p_{12} - p_2 t_2)\}\frac{T^2 - t_2^2}{2} - k_2 p_2 \frac{T^3 - t_2^3}{3} + k_0 p_0 (t_s - T)$$

With these expressions, TAP function is obtained and, as in other cases, maximized with respected to t_s and H.

GENETIC ALGORITHM

Now, the objective function TAP (t_s, H) is optimized using the genetic algorithm (GA). The development of GA for the present problem is as follows :

Representation :

We use binary string as a chromosome to represent real values of the variable t_s. The floating point number is represented in a standard way by a sequence of bits $< b_j b_{j-1}$ ...... $b_0 >$. For the present experiment, we have used j = 19. The mapping from a binary string $< b_j b_{j-1}$ .... $b_0 >$ into a real number t_s from the range [l..u] is straight forward and is completed in two steps :
a) Convert the binary string $< b_j b_{j-1} ...... b_0 >$ from base 2 to base 10 :

$$(< b_j b_{j-1} ...... b_0 >)_2 = (\sum_{i=0}^{j} b_i \times 2^i)_{10} = x'$$

b) Find a corresponding real number t_s such that

$$t_s = 1 + x^1 \times \frac{u - l}{2^{l+1} - 1}$$

Initialization :

The initialization process adopted here is very simple. We create a population of chromosome where each chromosome is a binary vector of 20 bits. All 20 bits for each chromosome are initialized randomly, i.e., the random number generator generates random bits (0 or 1). However, in the cases where the ratio between sizes of the feasible part of the search space and the whole search space is too low, initialization process can incorporate some additional heuristic (based on the problem's specific knowledge) to increase the number of initial feasible individuals. In the scenario - 1, to increase feasibility with respect to q_T, initially, in the first four places, there are 1 bits and the remaining bits are generated randomly. In all other scenarios, all twenty bits are generated randomly.

Evaluation function :

Each individual in the population represents a potential solution to the problem. Evaluation function is responsible for rating these potential solutions by assigning a real as a measure of their fitness. Evaluation function, **evel,** for binary string v is evaluated as follows :

i) a binary chromosome v is converted into a real value t_s

ii) H is computed by (19) as discussed in section - 3.

iii) evaluation function, **eval** for binary vector v is eval (v) = TAP as discussed in section - 3.

Selection :

In our experiments, we have used a roulette wheel selection for selecting individuals for reproduction. We construct such a roulette wheel as follows :

a) Calculate the fitness value eval (v_i) for each chromosome v_i (i = 1, 2, ... popsize).

b) Find the total fitness of the population

$$F = \sum_{i=1}^{popsize} eval\,(v_i).$$

c) Calculate the probability of a selection, p_i = eval (v_i)/F

d) Calculate a cumulative probability q_i for each chromosome v_i (i = 1, 2, ... popsize) :

$$q_i = \sum_{j=1}^{i} p_j$$

The selection process is based on spining the roulette wheel popsize times ; each time we select a single chromosome for a new population in the following way :
i) Generate a random (float) number r from the range [0..1].
ii) If $r < q_1$ then select the first chromosome (v_1), otherwise select the i-th chromosome $v_i (2 \le i \le popsize)$ such that $q_{i-1} < r < q_i$

Crossover :

The chromosomes, which survive the selection step, undergo genetic operation - crossover. Crossover exchange genetic material by selecting a random cutting point and creates two offsprings. In our experiments, we have used the classical 1 - point crossover. The probability of crossover $p_c = 0.42$ and 0.2, so that we expect that (on average) 42 and 20 out of 100 chromosomes undergo crossover. For this, we proceed in the following way: For each chromosome selected in selection process, we generate a random number r from the range[0..1] ; if $r < p_c$, we select a given chromosome for crossover. If the number of selected chromosomes is even, then we pair them easily and if the number of chromosomes were odd, we remove one selected chromosome. Now we mate selected chromosomes sequentially i.e. first two pairs and then the next two pairs and so on. In this way, the chromosomes are coupled together. For each of these two pairs, we generate a random integer number **pos** from the range [1...20] (20 is the total length-number of bits in a chromosome). The number pos indicates the position of the crossing point.

Mutation :

The mutation is performed on a bit-by-bit basis. The probability of mutation is $p_m = 0.05$. The number of chromosomes, **nmutations**, on which the mutation will be performed is calculated by
nmutation = (int) ((double)(POPSIZE)*NVARS)*PMUTATION,
where NVARS is the number of variables and is equal to 2, PMUTATION is the probability of mutation. The chromosome on which the mutation will be performed, will be selected randomly generating a random number from the range [1...POPSIZE]. After that we generate a random integer number point from the range [1...20]. The number point indicates the position of the mutation point. In that position, if there is a bit 1, that will be changed to bit 0 and vice versa.

Termination :

All these processes are iterativly repeated till the total improvement of the last 10 best solutions is less than 0.01.

Numerical Illustration :

The inventory model as developed in section 2 has been illustrated for the following values :
$k_0 = 250, k_1 = 50, d_0 = 140, d_1 = 40, p_0 = 10, p_1 = 1.5, a_0 = 0.05, a_1 = 0.10, t_1 = 4.2, t_2 = 4.5,$
$T = 6, c_1 = \$0.1, c_3 = \$88, m = 1.2$

With these parametric values, the objective function, TAP for each scenario has been optimized by the GA algorithm presented earlier and the results are :

Scenarious	t_s	*h*	q_s	*No. of Optimal Generation*	*TAP*
1	4.19999981	6.207857	499.051619	508	107.8176
2	4.2	6.207857	499.051619	-	107.8176
3	4.5	7.037215	536.881435	1335	216.332236
4	4.5	7.037215	536.881435	-	216.332236
5	5.315049	8.88570	627.663848	486	313.2720
6	6	9.990655	687.021009	-	284.225410
7	6.00001144	9.992670	687.021875	847	284.224618

6 Discussion

The above table indicate that the maximum profit comes from the scenario - 5 i,e., when $t_2 < t_s < T$. Hence,a retailer will go for stock upto the time , t=5.315049. He should stop the procurment just little before T i.e., when the replenishment comes back to its constant value. We observe that the optimum result for the scenario - 5 do not change for different values of GA parameters. Hence, it may be concluded that the prasent derivation of G A is Quite Stable.

7 Declution

For the first time,a realistic model for one-time business of deteriorating seasonal product is devaloped keeping the pravailing,praserving and markiting condition in the devaloping countries like India,Napel,Bhutan etc. in mind,For the present problem,a heuristic ralation has been drawn up and based on that GA have devaloped to solve the present problem. Till now, none has concidered such Ga Derivation for Inventory problems. Normally , GA gives the globel optimum and hence use of it is very convinient for the solution of decision making problems in inventory system. Though we have derived and applied GA for a paricular inventory problem, the present devalopment of GA is quite generaland following this, GA can be successfully derived and applied for other inventory problem with price discount, inflation, two warehouses, etc.formulated in different enviroments like crips, fuzzy and stochastic enviroments.

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DISTANCES BETWEEN INTUITIONISTIC FUZZY MATRICES

Amiya K. Shyamal and Madhumangal Pal*
Department of Applied Mathematics with Oceanology and Computer Programming
Vidyasagar University, Midnapore - 721102, India.

Abstract :
Two basic distances, Hamming distance and Euclidean distance, between intuitionistic fuzzy matrices (IFMs) are introduced here. Idempotent, symmetric etc. IFMs are defined in term of distances. A number of properties are also presented on these distances.

INTRODUCTION

Szmidt and Kacprzyk [6] proposed some new definitions of distances between intuitionistic fuzzy sets. We also defined some terms of distances (four distances-Hamming distance, normalised Hamming distance, Euclidean distance, normalised Euclidean distance) between interval-valued intuitionistic fuzzy sets [5]. For the first time Pal [4] introduced intuitionistic fuzzy determinant. Khan and Pal defined intuitionistic fuzzy matrix (IFM) and presented a number of properties on it [1, 3]. Intuitionistic fuzzy tautological matrices are defined in [2].

In this paper we define distances between two IFMs and propose some relations among distances between IFMs. Here we also introduce some other distances M_1, M_2, M_3, M_4, M, M^* between IFMs. Either two IFMs are equal, nearly equal or different from each other we can easily decided by measuring the distance.

DEFINITIONS

An intuitionistic fuzzy matrix A of order $m \times n$ is defined as
$A = [a_{ij}]_{m\times n} = [< a_{ij\mu}, a_{ij\nu} >]_{m\times n}$ where $a_{ij\mu}$ and $a_{ij\nu}$ are called membership and non-membership values of a_{ij} in A, which maintaining the condition $0 \leq a_{ij\mu} + a_{ij\nu} \leq 1$.

For simplicity we write $A = [a_{ij}]$ or $[< a_{ij\mu}, a_{ij\nu} >]$. All the elements of an IFM are the members of $< F >$, where $< F > = \{< a, b > : 0 \leq a + b \leq 1, a, b \in [0, 1]\}$.

Now we define some operations for intuitionistic fuzzy matrices (IFMs) $A = [a_{ij}]$ and $B = [b_{ij}]$ of order $m \times n$ as follows :

*e-mail : mmpal@vidyasagar.ac.in

(i) $A \vee B = [< \max\{a_{ij\mu}, b_{ij\mu}\}, \min\{a_{ij\nu}, b_{ij\nu}\}>]$.

(ii) $A \wedge B = [< \min\{a_{ij\mu}, b_{ij\mu}\}, \max\{a_{ij\nu}, b_{ij\nu}\}>]$.

(iii) $A \oplus B = [< a_{ij\mu} + b_{ij\mu} - a_{ij\mu}.b_{ij\mu}, a_{ij\nu}.b_{ij\nu} >]$.

(iv) $A \odot B = [< a_{ij\mu}.b_{ij\mu}, a_{ij\nu} + b_{ij\nu} - a_{ij\nu}.b_{ij\nu} >]$.

[The operators '+', '–' and '.' are ordinary addition, subtraction and multiplication.]

(v) $[k + 1]A = [k]A \oplus A$ and $[1]A = A, k = 1, 2, \ldots$

(vi) $A^{[k+1]} = A^{[k]} \odot A$ and $A^{[1]} = A, k = 1, 2, \ldots$

(vii) $A^c = [a^c_{ij}] = [< a_{ij\nu}, a_{ij\mu}>]$, complement of A.

(viii) $A' = [a_{ji}] = [< a_{ji\mu}, a_{ji\nu}>]$, transpose of A.

(ix) $\Box A = [< a_{ij\mu}, 1 - a_{ij\mu}>]$.

(x) $\Diamond A = [< 1 - a_{ij\nu}, a_{ij\nu}>]$.

(xi) $A@B = \left[< \dfrac{a_{ij\mu} + b_{ij\mu}}{2}, \dfrac{a_{ij\nu} + b_{ij\nu}}{2} > \right]$.

(xii) A is idempotent if $A^2 = A$.

For an IFM A of order $n \times n$ the following terms are defined.

(xiii) A is symmetric iff $A' = A$.

(xiv) A is reflexive iff $a_{ii} = < 1, 0 >$ for all i.

(xv) A is irreflexive iff $a_{ii} = < 0, 1 >$ for all i.

(xvi) A is diagonal if $a_{ij} = < 0, 1 >$ for all $i \neq j$.

(xvii) A is identity if $a_{ij} = < 0, 1 >$ for all $i \neq j$ and $a_{ii} = < 1, 0 >$. An identity IFM of order $n \times n$ is denoted by I_n.

We introducing two functions $\mu(A)$ and $\nu(A)$ for an IFM A. The function $\mu(A) = [a_{ij\mu}]$ is a fuzzy matrix whose entries are the membership values of A and $\nu(A) = [a_{ij\nu}]$ is also a fuzzy matrix whose entries are non-membership values of A.

DISTANCES BETWEEN IFMs

In this section we define four basic distances and other distances between IFMs. The distance δ between two IFMs A and B is a mapping from the set of IFMs (M) to the set of real numbers (R), i.e., $\delta : M \times M \to R \times R$.

1. Hamming distance

The Hamming distance between two IFMs A and B of order $m \times n$ is

$$H(A, B) = \Big(\sum_{i=1}^{m}\sum_{j=1}^{n} | a_{ij\mu} - b_{ij\mu} |, \sum_{i=1}^{m}\sum_{j=1}^{n} | a_{ij\nu} - b_{ij\nu} | \Big) = (h_{\mu}(A, B), h_{\nu}(A, B)),$$

where $h_{\mu}(A, B) = \sum_{i=1}^{m}\sum_{j=1}^{n} | a_{ij\mu} - b_{ij\mu} |$ and $h_{\nu}(A, B) = \sum_{i=1}^{m}\sum_{j=1}^{n} | a_{ij\nu} - b_{ij\nu} |$.

It is obvious that $0 \le h_{\mu}(A, B) \le mn$ and $0 \le h_{\nu}(A, B) \le mn$.

If

$$\left.\begin{array}{c} a_{ij\mu} = 1, a_{ij\nu} = 0 \,;\, b_{ij\mu} = 0, b_{ij\nu} = 1 \\ \text{or} \\ a_{ij\mu} = 0, a_{ij\nu} = 1 \,;\, b_{ij\mu} = 0, b_{ij\nu} = 0 \end{array}\right] \qquad (1)$$

then $h_{\mu}(A, B) = mn = h_{\nu}(A, B)$.

If $a_{ij\mu} = b_{ij\mu}$ and $a_{ij\nu} = b_{ij\nu}$ (2)

then $h_{\mu}(A, B) = 0 = h_{\nu}(A, B)$.

The hamming distance $H : M \times M \to R \times R$ satisfies the following conditions :

(i) $H(A, B) \ge (0,0)$ for all $A, B \in M$,

(ii) $H(A, B) = (0,0)$ iff $A = B$ for all $A, B \in M$,

(iii) $H(A, B) = H(B, A)$ for all $A, B \in M$ (symmetry),

(iv) $H(A, B) \le H(A, C) + H(C, B)$ for all $A, B, C \in M$ (triangular inequality).

The distance H is also called a metric on M.

2. Normalised Hamming distance

The normalised Hamming distance is denoted by $H_N(A, B)$ and is defined as

$$H_N(A, B) = \Big(\frac{h_{\mu}(A, B)}{mn}, \frac{h_{\nu}(A, B)}{mn} \Big) = \big(h^{*}_{\mu}(A, B), h^{*}_{\nu}(A, B) \big),$$

where $h^{*}_{\mu}(A, B) = \dfrac{h_{\mu}(A, B)}{mn}$ and $h^{*}_{\nu}(A, B) = \dfrac{h_{\nu}(A, B)}{mn}$.

It may be noted that $0 \le h^{*}_{\mu}(A, B) \le 1$ and $0 \le h^{*}_{\nu}(A, B) \le 1$.

3. Euclidean distance

The Euclidean distance between two IFMs A and B of order $m \times n$ is defined below :

$$E(A, B) = \left(\left\{ \sum_{i=1}^{m} \sum_{j=1}^{n} (a_{ij\mu} - b_{ij\mu})^2 \right\}^{\frac{1}{2}}, \left\{ \sum_{i=1}^{m} \sum_{j=1}^{n} (a_{ij\nu} - b_{ij\nu})^2 \right\}^{\frac{1}{2}} \right) = (e_\mu(A, B), e_\nu(A, B)),$$

where $e_\mu(A, B) = \left\{ \sum_{i=1}^{m} \sum_{j=1}^{n} (a_{ij\mu} - b_{ij\mu})^2 \right\}^{\frac{1}{2}}$ and $e_\nu(A, B) = \left\{ \sum_{i=1}^{m} \sum_{j=1}^{n} (a_{ij\nu} - b_{ij\nu})^2 \right\}^{\frac{1}{2}}$.

Here also $0 \le e_\mu(A, B) \le mn$ and $0 \le e_\nu(A, B) \le mn$. The sign of equality holds when (1) and (2) are satisfied.

The Euclidean distance E is also a metric on M.

4. Normalised Euclidean distance

The normalised Euclidean distance between two IFMs A and B is obtained by dividing both $e_\mu(A, B)$ and $e_\nu(A, B)$ by mn. That is $E_N(A, B) = (e^*_\mu(A, B), e^*_\nu(A, B))$, where

$$e^*_\mu(A, B) = \frac{e_\mu(A, B)}{mn} \text{ and } e^*_\nu(A, B) = \frac{e_\nu(A, B)}{mn}$$

It is easy to prove that $0 \le e^*_\mu(A, B) \le 1$ and $0 \le e^*_\nu(A, B) \le 1$.

Depending on the values of h_μ and h_ν we can define the following terms.

(i) If $h_\mu(A, B) = h_\nu(A, B) = mn$ or $e_\mu(A, B) = e_\nu(A, B) = mn$ then A and B are totally opposite.

(ii) If $h_\mu(A, B) = h_\nu(A, B) = \sqrt{mn}$ or $e_\mu(A, B) = e_\nu(A, B) = \sqrt{mn}$ then A and B are nearly equal.

(iii) If $h_\mu(A, B) = h_\nu(A, B) = (mn)^{1/4}$ or $e_\mu(A, B) = e_\nu(A, B) = (mn)^{1/4}$ then A and B are almost equal.

(iv) If $h_\mu(A, B) = h_\nu(A, B) = 0$ or $e_\mu(A, B) = e_\nu(A, B) = 0$ then A and B are exactly equal.

5. Other distances

The other distance between two IFMs $A = [\langle a_{ij\mu}, a_{ij\nu} \rangle]$ and $B = [\langle b_{ij\mu}, b_{ij\nu} \rangle]$ of same order are defined as follows.

$$M_1(A, B) = \max_{\substack{a_{ij\mu} \in \mu(A) \\ b_{ij\mu} \in \mu(B)}} |a_{ij\mu} - b_{ij\mu}|.$$

$$M_2(A, B) = \max_{\substack{a_{ij\nu} \in \nu(A) \\ b_{ij\nu} \in \nu(B)}} |a_{ij\nu} - b_{ij\nu}|.$$

$$M_3(A, B) = \min_{\substack{a_{ij\mu} \in \mu(A) \\ b_{ij\mu} \in \mu(B)}} |a_{ij\mu} - b_{ij\mu}|.$$

$$M_4(A, B) = \min_{\substack{a_{ij\nu} \in \nu(A) \\ b_{ij\nu} \in \nu(B)}} |a_{ij\nu} - b_{ij\nu}|.$$

$$M(A, B) = \frac{1}{2}\{M_1(A, B) + M_2(A, B)\}.$$

$$M^*(A, B) = \frac{1}{2} \max_{\substack{a_{ij} \in A \\ b_{ij} \in B}} \{|a_{ij\mu} - b_{ij\mu}| + |a_{ij\nu} - b_{ij\nu}|\}.$$

It is clear that the values of M_1, M_2, M_3, M_4, M and M^* are lies between 0 and 1. For any two IFMs A and B we can easily came to the following conclusions.

If $M_1(A, B)$ and $M_2(A, B)$ are both zero then A and B are exactly equal. When both the values of $M_1(A, B)$ and $M_2(A, B)$ are very small i.e., tends to zero, then A and B are nearly equal. Similarly, when both of $M_3(A, B)$ and $M_4(A, B)$ are large i.e., tends to 1 then A and B are totally opposite.

In the following some common terms are defined with help of distances.

Lemma 1 *Two IFMs A and B are exactly equal iff* $H(A, B) = (0,0)$ *or* $E(A, B) = (0,0)$.

Lemma 2 *For any IFM A, if* $H(A, A^2) = (0,0)$ *or,* $E(A, A^2) = (0,0)$ *then A is an idempotent IFM.*

Lemma 3 *A and B be two IFMs of same order such that* $B = A'$. *If* $H(A, B) = (0,0)$ *or,* $E(A,B) = (0,0)$ *then A is a symmetric IFM.*

Lemma 4 *If A is a symmetric IFM then (i)* $H(A, A \wedge A') = (0,0)$ *and (ii)* $H(A, A \vee A') = (0,0)$.

Here we define some relational operators for IFMs A, B, C, D.

(i) If $h_\mu(A, B) \leq h_\mu(C, D)$ and $h_\nu(A, B) \leq h_\nu(C, D)$ then $H(A, B) \leq H(C, D)$.

(ii) If $h_\mu(A, B) \leq h_\mu(C, D)$ and $h_\nu(A, B) \geq h_\nu(C, D)$ then $H(A, B) \preceq H(C, D)$.

(iii) $H(A, B) = H(C, D)$ iff $h_\mu(A, B) = h_\mu(C, D)$ and $h_\nu(A, B) = h_\nu(C, D)$.

$H(A, B) \leq H(C, D)$ implies $H(C, D) \geq H(A, B)$ and similarly, $H(A, B) \preceq H(C, D)$ implies $H(C, D) \succeq H(A, B)$

The relational operators $\leq$, $\geq$, $\succeq$, $\preceq$ are also similar for Euclidean distances.

Property 1 *For any IFM A,*

(i) $H(A, A^{[2]}) \succeq H(A, A^{[k]})$, $k = 2, 3, \ldots$

(ii) $H(A, [2]A) \leq H(A, [k]A)$, $k = 2, 3, ...$
(iii) $E(A, A^{[2]}) \leq E(A, A^{[k]})$, $k = 2, 3, ...$
(iv) $E(A, [2]A) \leq E(A, [k]A)$, $k = 2, 3, ...$
(v) $H(A, A^{[2]}) = H(A, [2]A)$
(vi) $E(A, A^{[2]}) = E(A, [2]A)$

Proof. (i) Let $A = [< a_{ij\mu}, a_{ij\nu} >]$.
Then $A^{[2]} = A \odot A = [< a^2_{ij\mu}, 2a_{ij\nu} - a^2_{ij\nu} >] = [< a^2_{ij\mu}, 1 - (1 - a_{ij\nu})^2 >]$.

For any positive integer k, $A^{[k]} = [< a^k_{ij\mu}, 1 - (1 - a_{ij\nu})^k >]$.

Since $0 \leq a_{ij\mu} \leq 1$ and $0 \leq a_{ij\nu} \leq 1$ then, $a^2_{ij\mu} \geq a^3_{ij\mu} \geq ... \geq a^k_{ij\mu}$, for any positive integer k and $(1 - a_{ij\nu})^2 \geq (1 - a_{ij\nu})^3 \geq .... \geq (1 - a_{ij\nu})^k$
or, $1 - (1 - a_{ij\nu})^2 \leq 1 - (1 - a_{ij\nu})^3 \leq .... \leq 1 - (1 - a_{ij\nu})^k$.
Hence, $H(A, A^{[2]}) \leq H(A, A^{[k]})$.
Proofs of (ii), (iii), (iv) are similar to (i).

(v) $H(A, A^{[2]}) = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} | a_{ij\mu} - a^2_{ij\mu} |, \sum_{i=1}^{n} \sum_{j=1}^{n} | a_{ij\nu} - 1 + (1 - a_{ij\nu})^2 | \right)$ and

$$H(A, [2]A) = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} | a_{ij\mu} - 1 + (1 - a_{ij\mu})^2 |, \sum_{i=1}^{n} \sum_{j=1}^{n} | a_{ij\nu} - a^2_{ij\nu} | \right).$$

We have $(a - 1) + (1 - a)^2 = a(a - 1)$ and $a - a^2 = a(1 - a)$.

Therefore, $|a_{ij\mu} - a^2_{ij\mu}| = |a_{ij\mu} - 1 + (1 - a_{ij\mu})^2|$.

Hence, $H(A, A^{[2]}) = H(A, [2]A)$
(vi) Proof is similar to (v).

Property 2 *For any IFM* $A = [< a_{ij\mu}, a_{ij\nu} >]$,
(i) $H(A, \Box A) = H(A^c, (\Diamond A)^c)$
(ii) $H(A, \Diamond A) = H(A^c, (\Box A)^c)$
(iii) $H(\Box A, \Diamond A) = H((\Box A)^c, (\Diamond A)^c)$
(iv) $H(\Box A, (\Box A)^c) = H(\Diamond A, \Diamond A^c)$
(v) $H(\Box A, (\Box A)^c) = H(A, A^c)$
(vi) $H((\Box A)^c, \Diamond A^c) = (0, 0)$
(vii) $H((\Diamond A)^c, \Box A^c) = (0, 0)$
(viii) $E(A, \Box A) = E(A^c, (\Diamond A)^c)$
(ix) $E(A, \Diamond A) = E(A^c, (\Box A)^c)$
(x) $E(\Box A, \Diamond A) = E((\Box A)^c, (\Diamond A)^c)$

(xi) $E(\Box A, (\Box A)^c) = E(\Diamond A, \Diamond A^c)$
(xii) $E(\Box A, (\Box A)^c) = E(A, A^c)$
(xiii) $E((\Box A)^c, \Diamond A^c) = (0, 0)$
(xiv) $E((\Diamond A)^c, \Box A^c) = (0, 0)$.

Proof. (i) Here $A = [< a_{ij\mu}, a_{ij\nu} >]$ then we have $\Box A = [< a_{ij\mu}, 1 - a_{ij\mu} >]$
Therefore, $H(A, \Box A) = (\sum\sum | a_{ij\mu} - a_{ij\mu}|, \sum\sum | a_{ij\nu} - 1 + a_{ij\mu}|)$
$= (0, \sum\sum|1 - (a_{ij\mu} + a_{ij\nu})|)$
Again, $H(A^c, (\Diamond A)^c) = (\sum\sum | a_{ij\nu} - a_{ij\nu}|, \sum\sum | a_{ij\mu} - 1 + a_{ij\nu}|)$
$= (0, \sum\sum|1 - (a_{ij\mu} + a_{ij\nu})|)$.
Hence, $H(A, \Box A) = H(A^c, (\Diamond A)^c)$.

Property 3 *For any IFM* $A = [< a_{ij\mu}, a_{ij\nu} >]$,
(i) $H(A, \Box A) \leq H(\Box A, \Diamond A)$
(ii) $H(A, \Diamond A) \leq H(\Box A, \Diamond A)$
(iii) $H(A, \Box A) \preceq H(A, \Diamond A)$
(iv) $E(A, \Diamond A) \leq E(\Box A, \Diamond A)$
(v) $E(A, \Box A) \leq E(\Box A, \Diamond A)$
(vi) $E(A, \Box A) \preceq E(A, \Diamond A)$

Proof. (iii) We have, $H(A, \Box A) = (\sum\sum | a_{ij\mu} - a_{ij\mu}|, \sum\sum | a_{ij\nu} - 1 + a_{ij\mu}|)$
$= (0, \sum\sum|1 - (a_{ij\mu} + a_{ij\nu})|)$
Again, $H(A, \Diamond A) = (\sum\sum | a_{ij\mu} - 1 + a_{ij\nu}|, \sum\sum | a_{ij\nu} - a_{ij\nu}|)$
$= (\sum\sum|1 - (a_{ij\mu} + a_{ij\nu})|, 0)$
Therefore, $H(A, \Box A) \preceq H(A, \Diamond A)$.

(v) We have, $E(A, \Box A) = \left(\left\{\sum\sum (a_{ij\mu} - a_{ij\mu})^2\right\}^{\frac{1}{2}}, \left\{\sum\sum (a_{ij\mu} - 1 + a_{ij\nu})^2\right\}^{\frac{1}{2}}\right.$
$= \left(0, \left\{\sum\sum (1 - a_{ij\mu} - a_{ij\nu})^2\right\}^{\frac{1}{2}}\right).$
Now, $E(\Box A, \Diamond A) = \left(\left\{\sum\sum (1 - a_{ij\mu} - a_{ij\nu})^2\right\}^{\frac{1}{2}}, \left\{\sum\sum (1 - a_{ij\mu} - a_{ij\nu})^2\right\}^{\frac{1}{2}}\right).$
Hence, $E(A, \Box A) \leq E(\Box A, \Diamond A)$.

Property 4 *Let* I_n *be an identity IFM of order* $n \times n$ *and* $A = [< a_{ij\mu}, a_{ij\nu} >]$ *be any IFM of the same order then,*
(i) $\Box I_n$ *and* $\Diamond I_n$ *be an identity IFM.*

(ii) I_n^c *are irreflexive IFMs.*
(iii) $I_n \vee A$ *and* $I_n \oplus A$ *are reflexive IFMs.*
(iv) $I_n \wedge A$ *and* $I_n \odot A$ *are diagonal IFMs.*

Property 5 *If* I_n *be an identity IFM of order* $n \times n$ *and* A *be any IFM of the same order then,*
(i) $H(I_n, \Box I_n) = (0, 0)$
(ii) $H(I_n, \Diamond I_n) = (0, 0)$
(iii) $H(\Box I_n, \Diamond I_n) = (0, 0)$
(iv) $H(I_n \odot A, I_n \wedge A) = (0, 0)$
(v) $H(I_n \oplus A, I_n \vee A) = (0, 0)$
(vi) $H(\Box I_n, I_n^c) = (n^2, n^2)$
(vii) $H(I_n, I_n^c) = (n^2, n^2)$
(viii) $H(\Diamond I_n, I_n^c) = (n^2, n^2)$
(ix) $H(I_n, I_n \odot A) \le (n, n)$
(x) $H(I_n, I_n \wedge A) \le (n, n)$
(xi) $H(I_n, I_n \oplus A) \le (n^2 - n, n^2 - n)$
(xii) $H(I_n, I_n \vee A) \le (n^2 - n, n^2 - n)$.

Proof. Proofs of (i) to (v) are trivial.
(vi) Since $\Box I_n$ is identity IFM of order $n \times n$ then it is obvious that all the diagonal elements of $\Box I_n$ are < 1,0 > and all the non-diagonal elements of $\Box I_n$ are < 0,1 >. Again, I_n^c is irreflexive IFM of the same order whose diagonal elements are < 0,1 > and all the non-diagonal elements are < 1,0 >.

Hence, $H(\Box I_n, I_n^c) = (n^2, n^2)$.

The proofs of (vii) and (viii) are same as (vi).
(ix) All the non-diagonal elements of the IFMs I_n and $I_n \odot A$ are same, they are < 0,1 >, since I_n is an identity and $I_n \odot A$ is a diagonal IFMs. The diagonal elements of $I_n \odot A$ and I_n are different. Here the number of diagonal elements is n.

Hence, $H(I_n, I_n \odot A) \le (n, n)$.

(x) Same as (ix).
(xi) In this case all the diagonal elements (n in number) of I_n and $I_n \oplus A$ are same, these are < 1,0 >, since $I_n \oplus A$ is reflexive IFM. The non-diagonal elements of $I_n \oplus A$ are different from I_n. Here the number of non-diagonal elements of IFMs I_n and $I_n \oplus A$ are $n^2 - n$. Therefore, $H(I_n, I_n \oplus A) \le (n^2 - n, n^2 - n)$. The sign of equality holds only when all the non-diagonal elements of A are < 1,0 >.
Proof of (xii) is similar to (xi).

Property 6 *For any two IFMs* $A = [< a_{ij\mu}, a_{ij\nu} >]$ *and* $B = [< b_{ij\mu}, b_{ij\nu} >]$,

(i) $H(A, A \wedge B) = H(B, A \vee B)$

(ii) $H(A, A \vee B) = H(B, A \wedge B)$

(iii) $H(A, A \oplus B) = H(B, A \odot B)$

(iv) $H(A, A \odot B) = H(A, A \oplus B)$

(v) $H(A \oplus B, A \vee B) = H(A \odot B, A \wedge B)$

(vi) $H(A , B) = H(A \vee B, A \wedge B)$

(vii) $H(A \odot B, A \vee B) = H(A \oplus B, A \wedge B)$

(viii) $H(A, A@B) = H(B, A@B)$

(ix) $H(A@B, A \wedge B) = H(A@B, A \vee B)$

(x) $H(A \oplus B, A@B) = H(A \odot B, A@B)$

(xi) $H(A, A@B) = H(A@B, A \wedge B)$

Proof. Proofs of (i) to (iv) are trivial.

(v) We have, $H(A \oplus B, A \vee B)$

$= (\Sigma\Sigma\, |a_{ij\mu} + b_{ij\mu} - a_{ij\mu} \cdot b_{ij\mu} - \max\{a_{ij\mu}, b_{ij\mu}\}|, \Sigma\Sigma\, |a_{ij\nu} \cdot b_{ij\nu} - \min\{a_{ij\nu}, b_{ij\nu}\}|)$ and also,

$H(A \odot B, A \wedge B)$

$= (\Sigma\Sigma\, |a_{ij\mu} \cdot b_{ij\mu} - \min\{a_{ij\mu}, b_{ij\mu}\}|, \Sigma\Sigma\, |a_{ij\nu} + b_{ij\nu} - a_{ij\nu} \cdot b_{ij\nu} - \max\{a_{ij\nu}, b_{ij\nu}\}|)$. The proof consists of several cases.

Case - I : When $a_{ij\mu} \geq b_{ij\mu}$ and $a_{ij\nu} \geq b_{ij\nu}$ then,

$H(A \oplus B, A \vee B) = (\Sigma\Sigma\, |b_{ij\mu} - a_{ij\mu} \cdot b_{ij\mu}|, \Sigma\Sigma\, |a_{ij\nu} \cdot b_{ij\nu} - b_{ij\nu}|)$ and

$H(A \odot B, A \wedge B) = (\Sigma\Sigma\, |a_{ij\mu} \cdot b_{ij\mu} - b_{ij\mu}|, \Sigma\Sigma\, |b_{ij\nu} - a_{ij\nu} \cdot b_{ij\nu}|)$.

Case - II : When $a_{ij\mu} \geq b_{ij\mu}$ and $b_{ij\nu} \geq a_{ij\nu}$ then,

$H(A \oplus B, A \vee B) = (\Sigma\Sigma\, |b_{ij\mu} - a_{ij\mu} \cdot b_{ij\mu}|, \Sigma\Sigma\, |a_{ij\nu} \cdot b_{ij\nu} - a_{ij\nu}|)$ and

$H(A \odot B, A \wedge B) = (\Sigma\Sigma\, |a_{ij\mu} \cdot b_{ij\mu} - b_{ij\mu}|, \Sigma\Sigma\, |a_{ij\nu} - a_{ij\nu} \cdot b_{ij\nu}|)$.

Case - III : When $b_{ij\mu} \geq a_{ij\mu}$ and $b_{ij\nu} \geq a_{ij\nu}$ then,

$H(A \oplus B, A \vee B) = (\Sigma\Sigma\, |a_{ij\mu} - a_{ij\mu} \cdot b_{ij\mu}|, \Sigma\Sigma\, |a_{ij\nu} \cdot b_{ij\nu} - a_{ij\nu}|)$ and

$H(A \odot B, A \wedge B) = (\Sigma\Sigma\, |a_{ij\mu} \cdot b_{ij\mu} - a_{ij\mu}|, \Sigma\Sigma\, |a_{ij\nu} - a_{ij\nu} \cdot b_{ij\nu}|)$.

Case - IV : When $b_{ij\mu} \geq a_{ij\mu}$ and $a_{ij\nu} \geq b_{ij\nu}$ then,

$H(A \oplus B, A \vee B) = (\Sigma\Sigma\, |a_{ij\mu} - a_{ij\mu} \cdot b_{ij\mu}|, \Sigma\Sigma\, |a_{ij\nu} \cdot b_{ij\nu} - b_{ij\nu}|)$ and

$H(A \odot B, A \wedge B) = (\Sigma\Sigma\, |a_{ij\mu} \cdot b_{ij\mu} - a_{ij\mu}|, \Sigma\Sigma\, |b_{ij\nu} - a_{ij\nu} \cdot b_{ij\nu}|)$.

Hence from all the cases we have, $H(A \oplus B, A \vee B) = H(A \odot B, A \wedge B)$.

The proofs of remaining parts are same.

Property 7 *For any two IFMs,* $A = [<a_{ij\mu}, a_{ij\nu}>]$ *and* $B = [<b_{ij\mu}, b_{ij\nu}>]$,

(i) $E(A, A \wedge B) = E(B, A \vee B)$
(ii) $E(A, A \vee B) = E(B, A \wedge B)$
(iii) $E(A, A \oplus B) = E(B, A \odot B)$
(iv) $E(A, A \odot B) = E(B, A \oplus B)$
(v) $E(A \oplus B, A \vee B) = E(A \odot B, A \wedge B)$
(vi) $E(A \vee B, A \wedge B) = E(A, B)$
(vii) $E(A \odot B, A \vee B) = E(A \oplus B, A \wedge B)$
(viii) $E(A, A@B) = E(A@B, B)$
(ix) $E(A@B, A \wedge B) = E(A@B, A \vee B)$
(x) $E(A \oplus B, A@B) = E(A \odot B, A@B)$
(xi) $E(A, A@B) = E(A@B, A \wedge B)$.

Property 8 *For any two IFMs,* $A = [<a_{ij\mu}, a_{ij\nu}>]$ *and* $B = [<b_{ij\mu}, b_{ij\nu}>]$,

(i) $H(A, A \odot B) \geq H(A, A \wedge B)$
(ii) $H(A, A \odot B) \geq H(A \oplus B, A \vee B)$
(iii) $H(A, A \oplus B) \geq H(A, A \vee B)$
(iv) $H(A, A \oplus B) \geq H(A \oplus B, A \vee B)$
(v) $H(A \odot B, A \vee B) \geq H(A \vee B, A \wedge B)$
(vi) $H(A \vee B, A \wedge B) \geq H(A@B, A \vee B)$
(vii) $H(A \oplus B, A@B) \geq H(A@B, A \vee B)$
(viii) $H(A \odot B, A \vee B) \geq H(A \oplus B, A@B)$.

Proof. (i) Here we have, $H(A, A \odot B) = (\sum\sum |a_{ij\mu} - a_{ij\mu} \cdot b_{ij\mu}|, \sum\sum |b_{ij\nu} - a_{ij\nu} \cdot b_{ij\nu}|)$ and,
$H(A, A \wedge B) = (\sum\sum |a_{ij\mu} - \min(a_{ij\mu}, b_{ij\mu})|, \sum\sum |a_{ij\nu} - \max(a_{ij\nu}, b_{ij\nu})|)$.
The proof consists of several cases.

Case - I : When $a_{ij\mu} \geq b_{ij\mu}$ and $a_{ij\nu} \geq b_{ij\nu}$ then, $H(A, A \wedge B) = (\sum\sum |a_{ij\mu} - b_{ij\mu}|, 0)$.
Now, $b_{ij\nu} - a_{ij\nu} \cdot b_{ij\nu} \geq 0$ and $a_{ij\mu} \cdot b_{ij\mu} \leq b_{ij\mu}$ for all the values of $a_{ij\mu}$, $b_{ij\mu}$, $a_{ij\nu}$ and $b_{ij\nu}$ where, $0 \leq a_{ij\mu}, b_{ij\mu}, a_{ij\nu}, b_{ij\nu} \leq 1$.
Since, $a_{ij\mu} \cdot b_{ij\mu} \leq b_{ij\mu}$
or, $a_{ij\mu} - a_{ij\mu} \cdot b_{ij\mu} \geq a_{ij\mu} - b_{ij\mu}$.
Hence, $H(A, A \odot B) \geq H(A, A \wedge B)$.

Case - II : When $a_{ij\mu} \leq b_{ij\mu}$ and $a_{ij\nu} \leq b_{ij\nu}$ then, $H(A, A \wedge B) = (0, \sum\sum |a_{ij\nu} - b_{ij\nu}|)$.
Now, $a_{ij\nu} \cdot b_{ij\nu} \leq b_{ij\nu}$
or, $b_{ij\nu} - a_{ij\nu} \cdot b_{ij\nu} \geq b_{ij\nu} - a_{ij\nu}$.
Hence, $H(A, A \odot B) \geq H(A, A \wedge B)$.

Case - III : When $a_{ij\mu} \geq b_{ij\mu}$ and $a_{ij\nu} \leq b_{ij\nu}$ then,
$H(A, A \wedge B) = (\sum\sum |a_{ij\mu} - b_{ij\mu}|, \sum\sum |a_{ij\nu} - b_{ij\nu}|)$.
We have, $a_{ij\mu} - a_{ij\mu} \cdot b_{ij\mu} \geq a_{ij\mu} - b_{ij\mu}$ and also, $b_{ij\nu} - a_{ij\nu} \cdot b_{ij\nu} \geq a_{ij\nu} - b_{ij\nu}$.
Hence, $H(A, A \odot B) \geq H(A, A \wedge B)$.
Therefore from all the cases we have, $H(A, A \odot B) \geq H(A, A \wedge B)$.
All proofs of the remaining parts are similar to (i).

Property 9 *For any two IFMs,* $A = [< a_{ij\mu}, a_{ij\nu} >]$ *and* $B = [< b_{ij\mu}, b_{ij\nu} >]$,
(i) $E(A, A \odot B) \geq E(A, A \wedge B)$
(ii) $E(A, A \odot B) \geq E(A \oplus B, A \vee B)$
(iii) $E(A, A \oplus B) \geq E(A, A \vee B)$
(iv) $E(A, A \oplus B) \geq E(A \oplus B, A \vee B)$
(v) $E(A \odot B, A \vee B) \geq E(A \vee B, A \wedge B)$
(vi) $E(A \vee B, A \wedge B) \geq E(A@B, A \vee B)$
(vii) $E(A \oplus B, A@B) \geq E(A@B, A \vee B)$
(viii) $E(A \odot B, A \vee B) \geq E(A \oplus B, A@B)$.

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INTUITIONISTIC FUZZY TAUTOLOGICAL MATRICES

Susanta K. Khan and Madhumangal Pal*
Department of Applied Mathematics with Oceanology and Computer Programming
Vidyasagar University
Midnapore - 721102, India.

Abstract :
In this paper, the intuitionistic fuzzy tautological matrix (IFTM) is introduced. Various properties of IFTM are presented here. The implication operator (→) between two IFMs is also defined and studied several properties.
Keywords : Intuitionistic fuzzy matrix, intuitionistic fuzzy tautological matrix.

INTRODUCTION

Several authors presented a number of results on fuzzy matrices. Kim and Roush [3] studied the canonical form of an idempotent fuzzy matrix. Kolodziejczyk [4] presented the canonical form of a strongly transitive matrix. Xin [7, 8] studied controllable fuzzy matrices. Pal [5] introduced intuitionistic fuzzy determinant. The intuitionistic fuzzy matrices (IFMs) are defined and studied its various properties in [1, 2]. The distance between two IFMs is defined and presented a number of properties in [6].

In this paper, intuitionistic fuzzy tautological matrices (IFTMs), a varient of IFMs is introduced. The implication operator is also defined. A number of results on IFTMs with implication operator are presented here.

DEFINITION AND PRELIMINARIES

Def. 1 *An intuitionistic fuzzy matrix (IFM) A of order* $m \times n$ *is defined as* $A = [< a_{ij\mu}, a_{ij\nu} >]_{m \times n} = [a_{ij}]_{m \times n}$, *where* $a_{ij\mu}$ *and* $a_{ij\nu}$ *are called membership and non membership values of an element* a_{ij}, *which maintaining the condition* $0 \leq a_{ij\mu} + a_{ij\nu} \leq 1$.

Def. 2 *An intuitionistic fuzzy matrix is called an intuitionistic fuzzy tautological matrix (IFTM) if and only if* $a_{ij\mu} \geq a_{ij\nu}$, *for all* i, j.

*e-mail : mmpal@vidyasagar.ac.in

The IFM $A = \begin{bmatrix} <0.5, 0.2> & <0.7, 0.3> & <1.0, 0.0> \\ <0.2, 0.0> & <0.2, 0.1> & <0.8, 0.2> \\ <0.3, 0.5> & <0.1, 0.1> & <0.9, 0.1> \end{bmatrix}$ is an IFTM, but the IFM,

$B = \begin{bmatrix} <0.5, 0.1> & <0.3, 0.5> & <0.2, 0.5> \\ <0.1, 0.1> & <0.0, 1.0> & <0.3, 0.7> \\ <0.8, 0.1> & <0.7, 0.2> & <0.6, 0.5> \end{bmatrix}$ is not an IFTM.

Def. 3 *Let A and B are two IFMs of same order. The implication operator defined over A and B in the following way,*
$A \rightarrow B = [< \max\{a_{ij\nu}, b_{ij\mu}\}, \min\{a_{ij\mu}, b_{ij\nu}\} >]$ *and* $A \rightarrow B = B \leftarrow A$.

Def. 4 *If A and B are two IFMs of same order, where* $A = [< a_{ij\mu}, a_{ij\nu} >]$ *and* $B = [< b_{ij\mu}, b_{ij\nu} >]$ *then*

(i) $A < B$ *if and only if* $a_{ij\mu} < b_{ij\mu}$ *and* $a_{ij\nu} > b_{ij\nu}$ *for all i, j.*

(ii) $A > B$ *if and only if* $B < A$.

(iii) $\bar{A} = B$ *if and only if* $a_{ij\mu} = b_{ij\mu}$ *and* $a_{ij\nu} = b_{ij\nu}$.

(iv) $A = [< a_{ij\nu}, a_{ij\mu} >]$.

(v) $A \cup B = [< \max\{a_{ij\mu}, b_{ij\mu}\}, \min\{a_{ij\nu}, b_{ij\nu}\} >]$.

(vi) $A \cap B = [< \min\{a_{ij\mu}, b_{ij\mu}\}, \max\{a_{ij\nu}, b_{ij\nu}\} >]$.

(vii) $A@B = [< \frac{a_{ij\mu} + b_{ij\mu}}{2}, \frac{a_{ij\nu} + b_{ij\nu}}{2} >]$

(viii) $A\$B = [< \sqrt{a_{ij\mu} . b_{ij\mu}}, \sqrt{a_{ij\nu} . b_{ij\nu}} >]$

(ix) $A \# B = [< \frac{2a_{ij\mu} . b_{ij\mu}}{a_{ij\mu} + b_{ij\mu}}, \frac{2a_{ij\nu} . b_{ij\nu}}{a_{ij\nu} + b_{ij\nu}} >]$

(x) $\Diamond A = [< 1 - a_{ij\nu}, a_{ij\nu} >]$

(xi) $\Box A = [< a_{ij\mu}, 1 - a_{ij\mu} >]$

(xii) $A + B = [< a_{ij\mu} + b_{ij\mu} - a_{ij\mu} . b_{ij\mu}, a_{ij\nu} . b_{ij\nu} >]$

(xiii) $A.B = [< a_{ij\mu} . b_{ij\mu}, a_{ij\nu} + b_{ij\nu} - a_{ij\nu} . b_{ij\nu} >]$

(xiv) $A * B = [< \frac{a_{ij\mu} + b_{ij\mu}}{2(a_{ij\mu} . b_{ij\mu} + 1)}, \frac{a_{ij\nu} + b_{ij\nu}}{2(a_{ij\nu} . b_{ij\nu} + 1)} >]$

The following propositions are trivial.

Proposition 1 *If A and B are both IFTMs then the following are IFMs :*
$A@B$, $A\$B$, $A\#B$, $\Diamond A$, $(\overline{\Box\overline{A}})$, $A \cup B$, $A + B$ *and* $A * B$.

Proposition 2 *If A and B are both IFMs then*
$\Box A$, $A \cap B$, $\overline{A}$ *and* $A.B$ *are not IFTMs.*

Property 1 *Let A and B be two IFMs then the following expressions are IFTMs.*
(i) $A \rightarrow A$.
(ii) $\overline{\overline{A}} \rightarrow A$.
(iii) $A \rightarrow (B \rightarrow A)$.

Proof. (i) $A \rightarrow A = [<\max\{a_{ij\nu}, a_{ij\mu}\}, \min\{a_{ij\mu}, a_{ij\nu}\}>]$
Now, obviously $\max\{a_{ij\nu}, a_{ij\mu}\} \geq \min\{a_{ij\mu}, a_{ij\nu}\}$
So, $A \rightarrow A$ *is an IFTM.*
(ii) $A = [<a_{ij\mu}, a_{ij\nu}>]$
$\overline{A} = [<a_{ij\nu}, a_{ij\mu}>]$
$\overline{\overline{A}} = A$
Then $\overline{\overline{A}} \rightarrow A = A \rightarrow A$
Now proof of (ii) is similar to (i).
(iii) $B \rightarrow A = [<\max\{b_{ij\nu}, a_{ij\mu}\}, \min\{b_{ij\mu}, a_{ij\nu}\}>]$.
$A \rightarrow (B \rightarrow A) = [<\max\{a_{ij\nu}, b_{ij\nu}, a_{ij\mu}\}, \min\{a_{ij\mu}, b_{ij\mu}, a_{ij\nu}\}>]$.
We know, $\max\{a_{ij\nu}, \max\{b_{ij\nu}, a_{ij\mu}\}\} \geq \min\{a_{ij\nu}, \min\{b_{ij\mu}, a_{ij\mu}\}\}$
or, $\max\{a_{ij\nu}, b_{ij\nu}, a_{ij\mu}\} \geq \min\{a_{ij\nu}, b_{ij\mu}, a_{ij\mu}\}$.
Then, $\max\{a_{ij\nu}, b_{ij\nu}, a_{ij\mu}\} \geq \min\{a_{ij\mu}, b_{ij\mu}, a_{ij\nu}\}$.
Therefore, $A \rightarrow (B \rightarrow A)$ is an IFTM.

Property 2 *If A and B be two IFMs then*
(i) $A \rightarrow (A \cup B)$
(ii) $B \rightarrow (A \cup B)$ *and*
(iii) $A \rightarrow (B \rightarrow (A \cap B))$ *are IFTMs.*

Proof. (i) $A \cup B = [<\max\{a_{ij\mu}, b_{ij\mu}\}, \min\{a_{ij\nu}, b_{ij\nu}\}>]$.
$A \rightarrow A \cup B = [<\max\{a_{ij\nu}, \max\{a_{ij\mu}, b_{ij\mu}\}\}, \min\{a_{ij\mu}, \min\{a_{ij\nu}, b_{ij\nu}\}\}>]$
$= [<\max\{a_{ij\nu}, a_{ij\mu}, b_{ij\mu}\}, \min\{a_{ij\mu}, a_{ij\nu}, b_{ij\nu}\}>]$.

Now, $\max\{a_{ij\nu}, \max\{a_{ij\mu}, b_{ij\mu}\}\} \geq \min\{a_{ij\nu}, \min\{a_{ij\mu}, b_{ij\nu}\}\}$
Therefore, $\max\{a_{ij\nu}, a_{ij\mu}, b_{ij\mu}\} \geq \min\{a_{ij\mu}, a_{ij\nu}, b_{ij\nu}\}$.
So, $A \to A \cup B$ is an IFTM.
(ii) $B \to A \cup B = [< \max\{b_{ij\nu}, \max\{a_{ij\mu}, b_{ij\mu}\}\}, \min\{b_{ij\mu}, \min\{a_{ij\nu}, b_{ij\nu}\}\} >]$
$= [< \max\{b_{ij\nu}, a_{ij\mu}, b_{ij\mu}\}, \min\{b_{ij\mu}, a_{ij\nu}, b_{ij\nu}\} >]$.
As $\max\{b_{ij\nu}, a_{ij\mu}, b_{ij\mu}\} \geq \min\{b_{ij\mu}, a_{ij\nu}, b_{ij\nu}\}$, $B \to (A \cup B)$ is an IFTM.
(iii) $A \cap B = [< \min\{a_{ij\mu}, b_{ij\mu}\}, \max\{a_{ij\nu}, b_{ij\nu}\} >]$.
$B \to (A \cap B) = [< \max\{b_{ij\nu}, \min\{a_{ij\mu}, b_{ij\mu}\}\}, \min\{b_{ij\mu}, \max\{a_{ij\nu}, b_{ij\nu}\}\} >]$.
$A \to (B \to (A \cap B)) = [< \max\{a_{ij\nu}, \max\{b_{ij\nu}, \min\{a_{ij\mu}, b_{ij\mu}\}\}\},$
$\min\{a_{ij\mu}, \min\{b_{ij\mu}, \max\{a_{ij\nu}, b_{ij\nu}\}\}\} >]$
$= [< \max\{a_{ij\nu}, b_{ij\nu}, \min\{a_{ij\mu}, b_{ij\mu}\}\}, \min\{a_{ij\mu}, b_{ij\mu}, \max\{a_{ij\nu}, b_{ij\nu}\}\} >]$.
We know, $\max\{a_{ij\nu}, \min\{b_{ij\nu}, a_{ij\mu}\}\} \geq \min\{a_{ij\nu}, \max\{b_{ij\nu}, a_{ij\mu}\}\}$
$\geq \min\{b_{ij\nu}, \max\{a_{ij\nu}, a_{ij\mu}\}\}$
$\geq \min\{a_{ij\mu}, \max\{b_{ij\nu}, a_{ij\nu}\}\}$.
Similarly, $\max\{a_{ij\nu}, b_{ij\nu}, \min\{a_{ij\mu}, b_{ij\mu}\}\} \geq \min\{a_{ij\mu}, b_{ij\mu}, \max\{a_{ij\nu}, b_{ij\nu}\}\}$.
So, $A \to (B \to (A \cap B))$ is an IFTM.

Property 3 *If A and B are IFMs then*
(i) $A \cap B \to A$
(ii) $A \cap B \to B$ *are IFTMs.*

Proof. (i) $A \cap B = [< \min\{a_{ij\mu}, b_{ij\mu}\}, \max\{a_{ij\nu}, b_{ij\nu}\} >]$.
$A \cap B \to A = [< \max\{\max\{a_{ij\nu}, b_{ij\nu}\}, a_{ij\mu}\}, \min\{\min\{a_{ij\mu}, b_{ij\mu}\}, a_{ij\nu}\} >]$
$= [< \max\{a_{ij\nu}, b_{ij\nu}, a_{ij\mu}\}, \min\{a_{ij\mu}, b_{ij\mu}, a_{ij\nu}\} >]$.
Hence $A \cap B \to A$ is an IFTM.
(ii) $A \cap B = [< \min\{a_{ij\mu}, b_{ij\mu}\}, \max\{a_{ij\nu}, b_{ij\nu}\} >]$.
$A \cap B \to B = [< \max\{\max\{a_{ij\nu}, b_{ij\nu}\}, \min\{\min\{a_{ij\mu}, b_{ij\mu}\}, b_{ij\nu}\} >]$
$= [< \max\{a_{ij\nu}, b_{ij\nu}, b_{ij\mu}\}, \min\{a_{ij\mu}, b_{ij\mu}, b_{ij\nu}\} >]$.
Hence $A \cap B \to B$ is an IFTM.

Property 4 *For any three IFMs A, B, C the following expressions are IFTMs.*
(i) $(A \to C) \to ((B \to C) \to ((A \cup B) \to C))$
(ii) $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$
(iii) $(\bar{A} \to B) \to ((\bar{A} \to B) \to A)$.

Proof. (i) $A \to C = [< \max\{a_{ij\nu}, c_{ij\mu}\}, \min\{a_{ij\mu}, c_{ij\nu}\} >]$.

$B \to C = [< \max\{b_{ij\nu}, c_{ij\mu}\}, \min\{b_{ij\mu}, c_{ij\nu}\} >]$.

$A \cup B = [< \max\{a_{ij\mu}, b_{ij\mu}\}, \min\{a_{ij\nu}, b_{ij\nu}\} >]$.

$(A \cup B) \to C = [< \max\{\min\{a_{ij\nu}, b_{ij\nu}\}, c_{ij\nu}\}, \min\{\max\{a_{ij\mu}, b_{ij\mu}\}, c_{ij\nu}\} >]$.

$((B \to C) \to ((A \cup B) \to C)) = [< \max\{\min\{b_{ij\mu}, c_{ij\nu}\}, \min\{a_{ij\nu}, b_{ij\nu}\}, c_{ij\mu}\},$
$\min\{\max\{b_{ij\nu}, c_{ij\mu}\}, \max\{a_{ij\mu}, b_{ij\mu}\}, c_{ij\nu}\} >]$.

$(A \to C) \to ((B \to C) \to ((A \cup B) \to C))$
$= [< \max\{\min\{a_{ij\mu}, c_{ij\nu}\}, \max\{\min\{b_{ij\mu}, c_{ij\nu}\}, \min\{a_{ij\nu}, b_{ij\nu}\}, c_{ij\mu}\}\},$
$\min\{\max\{a_{ij\nu}, c_{ij\mu}\}, \min\{\max\{b_{ij\nu}, c_{ij\mu}\}, \max\{a_{ij\mu}, b_{ij\mu}\}, c_{ij\nu}\}\} >]$
$=[< \max\{\min\{a_{ij\mu}, c_{ij\nu}\},\{\min\{b_{ij\mu}, c_{ij\nu}\}, \min\{a_{ij\nu}, b_{ij\nu}\}, c_{ij\mu}\},$
$\min\{\max\{a_{ij\nu}, c_{ij\mu}\}, \max\{b_{ij\nu}, c_{ij\mu}\},\max\{a_{ij\mu}, b_{ij\mu}\}, c_{ij\nu}\} >]$.

Now, $\{\max\{a_{ij\mu}, c_{ij\mu}\},\min\{b_{ij\mu}, c_{ij\nu}\}, \min\{a_{ij\nu}, b_{ij\nu}\}, c_{ij\mu}\}$
$\geq \max\{c_{ij\nu},\min\{a_{ij\mu}, c_{ij\mu}\ b_{ij\mu},\},\min\{a_{ij\nu}, b_{ij\mu}\}\}$
$\geq \max\{c_{ij\nu},\min\{a_{ij\mu}, b_{ij\mu}, c_{ij\mu}\}\}$
$\leq \min\{c_{ij\nu},\max\{a_{ij\mu}, b_{ij\mu}, c_{ij\mu}\}\}$
$\geq \min\{c_{ij\mu},\max\{a_{ij\mu}, b_{ij\mu}\}, \max\{c_{ij\mu}, b_{ij\nu}\}, \max\{a_{ij\nu}, c_{ij\mu}\}\}$.
$\geq \min\{c_{ij\nu}, \max\{a_{ij\mu}, b_{ij\mu}\}, \max\{a_{ij\mu}, b_{ij\nu}\}, \max\{a_{ij\nu}, c_{ij\nu}\}\}$.

Hence, $(A \to C) \to ((B \to C) \to ((A \cup B) \to C))$ is an IFTM.

(ii) $(B \to C) = [< \max\{b_{ij\nu}, c_{ij\mu}\}, \min\{b_{ij\mu}, c_{ij\nu}\}>]$.

$(A \to (B \to C)) = [< \max\{a_{ij\nu}, \max\{b_{ij\nu}, c_{ij\mu}\}\}, \min\{a_{ij\mu}, \min\{b_{ij\mu}, c_{ij\nu}\}\}>]$
$= [<\max\{a_{ij\nu}, b_{ij\nu}, c_{ij\mu}\}, \min\{a_{ij\mu}, b_{ij\mu}, c_{ij\nu}\} >]$.

$(A \to B) = [< \max\{a_{ij\nu}, b_{ij\mu}\}, \min\{a_{ij\mu}, b_{ij\nu}\} >]$.

$(A \to C) = [< \max\{a_{ij\nu}, b_{ij\mu}\}, \min\{a_{ij\mu}, b_{ij\nu}\} >]$.

$((A \to B) \to (A \to C)) = [< \max\{\min\{a_{ij\mu}, b_{ij\nu}\} \max\{a_{ij\nu}, c_{ij\nu}\}\},$
$\min\{\max\{a_{ij\nu}, b_{ij\mu}\}, \min\{a_{ij\mu}, c_{ij\nu}\}\} >]$
$=[< \max\{\min\{a_{ij\mu}, b_{ij\nu}\}, a_{ij\nu}, c_{ij\mu}\}, \min\{\max\{a_{ij\nu}, b_{ij\mu}\}\ a_{ij\mu}, c_{ij\nu}\} >]$.

$(A \to (B \to C)) \to ((A \to B) \to (A \to C))$
$[< \max\{\min\{a_{ij\mu}, b_{ij\mu}, c_{ij\nu}\}, \min\{a_{ij\mu}, b_{ij\mu}\}, a_{ij\nu}, c_{ij\mu}\},$
$\min\{\max\{a_{ij\nu}, b_{ij\nu}, c_{ij\mu}\} \max\{a_{ij\nu}, b_{ij\mu}\}, a_{ij\mu}, c_{ij\nu}\} >]$.

Now, $\max\{\min\{a_{ij\mu}, b_{ij\mu}, c_{ij\nu}\}, \min\{a_{ij\mu}, b_{ij\nu}\}, a_{ij\nu}, c_{ij\mu}\}$
$\geq \max\{a_{ij\nu}, c_{ij\mu}, \min\{b_{ij\mu}, c_{ij\nu}, a_{ij\mu}\}\}$
$\geq \min\{\max\{a_{ij\nu}, b_{ij\mu}, c_{ij\nu u}\}, a_{ij\mu}, c_{ij\nu}\}$
$\geq \min\{\max\{a_{ij\nu}, b_{ij\nu}, c_{ij\mu}\}, \max\{a_{ij\nu}, b_{ij\mu}\}, a_{ij\mu}, c_{ij\nu}\}$.

So, $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$ is an IFTM.

(iii) $\overline{A} \to \overline{B} = [< \max\{a_{ij\mu}, b_{ij\nu}\}, \min\{a_{ij\nu}, b_{ij\mu}\} >]$.

$\overline{A} \to B = [< \max\{a_{ij\mu}, b_{ij\mu}\}, \min\{a_{ij\nu}, b_{ij\nu}\} >]$.

$(\overline{A} \to B) \to A) = [< \max\{\min\{a_{ij\nu}, b_{ij\nu}\}, a_{ij\mu}\}, \min\{\max\{a_{ij\mu}, b_{ij\mu}\}, a_{ij\nu}\} >]$.

$(\overline{A} \to \overline{B}) \to ((A \to B) \to A)$

$= [< \max\{\min\{a_{ij\nu}, b_{ij\mu}\} \max\{\min\{a_{ij\nu}, b_{ij\nu}\} a_{ij\mu}\}\},$

$\min\{\max\{a_{ij\mu}, b_{ij\nu}\}, \min\{\max\{a_{ij\mu}, b_{ij\mu}\}, a_{ij\nu}\}\} >]$

$= [< \max\{\min\{a_{ij\nu}, b_{ij\mu}\} \min\{a_{ij\nu}, b_{ij\nu}\}, a_{ij\mu}\}, \min\{\max\{a_{ij\mu}, b_{ij\nu}\}, \max\{a_{ij\mu}, b_{ij\mu}\} a_{ij\nu}\} >]$.

Now, $\max\{\min\{a_{ij\nu}, b_{ij\mu}\} \min\{a_{ij\nu}, b_{ij\nu}\}, a_{ij\mu}\}$

$\geq \max\{a_{ij\mu}, \min\{a_{ij\nu}, b_{ij\mu}\}\}$

$\geq \min\{\max\{a_{ij\mu}, b_{ij\mu}\}, a_{ij\nu}\}$

$\geq \min\{\max\{a_{ij\mu}, b_{ij\nu}\}, \max\{a_{ij\mu}, b_{ij\mu}\}, a_{ij\mu}\}$.

So, $(\overline{A} \to \overline{B}) \to ((\overline{A} \to B) \to A)$ is an IFTM

Property 5 *If $A < B$ then $A \to B$ is an IFTM.*

Proof. Given $A < B$, then $A \to B = [< \max\{a_{ij\nu}, b_{ij\mu}\}, \min\{a_{ij\mu}, b_{ij\nu}\} >]$ for $A < B$ implies $a_{ij\mu} < b_{ij\mu}, a_{ij\nu} > b_{ij\nu}$.

Therefore, $\max\{a_{ij\nu}, b_{ij\mu}\} \geq \max\{a_{ij\nu}, a_{ij\mu}\} \geq \min\{b_{ij\nu}, a_{ij\mu}\}$.

So. $A \to B$ is an IFTM.

Property 6 *If A, B are IFMs then the following expressions are IFTMs.*

(i) $(A \cap (A \to B)) \to B$

(ii) $((A \to B) \cap \overline{B}) \to \overline{A}$.

Proof. (i) $A \to B = [< \max\{a_{ij\nu}, b_{ij\mu}\}, \min\{a_{ij\mu}, b_{ij\nu}\} >]$.

$(A \cap (A \to B)) = [< \min\{a_{ij\mu}, \max\{a_{ij\nu}, b_{ij\mu}\}, \max\{a_{ij\nu}, \min\{a_{ij\mu}, b_{ij\nu}\}\} >]$.

$(A \cap (A \to B)) \to B$

$= [< \max\{a_{ij\nu}, \min\{a_{ij\mu}, b_{ij\nu}\}, b_{ij\mu}\}, \min\{a_{ij\mu}, \max\{a_{ij\nu}, b_{ij\mu}\}, b_{ij\nu}\} >]$

$= [< \max\{a_{ij\nu}, \min\{a_{ij\mu}, b_{ij\nu}\}, b_{ij\mu}\}, \min\{a_{ij\mu}, \max\{a_{ij\nu}, b_{ij\mu}\}, b_{ij\mu}\} >]$.

Now, $\max\{a_{ij\nu}, \min\{a_{ij\mu}, b_{ij\nu}\}, b_{ij\mu}\} - \min\{a_{ij\mu}, \max\{a_{ij\nu}, b_{ij\mu}\}, b_{ij\nu}\} >]$

$\geq \max\{a_{ij\nu}, b_{ij\mu}\} - \max\{a_{ij\nu}, b_{ij\mu}\} = 0$.

So, $\max\{a_{ij\nu}, \min\{a_{ij\mu}, b_{ij\nu}\}, b_{ij\mu}\} - \min\{a_{ij\mu}, \max\{a_{ij\nu}, b_{ij\mu}\}, b_{ij\nu}\} >] \geq 0$.

Therefore, $(A \cap (A \to B)) \to B$ is an IFTM.

(ii) $A \to B = [< \max\{a_{ij\nu}, b_{ij\mu}\}, \min\{a_{ij\mu}, b_{ij\nu}\} >]$.

$(A \to B) \cap \overline{B} = [< \min\{ \max\{ a_{ij\nu}, b_{ij\mu}\}, b_{ij\nu}\}, \max\{\min\{ a_{ij\mu}, b_{ij\nu}\}, b_{ij\mu}\} >]$.

$((A \to B) \cap \overline{B}) \to \overline{A}$

$= [< \max\{\max\{\min\{ a_{ij\mu}, b_{ij\nu}\}, b_{ij\mu}\}, a_{ij\nu}\}, \min\{ \min\{ \max\{ a_{ij\nu}, b_{ij\mu}\}, b_{ij\nu}\}, a_{ij\nu}\} >]$

$= [< \max\{\min\{ a_{ij\mu}, b_{ij\nu}\}, b_{ij\mu}, a_{ij\nu}\}, \min\{ \max\{ a_{ij\nu}, b_{ij\mu}\}, b_{ij\nu}, a_{ij\mu}\} >]$.

Now, $\max\{ \min\{ a_{ij\mu}, b_{ij\nu}\}, b_{ij\mu}, a_{ij\nu}\} - \min\{ \max\{ a_{ij\nu}, b_{ij\mu}\}, b_{ij\nu}, a_{ij\mu}\}$

$\geq \min\{ a_{ij\mu}, b_{ij\nu}\} - \min\{ b_{ij\nu}, a_{ij\mu}\} = 0$.

So, $\max\{\min\{ a_{ij\mu}, b_{ij\nu}\}, b_{ij\mu}, a_{ij\nu}\} - \min\{ \max\{ a_{ij\nu}, b_{ij\mu}\}, b_{ij\nu}, a_{ij\mu}\} \geq 0$.

So, $((A \to B) \cap \overline{B}) \to \overline{A}$ *is an* IFTM.

Property 7 *For any three IFMs A, B, C the following are IFTMs.*

(i) $((A \to B) \cap (B \to C)) \to (A \to C)$

(ii) $((((A \to B) \to (\overline{C} \to \overline{D})) \to C) \to E) \to ((E \to A) \to (D \to A))$.

Proof. (i) $A \to B = [< \max\{ a_{ij\nu}, b_{ij\mu}\}, \min\{ a_{ij\mu}, b_{ij\nu}\} >]$.

$B \to C = [< \max\{ b_{ij\nu}, c_{ij\mu}\}, \min\{ b_{ij\mu}, c_{ij\nu}\} >]$.

$A \to C = [< \max\{ a_{ij\nu}, c_{ij\mu}\}, \min\{ a_{ij\mu}, c_{ij\nu}\} >]$.

$((A \to B) \cap (B \to C)) = \min\{ \max\{ a_{ij\nu}, b_{ij\mu}\}, \max\{ b_{ij\nu}, c_{ij\mu}\}\}$,

$\max\{\min\{ a_{ij\mu}, b_{ij\nu}\}, \min\{ b_{ij\mu}, c_{ij\nu}\}\} >]$.

$((A \to B) \cap (B \to C)) \to (A \to C)$

$=[< \max\{ \max\{\min\{ a_{ij\mu}, b_{ij\nu}\}, \min\{ b_{ij\mu}, c_{ij\nu}\}\}, \max\{ a_{ij\nu}, c_{ij\mu}\}\}$,

$\min\{ \min\{ \max\{ a_{ij\nu}, b_{ij\mu}\}, \max\{ b_{ij\nu}, c_{ij\mu}\}\}, \min\{ a_{ij\mu}, c_{ij\nu}\}\} >]$

$= [< \max\{\min\{ a_{ij\mu}, b_{ij\nu}, b_{ij\mu}, c_{ij\nu}\}, c_{ij\mu}, a_{ij\nu}\}, \min\{ \max\{ a_{ij\nu}, b_{ij\mu}, b_{ij\nu}, c_{ij\mu}\}, c_{ij\nu}, a_{ij\mu}\} >]$.

Now, $\max\{\min\{ a_{ij\mu}, b_{ij\nu}, b_{ij\mu}, c_{ij\nu}\}, c_{ij\mu}, a_{ij\nu}\}$ $\min\{ \max\{ a_{ij\nu}, b_{ij\mu}, b_{ij\nu}, c_{ij\mu}\}, c_{ij\nu}, a_{ij\mu}\}$

$= \min\{ \max\{ c_{ij\mu}, a_{ij\nu}, b_{ij\mu}, b_{ij\nu}\}, \max\{ c_{ij\mu}, a_{ij\nu}, a_{ij\mu}, c_{ij\nu}\}\} - \min\{ a_{ij\mu}, c_{ij\nu}, \max\{ a_{ij\nu}, b_{ij\mu}, b_{ij\nu}, c_{ij\mu}\}\}$

$\geq \max\{ a_{ij\nu}, b_{ij\nu}, b_{ij\mu}, c_{ij\mu}\} - \max\{ a_{ij\nu}, b_{ij\mu}, b_{ij\nu}, c_{ij\mu}\} = 0$.

So, $\max\{\min\{ a_{ij\mu}, b_{ij\nu}, b_{ij\mu}, c_{ij\nu}\}, c_{ij\mu}, a_{ij\nu}\} \geq \min\{ \max\{ a_{ij\nu}, b_{ij\nu}, b_{ij\nu}, c_{ij\mu}\}, c_{ij\nu}, a_{ij\mu}\}$.

Therefore, $((A \to B) \cap (B \to C)) \to (A \to C)$ is an IFTM.

(ii) $A \to B = [< \max\{ a_{ij\nu}, b_{ij\mu}\}, \min\{ a_{ij\mu}, b_{ij\nu}\} >]$.

$C \to D = [< \max\{\min\{ a_{ij\mu}, b_{ij\nu}\}, c_{ij\mu}, d_{ij\nu}\}, \min\{ \max\{ a_{ij\nu}, b_{ij\mu}\}, c_{ij\nu}, d_{ij\mu}\} >]$.

$(((A \to B) \to (\overline{C} \to \overline{D})) \to C)$

$= [< \max\{ \min\{ \max\{ a_{ij\nu}, b_{ij\mu}\}, c_{ij\nu}, d_{ij\mu}\}, c_{ij\mu}\}, \min\{ \max\{ \min\{ a_{ij\mu}, b_{ij\nu}\}, c_{ij\mu}, d_{ij\nu}\}, c_{ij\nu}\} >]$.

$((((A \to B) \to (\overline{C} \to \overline{D})) \to C) \to E)$

$= [< \max \{ \min \{ \max \{ \min \{ a_{ij\mu}, b_{ij\nu} \}, c_{ij\mu}, d_{ij\nu} \}, c_{ij\nu} \}, e_{ij\mu} \},$
$\min\{ \max \{ \min \{\max \{ a_{ij\nu}, b_{ij\mu} \}, c_{ij\nu}, d_{ij\mu} \}, c_{ij\mu} \} e_{ij\nu} \} >]$
and, $E \rightarrow A = [< \max \{ e_{ij\nu}, a_{ij\mu} \}, \min \{ e_{ij\mu}, a_{ij\nu} \} >]$.
$D \rightarrow A = [< \max \{ d_{ij\nu}, a_{ij\mu} \}, \min \{ d_{ij\mu}, a_{ij\nu} \} >]$.
$(E \rightarrow A) \rightarrow (D \rightarrow A) = [< \max \{ \min \{ e_{ij\mu}, a_{ij\nu} \}, \{ d_{ij\nu}, a_{ij\mu} \}, \min \{ \max \{ e_{ij\nu}, a_{ij\mu} \}, \{ d_{ij\mu}, a_{ij\nu} \}>]$.
Therefore, $((((A \rightarrow B) \rightarrow (\overline{C} \rightarrow \overline{D})) \rightarrow C) \rightarrow E) \rightarrow ((E \rightarrow A) \rightarrow (D \rightarrow A))$
$= [< \max \{ \min \{ \max \{ \min\{ \max \{ a_{ij\nu}, b_{ij\mu} \}, c_{ij\nu}, d_{ij\mu} \}, c_{ij\mu} \}, e_{ij\nu} \}, \min \{ e_{ij\mu}, a_{ij\nu} \}, d_{ij\mu}, a_{ij\mu} \},$
$\min\{ \max \{ \min \{\max\{ \min \{ a_{ij\mu}, b_{ij\nu} \}, c_{ij\mu}, d_{ij\nu} \}, c_{ij\nu} \}, e_{ij\mu} \}, \max \{ e_{ij\nu}, a_{ij\mu} \}, d_{ij\mu}, a_{ij\nu} \} >]$.

Now,

$\max \{ \min \{ \max \{ \min\{ \max \{ a_{ij\nu}, b_{ij\mu} \}, c_{ij\nu}, d_{ij\mu} \}, c_{ij\mu} \}, e_{ij\nu} \}, \min \{ e_{ij\mu}, a_{ij\nu} \}, d_{ij\nu}, a_{ij\mu} \}$
$-\min\{ \max \{ \min \{\max\{ \min \{ a_{ij\mu}, b_{ij\nu} \}, c_{ij\mu}, d_{ij\nu} \}, c_{ij\mu} \}, e_{ij\mu} \}, \max \{ e_{ij\nu}, a_{ij\mu} \}, d_{ij\mu}, a_{ij\nu} \}$
$\geq \max \{a_{ij\mu}, \min\{e_{ij\mu}, a_{ijV} \}\} - \min \{a_{ij\mu}, \max \{e_{ij\mu}, a_{ij\mu} \}\}$
$= \min\{ \max \{ a_{ij\mu}, a_{ij\nu} \}, \max \{ a_{ij\nu}, e_{ij\nu} \}\} - \max\{ \min \{ a_{ij\mu}, a_{ij\nu} \}, \min \{ a_{ij\nu}, e_{ij\nu} \}\} \geq 0$.
For supposing the condition $\max \{ a_{ij\mu}, e_{ij\mu} \} \geq \{ a_{ij\nu}, e_{ij\nu} \}$ and
using the inequality $\max \{ a_{ij\mu}, a_{ij\nu} \} \geq \{ a_{ij\mu}, a_{ij\nu} \}$.
Again, we also let $\max \{ a_{ij\mu}, e_{ij\mu} \} < \min\{ a_{ij\nu}, e_{ij\nu} \}$ then
$a_{ij\mu} < a_{ij\nu},\ a_{ij\mu} < e_{ij\nu}, e_{ij\mu} < a_{ij\nu}, e_{ij\mu} < e_{ij\nu}$
If $e_{ij\nu} \leq \max\{ c_{ij\mu}, \min \{c_{ij\nu}, d_{ij\mu}, \max \{ a_{ij\nu}, b_{ij\mu} \}\}\}$, then
$\max \{d_{ij\nu}, a_{ij\mu}, e_{ij\mu}, \min\{\max\{\min \{c_{ij\nu}, d_{ij\mu}, \max\{a_{ij\nu}, b_{ij\mu} \}\}, c_{ij\mu} \}, e_{ij\nu} \}\}$
$-\min \{d_{ij\mu}, a_{ij\nu}, e_{ij\nu}, \max\{\min\{\max \{c_{ij\mu}, d_{ij\nu}, \min \{a_{ij\mu}, b_{ij\nu} \}\}, c_{ij\nu} \}, e_{ij\mu} \}\}$
$\geq \max \{d_{ij\nu}, a_{ij\mu}, e_{ij\mu}, e_{ij\nu} \} - \min \{d_{ij\mu}, a_{ij\nu}, e_{ij\nu} \} \geq 0$.
If $e_{ij\mu} \geq \min\{ \max \{c_{ij\mu}, d_{ij\nu} \}, \min \{a_{ij\mu}, b_{ij\nu} \}\}, c_{ij\mu} \}$, then
$\max \{d_{ij\nu}, a_{ij\mu}, e_{ij\mu}, \min \{\max\{ \min \{ c_{ij\nu}, d_{ij\mu}, \max \{ a_{ij\nu}, b_{ij\mu} \}\}, c_{ij\mu}, e_{ij\nu} \}\}\}$
$-\min \{d_{ij\mu}, a_{ij\nu}, e_{ij\nu}, \max \{\min\{ \max \{ c_{ij\mu}, d_{ij\nu}, \min \{ a_{ij\mu}, b_{ij\nu} \}\}, c_{ij\nu}, e_{ij\mu} \}\}\}$
$\geq \max \{d_{ij\nu}, a_{ij\mu}, e_{ij\mu} \} - \min \{d_{ij\mu}, a_{ij\nu}, e_{ij\mu}, e_{ij\nu} \} \geq 0$.
So, if $e_{ij\nu} \geq \max \{c_{ij\mu}, \min \{c_{ij\nu}, d_{ij\mu}, \max \{ a_{ij\nu}, b_{ij\mu} \}\}\}$
and $e_{ij\mu} < \min\{ \max \{c_{ij\mu}, d_{ij\nu}, \min \{ a_{ij\mu}, b_{ij\nu} \}\}, c_{ij\nu} \}$, then
$e_{ij\nu} > e_{ij\mu}, e_{ij\nu} > \min \{e_{ij\nu}, d_{ij\nu}, \max \{ a_{ij\nu}, b_{ij\nu} \}\}$ and
$e_{ij\mu} > e_{ij\nu}, e_{ij\mu} > \max \{e_{ij\mu}, d_{ij\nu}, \min \{ a_{ij\mu}, b_{ij\nu} \}\}$.
Therefore, $\max \{d_{ij\nu}, a_{ij\mu}, e_{ij\mu}, c_{ij\mu}, \min \{ c_{ij\nu}, d_{ij\mu}, \max \{ a_{ij\nu}, b_{ij\mu} \}\}\}$

$$-\min\{d_{ij\mu}, a_{ij\nu}, e_{ij\nu}, c_{ij\nu}, \max\{c_{ij\mu}, d_{ij\nu}, \min\{a_{ij\mu}, b_{ij\nu}\}\}\}$$
$$\geq \min\{c_{ij\nu}, d_{ij\mu}, \max\{a_{ij\nu}, b_{ij\mu}\}\} - \min\{d_{ij\mu}, a_{ij\nu}, e_{ij\nu}, c_{ij\nu}, \max\{c_{ij\mu}, d_{ij\nu}, \min\{a_{ij\mu}, a_{ij\nu}\}\}\}$$
$$\geq \min\{c_{ij\nu}, d_{ij\mu}, a_{ij\nu}\} - \min\{d_{ij\mu}, a_{ij\nu}, e_{ij\nu}, c_{ij\nu}, \max\{c_{ij\mu}, d_{ij\nu}, \min\{a_{ij\mu}, b_{ij\nu}\}\}\} \geq 0.$$
Hence the given IFM is IFTM.

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TWO STORAGE FUZZY INVENTORY MODEL FOR THE DEMAND DEPENDENT ON THE DISPLAYED INVENTORY LEVEL

B. Mondal[a], S. Kar[b], A. K. Bhunia[c] and M. Maiti[d]
[a]Raja N.L. Khan Women's College
West Midnapore - 721102, West Bengal
[b]Haldia Institute of Technology
Debhog - 721657, Haldia,
East Midanpore, West Bengal
[c]Department of Mathematics
Burdwan University
Burdwan, West Bengal
[d]Department of Applied Mathematics
Vidyasagar University
West Midnapore - 721102, West Bengal

Abstract :
A profit maximization multi-item inventory model is developed for several items under some limitations for which two separate storage facilities are required due to the limitations of the existing storage (showroom) capacity. In this paper, we consider the demand which is a deterministic linear function of the stock level of the show-room (exiting storage) and the selling price of the item. The replenishment rate is infinite and shortages are not allowed . The problem has been formulated as a fuzzy optimization problem associating fuzziness with the objective and constraint goals. The model is illustrated with a numerical example, solved by Fuzzy Nonlinear Programming (FNLP) method and the results for the fuzzy and crisp models are compared.

INTRODUCTION

Generally classical inventory models are developed under the basic assumptions that the management purchases (or produces) the single item. However, in many real-life situations, this assumption is not correct. Instead of a single item, many companies or enterprises or retailers are motivated to store several items in their show-room for more profitable business affair. Another cause of their motivation is to attract the customers to purchase several items in one show-room / shop. Multi-item classical inventory models under different resource constraints such as available floor / shelf-space, capital investment and average number of inventory etc. are presented in the well-known books [8,15,25,34] of this subject. Padmanabhan and Vrat [26] developed a multi-item multi-objective inventory model of deteriorating items with stock dependent demand by a non-linear goal programming method. Considering two constraints, Ben-Daya and Raouf [4] discussed a multi-item inventory model with stochastic demand.

In the field of inventory management, an important problem associated with the inventory maintenance is to decide where to stock the goods. The basic assumption in the traditional inventory models is that the management owns a storage (warehouse) with unlimited capacity. However, in some situation, it is not always true (e.g. in an important supermarket or in an important business place, the storage space of a showroom is limited). As a result, due to the procurement of large number of items and the limited capacity of the existing storage (viz. the own warehouse, OW), an extra storage (viz. the rented warehouse, RW) is hired on rental basis to store the excess items. This rented warehouse is located away from OW. In practice, large stock attracts the management anattractive price discount for bulk purchases or higher aequmisition cost.then the inventory related other costs or storing the highly demandable several items. The actual service to the customer is done at OW only. As the holding cost in RW is greater than that in OW, transporting the stocks from RW to OW in order to reduce the holding cost empties the stocks of RW first.

In the last few years, several authors have been developed two warehouses inventory models. An early discussion on this area was represented by Hartely [16]. He developed this type of model under the assumption that the cost of transporting a unit from RW to OW is not significantly high. Recently, several researchers reformulated the model of Hartley [16] changing the assumptions which makes it more realistic. To have an overview of the trend of research in this areas may refer to the article of Sarma [33], Murdeswar and Sethi [24], Dave [11], Pakkala and Achery [28], Goswami and Chaudhuri [12], Bhunia and Maiti [6], Benkherouf [5] and others.

Most of the inventory models developed under the assumption for constant demand rate in both manufacturing and sales environment. However, in many situations particularly in certain consumer products, the on-hand stock levels may influence the demand rate. In this connection, according to Levin [22] "At times, the presence of inventory has a motivational effect on the people around it. It is a common belief that the large piles of goods displayed in a super market will lead the consumer to buy more". In the recent years, several models have been developed incorporating the effect of demand dependence on stock level. This type of model was first discussed by Gupta and Vrat [14] in 1986. They developed a model in which the demand rate is a function of the initial stock level (order quantity). In 1988, Baker and Urban [2] investigated the situation in which the demand rate is a function (polynomial type) of the on-hand stock level. Since then, this type of models was considered by Mandal and Phaujder [23], Dutta and Pal [10], Urban [39], Goswami and Chaudhuri [13], Padmanabhan and Vrat [27]. Bhunia and Maiti [7] etc. They developed their models assuming the different types of demand (viz. linear type, polynomial type etc.) dependent on stock level.

Again, in the present competitive market, the demand of an item is dependent on the selling price of it. Actually the selling price of an item is one of the decisive factors in selecting an item for use. It is common practice that the higher selling price causes the decrease in demand whereas the lower selling price has reverse effect. Hence, it can be conclude that the demand of an item is a function of selling price. Generally, this type of demand is seen for furnished goods. Incorporating the effects of selling price on demand, several researchers have developed the different types of inventory models. Whitin [40] was the first to develop this type of model. Since then, Abad [1], Lee and Rosenblett [21], Kunreuther and Richard [18], Subramanayam and Kumaraswamy [36], Urban [38] and the others investigated the inventory models taking price dependent demand for non-deteriorating items. Cohen [9], Mukherjee [32] also developed the deteriorating inventory model considering the price dependent demand. Recently, Bhunia and Maiti [7] developed an inventory model for deteriorating item with infinite rate of replenishment. In this model, demand is assumed to be dependent on selling price of an item, time and the frequency of advertisement.

Bellman and Zadeh [3] first introduced fuzzy set theory in decision making processes. Later, Tanaka, et al [37] applied the concepts of fuzzy set decision problems by considering the objectives as fuzzy goals over the α-cuts of a fuzzy constraint set and Zimmermann [41] showed that the classical algorithms can be used to solve multi-objective fuzzy linear programming problems. Fuzzy mathematical programming has been applied to several fields, for instance, project network, reliability optimization, transportation problem, media selection for advertising, air pollution regulation, etc. (Ref. Lai and Hwang [19,20]). However, it has not been much used in inventory models. Sommer [35] applied fuzzy dynamic programming to an inventory and production scheduling problem. Kacprzyk and Staniewski [17] considered a fuzzy inventory problem in which, instead of minimizing the total average cost, they reduced it to a multi-stage fuzzy-decision making problem and solved by branch and bound algorithm. Park [29] examined the EOQ formula in the fuzzy set theoretic perspective associating the fuzziness with cost data. Recently, Roy and Maiti [30] solved the classical EOQ model in fuzzy environment with fuzzy goal, fuzzy inventory cost and storage area by fuzzy non-linear programming method using different types of membership functions for inventory parameters. They [31] also examined the fuzzy EOQ model with demand dependent unit price and imprecise storage area by fuzzy non-linear programming method.

In this paper, a profit maximization multi-item inventory model is developed with two separate storage facilities. The storage capacity of OW is limited and that of RW is unlimited. Here, the items at RW are transferred to OW for sale. Demands are deterministic functions of selling price and inventory level at OW. Under the imprecise investment cost and imprecise average inventory the multi-item inventory model is formulated in fuzzy environment. Here, fuzzy parameters are represented by linear membership functions and after fuzzification, it is solved by fuzzy non-linear programming method. Both crisp and fuzzy models are illustrated numerically and their results are compared.

ASSUMPTIONS AND NOTATIONS

We use the following notations in proposed model :

m = number of items,
R = total investment cost for replenishment,
B = average inventory.

For i-th item (i = 1,2, ..., m)

$q_i(t)$ = stock level at time t,
S_i = the highest stock level,
$D_i(p_i, q_i)$ = demand rate (function of cost price and stock level),
W_i = capacity in OW, i.e. $W = \sum_{i=1}^{m} W_i$,
H_i = inventory holding cost per unit item in OW
F_i ($F_i > H_i$) = inventory holding cost per unit item in RW,
c_i = price per unit item,
p_i = selling price per unit,
C_{3i} = replenishment cost per cycle,
C_{ti} = transportation cost for transferring the items from RW to OW,

The basis assumptions of the model are as follows :

(i) Replenishment is instantaneous,
(ii) Lead time is zero,
(iii) Shortages are not allowed,
(iv) Demand $D_i(p_i, q_i)$ is a deterministic function of selling price p_i and the inventory level $q_i(t)$ in the own warehouse such that $D_i(p_i, q_i(t)) = a_i - b_i p_i + d_i q_i(t)$, $a_i, b_i, d_i > 0$,
(v) Selling price p_i per unit is determined by a mark-up rate r_i over unit purchase price p_i, i.e. $p_i = r_i c_i$, $r_i > 1$,
(vi) The items of RW are transferred to OW in n_i shipment, of which K_i ($K_i < W_i$) are transferred in each shipment,

(vii) The total storage capacity of OW is W and that of RW is unlimited,

(viii) The transportation cost of K_i units from RW to OW in each shipment is $a'_i + b'_i(K_i - P_i)$, where P_i ($<K_i$) is the maximum number of units which can be transported under a fixed charge a_i and for every additional unit after P_i, a fixed charge b_i is paid in each shipment.

MODEL DESCRIPTION AND ANALYSIS

In the development of the proposed model, we assume that a retailer purchases $S_i(S_i > W_i)$ units of i-th item of which W_i units are kept in OW and $(S_i - W_i)$ units are kept in RW. The demands are met using the stocks of OW until the stock level drops to $(W_i - K_i)$ units at the end of $t'_{i,1}$. At this stage, K_i ($K_i < W_i$) units are transported from RW to OW. As a result, the stock level of OW again becomes W_i and the stock of OW are used to meet the further demand. This process is continued until the stock in RW is fully exhausted. After the last shipment, only W_i units are used to satisfy the demand during the interval $[t'_{i,n}, T_i]$. A pictorial representation of the system is given in Fig. - 1.

The inventory level $q_i(t)$ for the i-th item in OW at time $t(t'_{i,j-1} < t < t'_{i,j})$ satisfies the differential equation

$$dq_i(t)/dt + D(p_i,q_i) = 0,\ i = 1, 2, 3 \ldots, m,\ J = 1, 2, \ldots, n_i \qquad \ldots(3.1)$$

subject to the condition that

$$q_i(t) = W_i \text{ at } t = t'_{i,j-1}, \quad \text{and } q_i(t) = W_i - K_i \text{ at } t = t'_{i,j}. \qquad \ldots(3.2)$$

Again, the inventory level $q_i(t)$ in OW at time t ($t'_{i,n_i} < t < T_i$) satisfies the differential equation

$$dq_i(t)/dt + D(p_i,q_i) = 0,\ i = 1, 2, 3 \ldots, m, \qquad \ldots(3.3)$$

subject to the condition that

$$q_i(t) = W_i \text{ at } t = t'_{i,n_i}, \text{ and } q_i(t) = 0 \text{ at } t = T_i. \qquad \ldots(3.4)$$

The relation between S_i and K_i is given by

$$S_i = n_i K_i + W_i \qquad \ldots(3.5)$$

Since the demand rate is a function of the stock-level of existing (own) warehouse, the total time period (cycle length) of i-th item would depend on the instantaneous stock-level of the exiting warehouse.

$$T_i = n_i \int_{W_i - K_i}^{W_i} \frac{dq_i}{D(p_i, q_i)} + \int_0^{W_i} \frac{dq_i}{D(p_i, q_i)} \qquad \ldots(3.6)$$

The time between (j – 1)-th and j-th shipment $t_{i,j}$ of i-th item is given by

$$t_{i,j} = t'_{i,j} - t'_{i,j-1} = \int_{W_i-K_i}^{W_i} \frac{dq_i}{D(p_i, q_i)} \qquad j = 1, 2, 3, ..., n_i \qquad ....(3.7)$$

The holding cost of i-th item in RW is

$$\sum_{j=1}^{n_i} F_i (n_i - j + 1) K_i t_{i,j}, \quad i = 1, 2, ..., m. \qquad ....(3.8)$$

Between (j-1)-th and j-th shipments, K_i units of i-th item in OW used to meet the demand for a period $t'_{i,j}$. So the holding cost for these items in OW is

$$H_i \int_{W_i-K_i}^{W_i} \frac{dq_i}{D(p_i, q_i)} \qquad ....(3.9)$$

The total holding cost in OW for these items is

$$n_i H_i \int_{W_i-K_i}^{W_i} \frac{dq_i}{D(p_i, q_i)} \qquad ....(3.10)$$

A quantity of $(W_i - K_i)$ units of i-th item is kept unused in OW for a period of t_{ij}, the holding cost in OW for the quantities is

$$H_i (W_i - K_i) \sum_{j=1}^{n_i} t_{i,j} \qquad ....(3.11)$$

When the last shipment K_i of i-th item arrives in OW, the on hand inventory in OW becomes W_i, the holding cost for these units during usage in OW is

$$H_i \int_0^{W_i} \frac{q_i dq_i}{D(p_i, q_i)} \qquad ....(3.12)$$

$$C_{it} = n_i[a'_i + b'_i(K_i - P_i)] \qquad \text{when } K_i > P_i$$
$$= n_i a'_i \qquad \text{when } K_i \le P_i$$

The total cost for i-th item [i=1,2,....,m] item of the system becomes

$$TC_i = A_i \sum_{j=1}^{n_i} F_i (n_i - j + 1)K_i t_{i,j} + n_i H_i \int_{W_i - K_i}^{W_i} \frac{q_i dq_i}{D(p_i, q_i)} + H_i(W_i - K_i) \sum_{j=1}^{n_i} t_{i,j}$$

$$+ H_i \int_0^{W_i} \frac{q_i dq_i}{D(p_i, q_i)} + C_{ii} \quad \text{....(3.13)}$$

Hence the average profit for i-th item of the system

$$Z_i = [(p_i - c_i)S_i - TC_i]/T_i \quad \text{....(3.14)}$$

Therefore, the total profit per unit time is given by

$$Z(S_1, S_2, ..., S_m) = \sum_{i=1}^{m} Z_i \quad \text{....(3.15)}$$

MATHEMATICAL FORMULATION OF THE CRISP MODEL

In this case, if R and B be the upper limits of investment and average number of units in inventory, then the problem finding the optimal policy $\{S_i\}$ subject to the restriction on both investment and average inventory available can be stated as follows :

$$\text{Maximise} Z(S_1, S_2, ..., S_m) \quad \text{....(4.1)}$$

subject to

$$\sum_{i=1}^{m} c_i S_i \leq R$$

$$\sum_{i=1}^{m} \frac{S_i}{2} \leq B$$

and $S_i > W_i$, $i = 1, 2, ..., m$.

MATHEMATICAL FORMULATION OF THE FUZZY MODEL

When the above profit goal, total investment and average inventory become fuzzy, then the above crisp model is transformed to

$$\text{Maximise}/(S_1, S_2, ..., S_m) \quad \text{....(5.1)}$$

Subject to

$$\sum_{i=1}^{m} c_i S_i \le \tilde{R}$$

$$\sum_{i=1}^{m} \frac{S_i}{2} \le \tilde{B}$$

and $S_i > W_i$, $i = 1, 2, \ldots, m$.

(here wave bar (~) represents the fuzzy characterisation).

In this fuzzy model, the fuzzy objective of the profit goal, fuzzy investment cost constraint and average inventory constraint are represented by their membership functions μ_Z, μ_R, μ_B which are assumed to be non-decreasing and non-increasing continuous linear membership functions as follows :

$$\mu_Z = \begin{cases} 1 \quad , & \text{for } Z > Z_0 \\ 1 - \dfrac{Z_0 - Z}{P_Z} \quad , & \text{for } Z_0 - P_Z \le Z \le Z_0 \\ 0 \quad , & \text{for } Z < Z_0 - P_Z \end{cases}$$

$$\mu_R = \begin{cases} 1 \quad , & \text{for } \sum_{i=1}^{m} c_i S_i < R \\ 1 - \dfrac{\sum_{i=1}^{m} c_i S_i - R}{P_R} \quad , & \text{for } R \le \sum_{i=1}^{m} c_i S_i \le R + P_R \\ 0 \quad , & \text{for } \sum_{i=1}^{m} c_i S_i > R + P_R \text{ ,} \end{cases}$$

and $$\mu_B = \begin{cases} 1 \quad , & \text{for } \sum_{i=1}^{m} \frac{S_i}{2} < B \\ 1 - \dfrac{\sum_{i=1}^{m} \frac{S_i}{2} - B}{P_B} \quad , & \text{for } B \le \sum_{i=1}^{m} \frac{S_i}{2} \le B + P_B \\ 0 \quad . & \text{for } \sum_{i=1}^{m} \frac{S_i}{2} > B + P_B \text{ .} \end{cases}$$

Here, objective goal, investment constraint and average inventory constraint are respectively Z_0, R and B having their respective tolerances P_Z , P_R and P_B.

By FNLP method, the proposed fuzzy model reduces to

Maximize α ....(5.2)

Subject to

$$Z > Z_0 - (1-\alpha)P_Z ,$$

$$\sum_{i=1}^{m} c_i S_i \le R + (1-\alpha)P_R ,$$

$$\sum_{i=1}^{m} \frac{S_i}{2} \le B + (1-\alpha)P_B ,$$

$$0 \le \alpha \le 1 ,$$

and $S_i > W_i$, $i = 1, 2, ..., m.$

OPTIMAL SOLUTION PROCEDURE

The following algorithm can be used to find the optimal ordering policy.

(a) Crisp Model :

Step-1 : Find the unconstrained optimal solution. This is obtained by the necessary condition that all first partial derivatives equal to zero, that is $\partial Z/\partial S_i = 0$, which gives the optimal order quantity $\{S_i^*\}$

If these $\{S_i^*\}$ satisfy both the constraints of (16). This is the optimal policy. Otherwise go to the next step.

Step-2 : Find the solution considering only the investment constraint using Lagrange methods.

Here the Lagrange function is

$$L(S_1, S_2, ..., S_m, \lambda) = Z(S_1, S_2, ..., S_m) + \lambda\left(R - \sum_{i=1}^{m} c_i S_i\right) \quad ...(6.1)$$

where λ is the Lagrange multiplier.

The necessary conditions for optimal solution are

$$\partial L/\partial S_i = 0 \text{ for } i = 1, 2, ..., m.$$

$$\partial L/\partial \lambda = 0 \quad ...(6.2)$$

If this solution satisfies the average inventory constraints, then stop. This is the optimal policy. Otherwise go to the next step.

Step-3 : Find the solution considering only the average inventory constraints and use the Lagrange method.

Here,

$$L(S_1, S_2, \ldots., S_m, \mu) = Z(S_1, S_2, \ldots., S_m) + \mu\left(B - \sum_{i=1}^{m} S_i / 2\right) \qquad \ldots(6.3)$$

where μ is the Lagrange multiplier.

The necessary conditions for optimal solution are

$$\partial L/\partial S_i = 0 \text{ for } i = 1, 2, \ldots., m.$$

$$\partial L/\partial \mu = 0 \qquad \ldots(6.4)$$

If this solution satisfies the investment constraint, then stop. This is the optimal policy. Otherwise go to the next step.

Step-4 : Find the solution considering both the average inventory constraint and investment constraint and use the Lagrange method.

Here the Lagrange function is

$$L(S_1, S_2, \ldots, S_m, \lambda, \mu) = Z(S_1, S_2, \ldots, S_m) + \lambda\left(R - \sum_{i=1}^{m} c_i S_i\right) + \mu\left(B - \sum_{i=1}^{m} \frac{S_i}{2}\right) \qquad \ldots(6.5)$$

The optimal solution satisfies :

$$\partial L/\partial S_i = 0 \text{ for } i = 1, 2, \ldots, m.$$

$$\partial L/\partial \lambda = 0$$

$$\partial L/\partial \mu = 0 \qquad \ldots(6.6)$$

This is the optimal policy.

(b) **Fuzzy Model :**

The objective function along with the constraints is optimized using an gradient based optimization algorithm --- generalised reduced gradient (GRG) method. The GRG method is based on the idea of elimination of variables using the equality constraints. Here, the change in the objective function is made using the generalised reduced gradient.

NUMERICAL RESULTS

To illustrate the inventory model, the parametric values for two items are taken in Table -1 as follows.

Table - 1 : Values of input parameters

Items	C_{3i} (\$)	F_i (\$)	H_i (\$)	W_i (\$)	c_i (\$)	r_i	a_i	b_i	d_i	a'_i	b'_i	p_i
1	300	2	1	100	12	1.5	50	0.2	3	12	0.2	20
2	350	2.5	1.5	125	16	1.45	75	0.25	3.5	10	0.25	25

R = \$2874, B = 260.

Optimal result for crisp model

Step - 1 :

The unconstrained solution for the system is

Table - 2

Profit (\$)	T_1	T_2	K_1	K_2	n_1	n_2	S_1	S_2	R	B
174.41	1.3289	1.2058	45.97	55.58	4	3	283.86	291.75	3169.81	287.81

This solution violates both the investment and average inventory constraints.
Hence we proceed step -2.

Step - 2 :

Now, we find the solution considering the investment constraint. The solution is

Table - 3

Profit ($)	T_1	T_2	K_1	K_2	n_1	n_2	S_1	S_2	R	B
166.66	1.2653	1.1024	42.53	64.44	4	2	270.13	253.89	2874.00	262.01

This solution violates the average inventory constraint.
Therefore we go to step - 3.

Step - 3 :

Now, we find the solution considering the average inventory constraint. The solution is

Table - 4

Profit ($)	T_1	T_2	K_1	K_2	n_1	n_2	S_1	S_2	R	B
166.42	1.1996	1.1363	48.62	49.71	3	3	245.87	274.13	2875.13	260.00

This solution violates the investment constraint
Hence we proceed the next step.

Step - 4 :

We find the solution considering both the investment and average inventory constraints. The solution is

Table - 5

Profit ($)	T_1	T_2	K_1	K_2	n_1	n_2	S_1	S_2	R	B
166.43	1.2011	1.1358	48.67	49.67	3	3	246.01	274.11	2874.00	260.00

This is the optimal policy.

Optimal result for fuzzy model

For fuzzy model, we consider the input data shown in Table - 1 alongwith the following fuzzy data :

$\tilde{Z} = (\$140, \$200)$, $\tilde{R} = (\$2874, \$3474)$, $\tilde{B} = (260, 320)$. For these data, the optimum result of the fuzzy model (17) is given in Table - 6.

Table - 6

α	Profit ($)	T_1	T_2	K_1	K_2	n_1	n_2	S_1	S_2	R	B
0.5711	174.27	1.3133	1.1911	45.14	54.36	4	3	280.55	288.09	3131.32	284.32

From the result of the table -6, it is observed that the fuzzy result is better than the crisp one.

Concluding Remarks

In the present paper, we have formulated and solved a fuzzy multi-item inventory model for two warehouses considering the limitation on investment and average inventory. The demand is taken to be linearly dependent on the selling price and current stock level. This feature of the model makes it the most general among the models referred to the introduction.

During formulation, two cases arise (i) $S_i \leq W_i$ and (ii) $S_i > W_i$. If $S_i \leq W_i$, the problem reduces to a single storage problem. We have rejected this condition as our models are two storage models. For the other case, $S_i > W_i$, the results of our models are feasible only when $K_i \leq W_i$.

A future study should incorporate more realistic assumptions into the proposed model, such as finite time horizon, shortages, all unit discount (AUD) and incremental quantity discount (IQD) etc. Moreover, present model can also be formulated in other environments such as probabilistic or mixed environments taking one or more system parameters as fuzzy and / or probabilistic.

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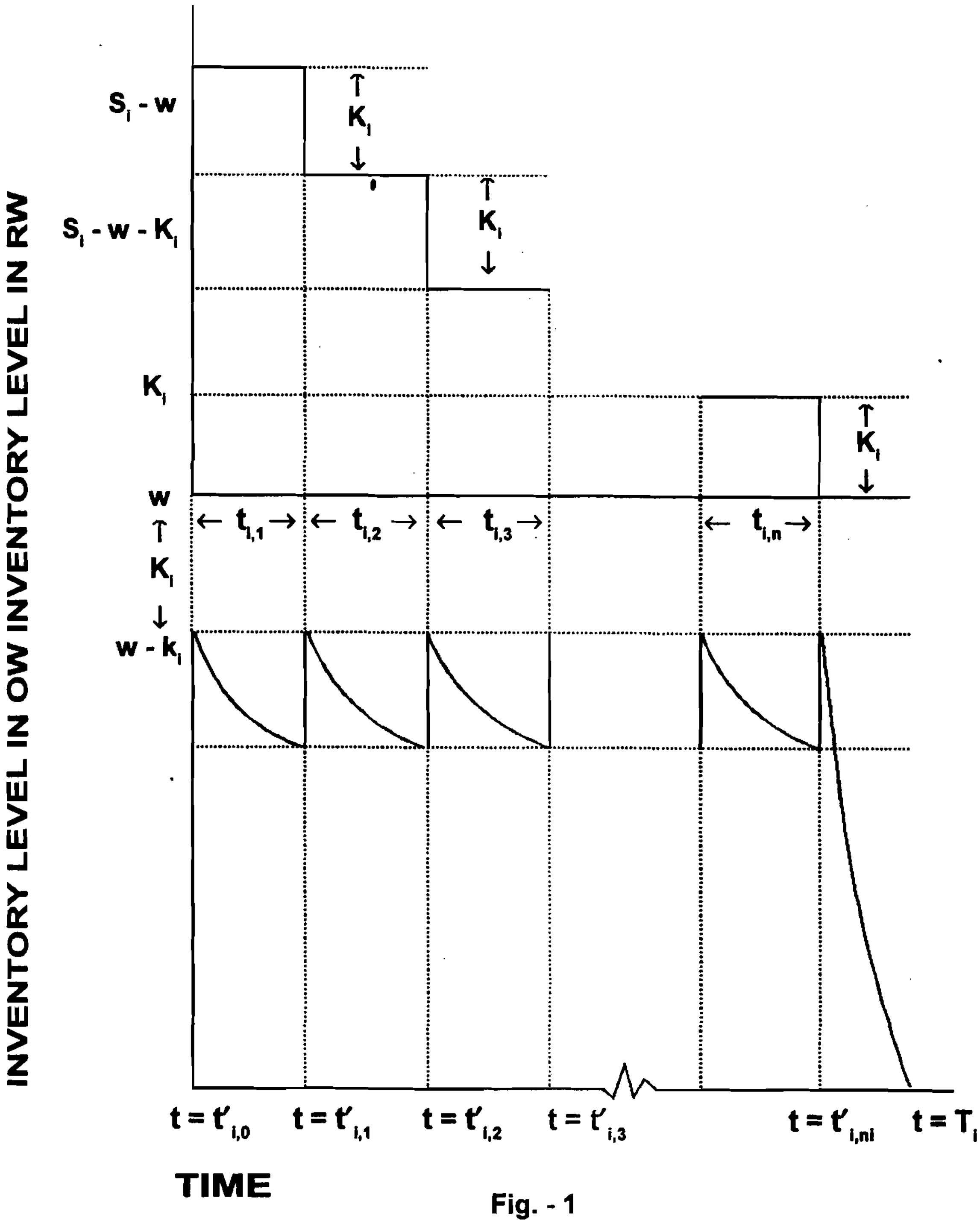

INVENTORY LEVEL IN OW INVENTORY LEVEL IN RW
S_i - w
S_i - w - K_i
K_i
w
K_i
w - k_i
t_i,1
t_i,2
t_i,3
t_i,n
t = t'_i,0
t = t'_i,1
t = t'_i,2
t = t'_i,3
t = t'_i,ni
t = T_i
TIME

Fig. - 1

SOME COMMON FIXED POINT THEOREMS IN METRIC SPACES -II

Kalishankar Tiwary and S. C. Debnath
Prafulla Chandra College
(City College South Calcutta)
23/49 Gariahat Road, Calcutta - 29

Abstract :
In this paper two theorems have been proved. First theorem deals with common fixed point for three mappings and the other deals with a Sequence of mappings. These theorems extend the results proved in {[4],[5],[8-9]}.

INTRODUCTION

Sastry et. al. [8] recently proved a fixed point theorem which extends the theorems proved by Khan, Swaleh and Sessa [4], Pathak and Sharma [5, Bohr and Ram etc. Sastry et. al. [8] noted that the inequality used in Khan et. al. [4] is not true for x = y because the term a(0) used therein is not defined. They further exhibited one example to show that the uniqueness part of the theorem of Pathak and Sharma [5] fails.

Let R^+ denote the set of all non-negative reals, N the set of natural numbers and Φ the set of all continuous self-maps ϕ of R^+ such that ϕ is monotonically increasing and $\phi(t) = 0$ if and only if $t = 0$. Using this function and altering the distance between the points Sastry et. al. [8] proved the following theorem.

Theorem 1 : Let (X,d) be complete metric Space. Assume that there exist non-negative constants a, b, c in [0,1] such that

$$\phi[d(Tx,Ty)] \leq a\,\phi(d(x,y)) + c[\phi(d(x,Ty)).\phi(d(y,Tx))]^{\frac{1}{2}} + \frac{b}{2}[\phi(d(x,Tx)) + \phi(d(y,Ty))]$$

for all $x, y \in X$. Then

(i) For $x \in X$, the sequence $\{T^n x\}$ converges to a fixed point of T if $a+b<1$.

(ii) T has almost one fixed point in X if $a+c<1$.

Sessa [10] weakend the notion of commutativity of two mappings by weakly commutativity in the following way.

Definition 1 : Two mappings $f, g : X \to X$ where X is a metric space, are said to be weakly commutative if and only if $d(fgx, gfx) \leq d(fx, gx)$ for all $x \in X$.

Every commutative pair is weakly commutative but the converse is not true [10].

In the subsequent Jungck [2] extended weakly commutativity by compatibility and showed that every weakly, commutativity pair is compatible but the converse is not true.

Definition 2 : Two mappings f, g : X → X where X is a metric space, are said to be compatible if and only if whenever $\{x_n\}$ is a sequence in X such that

$$fx_n,\ gx_n \to t \text{ then } d(fgx_n,\ gfx_n) \to 0.$$

The following proposition is collected from Jungck and Rhoades [3].

Proposition A : Let f, g be compatible self maps of a metric space (X,d).

1. If f(t) = g (t) then fg(t) = gf(t).
2. Suppose that $\lim fx_n = \lim gx_n = t$ for some $t \in X$ and $x_n \in X$. Then
   a) if f is continuous at t, $\lim gfx_n = f(t)$
   a) if f and g are continuous at t then f(t) = g (t) and fg(t) = gf(t).

We have proved the following theorems.

Theorem 2 : Let (X,d) be a complete metric space. Let f,g and h be self maps on X. Assume that there exist non-negative constants a,b,c in [0,1] such that

(A) $\phi(d(fx,gy)) \le a\phi(d(hx,hy)) + c[\phi(d(hx,gy)).\phi(d(hy,fx))]^{\frac{1}{2}} + \frac{b}{2}[\phi(d(hx,fx)) + \phi[d(hy,gy))]$

for all $x, y \in X$,

(B) $f(X) \cup g(X) \subset h(X)$,

(C) The pairs f, h and g, h are compatible and h is continuous. Then

(i) for $x = x_0 \in X$, the sequence $\{x_n\}$, defined by

$fx_{2n} = hx_{2n+1}$ and $gx_{2n+1} = hx_{2n+2}$, n = 0,1,2 ....

converges to a common fixed point of f,g and h if a+b<1 and a+c<1.

ii) f, g, h have almost unique common fixed point in X.

Proof : Let $x = x_0$ be an arbitrary element of X. Since $f(X) \cup g(X) \subset h(X)$, an element $x_1 \in X$ can be so selected that $fx_0 = hx_1$, and since $g(X) \subset h(X)$, an element $x_2 \in X$ can be selected such that $gx_1 = hx_2$. In this way we can construct a sequence $\{x_n\}$ in X so that

$fx_{2n} = hx_{2n+1}$ and $gx_{2n+1} = hx_{2n+2}$, n = 0,1,2....

We define $\alpha_n = d(hx_n, hx_{n+1})$ for n =0,1,2...

and $\beta_n = \phi(\alpha_n)$. Then we have

$$\beta_{2n+1} = \phi(\alpha_{2n+1}) = \phi(d(hx_{2n+1}, hx_{2n+2})) = \phi(d(fx_{2n}, gx_{2n+1}))$$

$$\le \phi(d(hx_{2n}, hx_{2n+1})) + c[\phi(d(hx_{2n}, gx_{2n+1})).\phi(d(hx_{2n+1}, fx_{2n}))]^{\frac{1}{2}}$$

$$+ \frac{b}{2}[\phi(d(hx_{2n}, fx_{2n})) + \phi(d(hx_{2n+1}, gx_{2n+1}))]$$

$$= a\phi(\alpha_{2n}) + c[\phi(d(hx_{2n}, hx_{2n+2})).\phi(d(hx_{2n+1}, hx_{2n+1}))]^{\frac{1}{2}} + \frac{b}{2}[\phi(\alpha_{2n}) + \phi(\alpha_{2n+1})]$$

i.e. $\beta_{2n+1} \leq \frac{2a+b}{2-b}\ \beta_{2n}$ .....(1)

and hence $\beta_{2n+1} < k\ \beta_{2n}$ where $k = \frac{2a+b}{2-b} < 1$, since $a+b<1$.

Hence

$$\beta_n \leq k^n \beta_0, \text{ so that } \beta_n \to 0 \text{ as } n \to \infty.$$

Since

$$\beta_n < \beta_{n-1}, \text{ so that } \alpha_n < \alpha_{n-1} = 1,2,3...$$

Thus

$$\alpha_n \to \alpha \text{ (say)}.$$

Then $\beta_n = \phi(\alpha_n) \to \phi(\alpha)$, since ϕ is continuous.
So that $\phi(\alpha) = 0$ and hence $\alpha = 0$.
Thus $\alpha_n \to 0$. ....(2)

We now show that $\{hx_n\}$ is a cauchy sequence. Then for every positive integer ε and for every positive integer k there exist two positive integers 2m(k) and 2n(k) such that

$$2m(k)>2n(k)>k \text{ and } d(hx_{2m(k)}, hx_{2n(k)}) > \varepsilon$$

Further let 2m(k) be the smallest even integer for which 2m(k)>2n(k)>k,

$$d(hx_{2m(k)}, hx_{2n(k)})>\varepsilon \text{ and } d(hx_{2m(k)-2}, hx_{2n(k)}) \leq \varepsilon.$$

Then

$$\begin{aligned}
\phi(d(hx_{2m(k)}, hx_{2n(k)})) &= \phi(d(fx_{2m(k)-1}, gx_{2n(k)-1})) \\
&\leq a\,\phi(d(hx_{2m(k)-1}, hx_{2n(k)-1})) + c[\phi(d(hx_{2m(k)}, gx_{2n(k)-1})).\phi(d(hx_{2n(k)-1}, hx_{2m(k)}))]^{\frac{1}{2}} \\
&\quad + \frac{b}{2}[\phi(d(hx_{2m(k)-1}, fx_{2m(k)-1})) + \phi(d(hx_{2n(k)-1}, hx_{2n(k)-1}))] \\
&= a\,\phi(d(hx_{2m(k)-1}, hx_{2n(k)-1})) + c[\phi(d(hx_{2m(k)-1}, hx_{2n(k)})).\phi(d(hx_{2n(k)-1}, hx_{2m(k)}))]^{\frac{1}{2}} \\
&\quad + \frac{b}{2}[\phi(d(hx_{2m(k)-1}, hx_{2m(k)})) + \phi(d(hx_{2n(k)-1}, hx_{2n(k)}))]
\end{aligned} \quad ...(3)$$

we have

$$\varepsilon < d(hx_{2n(k)}, hx_{2m(k)}) \leq d(hx_{2n(k)}, hx_{2m(k)-2}) + d(hx_{2m(k)-2}, hx_{2m(k)-1}) + d(hx_{2m(k)-1}, hx_{2n(k)})$$

$$\leq \varepsilon + \alpha_{2m(k)-2} + \alpha_{2m(k)-1},$$

so that from (2),

$$\lim_{k \to \infty} d(hx_{2n(k)}, hx_{2m(k)}) = \varepsilon \quad ...(4)$$

Using the triangle inequality,

$$|d(hx_{2m(k)}, hx_{2n(k)+1}) - d(hx_{2m(k)}, hx_{2n(k)})| \leq \alpha_{2n(k)}$$

$$|d(hx_{2m(k)+1}, hx_{2n(k)+1}) - d(hx_{2m(k)}, hx_{2n(k)+1})| \leq \alpha_{2n(k)}$$

and

$$|d(hx_{2m(k)+1}, hx_{2n(k)+2}) - d(hx_{2m(k)+1}, hx_{2n(k)+1})| \leq \alpha_{2n(k)+1}$$

Using (4) and the above inequalities

$$\begin{aligned} \varepsilon &= \lim_{k \to \infty} d(hx_{2m(k)}, hx_{2n(k)}) \\ &= \lim_{k \to \infty} d(hx_{2m(k)}, hx_{2n(k)}) \\ &= \lim_{k \to \infty} d(hx_{2m(k)}, hx_{2n(k)+2}). \end{aligned}$$

Letting $k \to \infty$ in (3), we get

$$\phi(\varepsilon) \leq a\phi(\varepsilon)+c\phi(\varepsilon) = (a+c)\phi(\varepsilon)<\phi(\varepsilon), \text{ since } a+c<1, \text{ a contradiction.}$$

So, $\{hx_n\}$ is a cauchy sequence.

Since X is a complete metric space, $\{hx_n\}$ converges.

Suppose $u = \lim_{n \to \infty} hx_n$.

Also, $\lim_{n \to \infty} fx_{2n} = \lim_{n \to \infty} gx_{2n+1} = u.$

Since h is continuous, we have

$$\phi(d(fhx_{2n}, gx_{2n+1})) \leq a\phi\, d(hx_{2n}, hx_{2n+1})+c[\phi(d(hhx_{2n}, gx_{2n+1})).\phi\, d(hx_{2n+1}, fhx_{2n}))]^{\frac{1}{2}}$$
$$+\frac{b}{2}[\phi(d(hhx_{2n}, fhx_{2n}))+\phi(d(hx_{2n+1}, gx_{2n+1}))]. \quad \ldots (5)$$

Since f and h are compatible.

$$\lim_{n \to \infty} d(fhx_{2n}, hfx_{2n}) = 0.$$

From (5), when $n \to \infty$, we have

$\phi(d(hu, u) \leq a\phi(d(hu, u) + c\phi(d(hu.u) = (a+c)\phi(d(hu, u)) < \phi(d(hu, u)),$

a contradiction, so hu = u. Now,

$$\phi(d(fu, gx_{2n+1})) \leq a\, \phi(d(hu, hx_{2n+1}))+c[\phi(d(hu, gx_{2n+1})).\phi\, d(hx_{2n+1}, fu))]^{\frac{1}{2}}$$
$$+\frac{b}{2}[\phi(d(hu, fu))+\phi(d(hx_{2n+1}, gx_{2n+1}))].$$

Taking limit as $n \to \infty$, we get

$$\phi(d(fu, u) \leq \frac{b}{2}\phi(d\,(u, fu) < \phi(d(u, fu)$$ and a contradiction.

So, $fu = u$. Thus $hu = fu = u$.
In a similar way it can be shown that $gu = hu = u$. Thus u becomes a common fixed point of f, g and h. It now remains to show that u is unique common fixed point of f, g and h. If possible, let v be another common fixed point of f, g and h. Then

$$\begin{aligned}\phi\,(d(u, v)) = \phi(d(fu,gv)) & \\ & \leq a\phi(d(hu\,,hv))+c[\phi(d(hu, gv)).\phi(d(hv, fu))]^{\frac{1}{2}} \\ & \qquad + \frac{b}{2}[\phi(d(hu, fu))+\phi(d(hv, gv))] \\ & = a\phi(d(u, v)) + c\phi(d(u, v)) \\ & = (a+c)\phi(d(u, v))<\phi(d(u, v)),\end{aligned}$$ a contradiction.

Hence $u = v$ and u is a unique common fixed point f, g and h. This completes the proof of the theorem.

Theorem 3 : Let f, g and h be self maps of a complete metric space which satisfy $f(X) \cup g(X) \subset h(X)$, the pairs f, h, and g, h are compatible. Suppose that f and g are continuous. If the inequality (A) of Theorem 2 holds then the sequence $\{x_n\}$ defined by

$$fx_{2n} = hx_{2n+1} \text{ and } gx_{2n+1} = hx_{2n-2},\ n = 0,1,2,3 \text{ .... and } x_0 \in X$$

converges to a common fixed point of f, g and h and f, g, and h have unique common fixed point in X.

Proof : Following the proof of Theorem 2 we obtain sequence $\{hx_n\}$,$\{fx_{2n}\}$ and $\{gx_{2n-1}\}$ converge to $u \in X$.
Since f is continuous and f and h are compatible,

$$ffx_{2n} \to fu \text{ and } fhx_{2n} = hfx_{2n} \to fu.$$

Similarly g is continuous and g and h are compatible,

$$ggx_{2n-1} \to gu \text{ and } ghx_{2n-1} = hgx_{2n-1} \to gu.$$

Then we have,

$$\begin{aligned}\phi(d(fx_{2n}, hx_{2n+1})) \leq a\phi(d(hfx_{2n}, hx_{2n+1})+c[\phi(d(hfx_{2n}, gx_{2n+1})).\phi(d(hx_{2n+1}, fx_{2n}))]^{\frac{1}{2}} \\ + \frac{b}{2}[\phi(d(hfx_{2n}, ffx_{2n}))+\phi(d(hx_{2n+1},gx_{2n+1}))]\,.\end{aligned}$$

Taking limit as $n \to \infty$,

$\phi(d(fu,u) \leq (a+c)\phi(d(fu,u))<\phi(d(fu,u))$, a contradiction. So $fu = u$.

Similarly, $gu = u$. So $fu = gu = u$.

Since $f(X)\subset h(X)$ and hence there exists a point v in X such that $u = fu = hv$.

Then we have

$$\phi(d(fv, gx_{2n+1}))\leq a\phi(d(hv, hx_{2n+1})+c[\phi(d(hv, gx_{2n+1})).\phi(d(hx_{2n+1}, fv))]^{\frac{1}{2}} + \frac{b}{2}[\phi(d(hv, fv))+\phi(d(hx_{2n+1},gx_{2n+1}))].$$

Taking limit as $n \to \infty$,

$\phi(d(fv,u)) \leq \frac{b}{2}\phi(d(fv,u))<\phi(d(fv,u))$, a contradiction. So $fv = u$.

Since f and h are compatible and $u = fv = hv$, so $fhv = hfv$ and so $fu = fhv = hfv = hu$,

Thus $u = fu = gu = hu$. It follows easily that u is a unique common fixed point of f, g and h.

Remark : If $f = g = T$ and $h = I$, the identity map, the Theorem 2 reduces to the Theorem 1 of Sastry et. al. [8].

Theorem 4 : Let (X,d) be a complete metric space, let f, g and h be self maps on X satisfying for some positive integers m, n and p,

(A) $$\phi[d(f^m x, g^n y)]\leq a\phi(d(h^p x, h^p y)]+c[\phi(d(h^p x, g^n y).\phi(d(h^p y, f^m x))]^{\frac{1}{2}} + \frac{b}{2}[\phi(d(h^p x, f^m x))+\phi[d(h^p y, g^n y))] \quad \text{... (B)}$$

for all $x, y\in X$,

(B) $f^m(X)\cup g^n(X)\subset h^p(X)$,

(C) $fh = hf$, $gh = hg$ and h is continuous, then

(i) For $x = x_0\in X$, the sequence $\{x_n\}$, defined by

$$f^m x_{2n} = h^p x_{2n}$$

and $g^n x_{2n+1} = h^p x_{2n+2}$, $n = 0,1,2 ....$

converges to a common fixed point of f, g and h if $a+b<1$ and $a+c<1$.

(ii) f, g and h have unique common fixed point in X.

Proof : Let $F = f^m$, $G = g^n$ and $H = h^p$. Since $fh = hf$, $gh = hg$, it follows that $FH = HF$, $GH = HG$ and H is continuous. Hence it follows from Theorem 2, that F, G and H have unique common fixed point z, say in X. We now wish to show that z is also unique common fixed point f, g and h. Now,

$$\phi(d(z,hz)) = \phi(d(f^m z, g^n hz)]+a\phi(d(h^p z, h^p hz)]+c[\phi(d(h^p z, g^n hz).\phi(d(h^p hz, f^m z))]^{\frac{1}{2}} + \frac{b}{2}[\phi(d(h^p z, f^m z))+\phi[d(h^p hz, g^n hz))]$$

i.e. $\phi(d(z,hz) \le (a+c)\phi(d(z,hz))<\phi(d(z,hz))$, a contradiction. So hz = z.
Next,

$$\phi(d(fz, gz)) = \phi[d(f^m fz, g^n gz)] = \phi(d(Ffz,Ggz)$$
$$\le a\phi(d(Hfz, Hgz))]+c[\phi(d(Hfz, Ggz).\phi(d(Hgz, Ffz))]^{\frac{1}{2}}$$
$$+\frac{b}{2}[\phi(d(Hfz, Ffz))+\phi[d(Hgz, Ggz))]$$

i.e. $\phi(d(fz,gz)) \le (a+c)\phi(d(fz,gz))<\phi(d(fz,gz))$, a contradiction. So fz = gz.
Similarly, we can show that fz = hz. So fz = gz = z. It follows easily that z is a unique common fixed point of f, g and h.

Note : If m = n = p and f = g = T and h = I, the identity map, the Theorem 4 is a Theorem 1 of Sastry et. al. [9]

Theorem 5 : Let f, g and h be self maps of a complete metric space which satisfy $f^m(X)\cup g^n(X)\subset h^p(X)$, for some positive integers m, n and p and fh = hf = hg. Suppose f and g are continuous. If the inequality (B) of Theorem 4 holds, the sequence $\{x^n\}$ defined in Theorem 4 converges to a common fixed point f, g and h and f, g and h have a unique common fixed point in X.

Proof : Let $F = f^m$, $G = g^n$ and $H = h^p$. Since fh = hf, gh = hg, it follows that FH = HF, GH = HG and hence compatible. Also F and G are continuous. Following the proof of Theorem 3, it follows that F, G and h have unique common fixed point z, say in X. It now remains to show that z is also a unique common fixed point of f, g and h.
Now,

$$\phi(d(fz,z)) = \phi(d(Ffz,Gz))$$
$$\le a\phi(d(Hfz,Hz))]+c[\phi(d(Hfz,Gz).\phi(d(Hz.Ffz))]^{\frac{1}{2}}$$
$$+\frac{b}{2}[\phi(d(Hfz, Ffz))+\phi[d(Hz, Gz))]$$

i.e. $\phi(d(fz,z)) \le (a+c)\phi(d(fz,z))<\phi(d(fz,z))$, a contradiction. So fz = z.
Similarly, we can show that gz = z. So, fz = gz = z.
Since $F(X)\subset H(x)$ and hence there exists a point u in X that z = fz = hu.
Then we have

$$\phi(d(fu,z)) = \phi(d(Ffu,Gz))\le a\phi(d(Hfu,Hz))]+c[\phi(d(Hfu,Gz).\phi(d(Hz,Ffu))]^{\frac{1}{2}}$$
$$+\frac{b}{2}[\phi(d(Hfu, Ffu))+\phi[d(Hz, Gz))]$$

i.e. $\phi(d(fu,z)) \le (a+c)\phi(d(fu,z)) < \phi(d(fu,z))$, a contradiction. So $fu = z$.
So, $z = fu = hu$ and so $fz = fhu = hfu = hz$. Thus $z = fz = gz = hz$.
It follows easily that z is unique common fixed point of f, g and h.

A common fixed point for a sequence of mappings

Theorem 6 : Let (X,d) be a complete metric space and let $\{F_i\}$ be a sequence self maps satisfying

(A) $$\phi(d(F_i x,F_j y)] \le a\phi[d(hx,hy)]+c[\phi(d(hx,F_j y).\phi(d(hy,F_i x))]^{\frac{1}{2}} + \frac{b}{2}[\phi(d(hx,F_i x))+\phi(d(hy,F_j y))]$$

for all $x, y \in X$,

(B) $F_i(X) \subset h(X)$, $i = 1,2,3...$

(C) Each F_i is compatible with h.

If one of the mappings F_i, $i = 1,2,3...$ and h be continuous then all the mappings F_i, $i = 1,2,3$ .... and h have unique common fixed point in X, provided $a+b<1$ and $a+c<1$.

Proof : Let us arbitrary select a pair of mappings F_i and F_j. Let x_0 be any point in X. Define sequences $\{x_n\}$ and $\{y_n\}$ in X given by the rule $y_{2n} = F_i x_{2n} = hx_{2n+1}$, and $y_{2n+1} = F_i x_{2n+1} = hx_{2n+2}$.
This can be done by virtue of (B).

If $F_i x_{2n} = F_j x_{2n+1}$ or $F_j x_{2n+1} = F_i x_{2n+2}$ for some value of n, the existence of fixed point is easily established. So let us assume that $F_i x_{2n} \ne F_j x_{2n+1}$ or $F_j x_{2n+1} \ne F_i x_{2n+2}$ for every value of n. Let $\beta_{2n} = \phi(d(hx_n,hx_{n+1}))$, $n = 1,2,3...$

Following the proof of the Theorem 2, u is a common fixed point of F_i,F_j and h.

Uniqueness of the common fixed point follows easily. This completes the proof of the theorem.

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THE ASIAN BROWN CLOUD CONTROVERSY

Ram Kumar Bhakat
Department of Botany & Forestry
Vidyasagar University
Midnapore - 721 102, India

The wide publicity given to the release of a United Nations Environment Programme (UNEP) report on the so-called Asian Brown Cloud and its multifarious impacts on health, agriculture and climate on both regional and global scales, has led to considerable concern. We find that the UNEP news release (and hence the media reports based on it) is a blend of observations and scientifically sound deductions on the one hand and sensational statements with little scientific basis on the other. The UNEP report is based on the findings of an international programme called the Indian Ocean Experiment (INDOEX). The term Asian Brown Cloud was coined by leaders of INDOEX to describe the brown haze occurring during the period January to March, over the South Asian region and the tropical Indian Ocean, Arabian Sea and Bay of Bengal. It is important to note that, the haze is not a permanent feature of the atmosphere over the Asian region and the surrounding seas. It occurs only during January - March, in the season following the southwest monsoon and northeast monsoon seasons.

It is suggested in the UNEP report that the impact of the haze assessed with the help of an atmospheric general circulation model is a decrease in rainfall in northwest Asia (including Saudi Arabia, Pakistan, Afghanistan). However, we find that the model simulation of the rainfall patterns over this region is particularly poor and hence the reliability of this projection is suspect. Also, the expected magnitude of the impact on crop yields is small and there is no basis for the statement in the UNEP news release that the 'vast blanket of pollution across South Asia is damaging agriculture'.

The above mentioned sentences are the executive summary of a research paper published in the recent issue of highly esteemed Indian journal[1] (Current Science, Vol. 83, 586 - 592, 2002) criticising the UNEP report and the subsequent press release in August 2002 on "Asian Brown Cloud"[2-3] which highlight: "A vast blanket of pollution stretching across South Asia is damaging agriculture, modifying rainfall patterns including those of the mighty Monsoon and putting hundreds of thousands of people at risk". The authors (of this paper), based on their own findings and others, then conclude saying, "There are major inconsistencies between the alleged adverse effects on the monsoon and agriculture mentioned in the news release, and the results of scientific studies discussed in the UNEP report. It is difficult to attribute the large discrepancies to oversight. It is important for the people in the Asian region to understand that implications in the UNEP news release, of immediate and catastrophic consequences of the haze on the monsoon and agriculture, have no specific basis".

The press release quotes the Executive Director of UNEP as saying that "the haze is the result of forest fires, the burning of agricultural wastes, dramatic increases in the burning of fossil fuels in vehicles, industries and power stations and emissions from millions of inefficient cookers, burning wood, cowdung and other biofuels".[4] The UNEP official is further quoted to go on adding an alarmist and striking note as saying : "There are also global implications not least because a pollution parcel like this, which stretches three kilometers high, can travel half way round the globe in a week." The release of this report on the eve of (barely a fortnight ago) the World Summit on Sustainable Development, held during August 26 - September 4, 2002 seems to be more politically driven than based on scientific facts. Upon receiving the report, the Union Government of India met a cross-section of Indian scientist to discuss the matter following which the government outrightly rejected the UNEP report and emphasised that there is no scientific truth for the alarmist proclamation by the Executive Director. There is neither evidence in the UNEP report that the haze leads to the disruption of the weather system or monsoons nor any scientific basis from observations for attributing the droughts over the western part of the Asian continent to the haze. The haze described in the Indian Ocean Experiment (INDOEX) is characteristic of Asian region and is observed over other regions like Europe, North America and East Asia too. However, neither the INDOEX nor the UNEP report deals with the wet monsoon period during June to September (the south-west monsoon) and October to November (north-east monsoon).

As such, the results of the report and the numerical experiments conducted cannot be applied to other seasons and drastic conclusions about the disruption of weather patterns, and floods and droughts being caused by haze are unfounded.

The rejection on the Indian part stems largely from the politics that dictates, though in an unsubstantiated manner, to link the Asian Brown Cloud, a relatively localised pollution episode, to serious local and global changes in weather patters, even long-term climatic repercussion. The denial comes not from the fact that there is no pollution over Asian countries nor it is the case of 'everybody's business is nobody's business'. The rejection also appears to stem from the statements of a research paper summarising the results of INDOEX published in the *Science* journal in the early 2001 which states, "Unless international control measures are taken, air pollution in the Northern Hemisphere will continue into a global plume across the developed and developing world."[5] And it is this premise that the UNEP report too attempts to establish despite limitations.[6] Rather than advocating international agreements to control pollution, the attitude appears to be of imposing an asymmetric control regime. The appearance of this well-documented study, authored by an international group of scientists including Indians, although appears to be a major contribution to the pollution problems of the Indian Oceanic region, but on closer scrutiny it assumes a new dimension following the news release of UNEP this year. Herefrom, the political jugglery sets in.

Understanding the factors that damage our environment is a legitimate concern of science and scientists. It is clear that the news release has created awareness about pollution. This should be reflected in our policy programmes. People living in Asia must be conscious about this haze because of adverse impacts on health. Similarly, development of technological infrastructure to minimise and contain the effects of environmental pollution is urgently needed. But this type of efforts acquire complex dimensions because of the interplay of local and global politics on environments. The issue of global warming is an excellent example in this direction. Discussions on greenhouse effect began a century ago, but the international negotiation to mitigate this is muddled every time by the political plays among the countries. The global politics of environment clearly involves complex issues of national self-interest and global equity. Even within India, the Cauvery water dispute is a case in point. Uneven development in India leads to stark disparities. The rejection by the United States of Kyoto Protocol on climate change which mandated reduction of carbon emissions by developed countries was perceived as a major threat to U.S. economy and a way of life. Therefore preaching austerity and self-restraint to the economically powerful hardly finds a niche in the discourses of international environmental negotiations.

The international politics of environment can be complex and seems to be a labyrinthine to ordinary citizens. The science of ozone depletion was published in 1970 but international agreement to restrict the use of ozone-depleting substances was negotiated only in 1987. From Stockholm to Rio to Johannesburg, international negotiations with preconceived political agendas frequently circumambulate the globe. But viewed in the context of World Summit on Sustainable Development, there is no denying the fact that the UNEP press release of a three-year old research finding is motivated more by politics than science. It is this axiom why the Indian scientists are becoming sceptic to an overall research finding commissioned by an internationally reputed agency.

The UNEP press release on Asian Brown Cloud alias Asian Brown Haze and the subsequent statements reported verbatim in the media are based on facts and fantasy. It is a blend of scientific deductions on the one hand and exaggerated and sensational remarks on the other. The presence of the blanket of haze belongs to the first category. But referring the haze as cloud (perhaps this terminology was chosen to convey an impression of the Asian countries choking under a thick cocktail of pollutions) and the reported impacts of the haze on monsoon, climate and agricultural productivity which are unsubstantiated and therefore need further scientific scrutiny seem unwarranted. Therefore, there is little doubt that politics has taken precedence over science.

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