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On the Exponential Diophantine Equation $27^x - 11^y = z^2$

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ABSTRACT

In this article, we study and establish the theorem of the exponential Diophantine equation $27^x - 11^y = z^2$, which has exactly two solutions where x, y and z are non-negative integers using Catalan's conjecture, modular arithmetic, factorial method and elementary mathematical concepts.

Keywords: Diophantine equation; Factoring method; Modular arithmetic method; Integer Solution; Catalan's Conjecture; Non-linear Equation.

AMS Mathematics Subject Classification (2010): 11D61

1. Introduction

A polynomial equation that takes only integer values known as a Diophantine equation. A large number of mathematicians have studied various forms of Diophantine equations in the last couple of decades and it is considered as one of the significant problems in the elementary number theory and the algebraic number theory. The most general form of the Diophantine equation $a^x + b^y = z^2$, have been studied by many mathematicians [5-12]. In 1844, the famous mathematician Eugne Charles Catalan have studied and suggested that the only one solution in natural number of $a^x - b^y = 1$ with is $(a, b, x, y) = (3, 2, 2, 3)$. Interestingly, in 2004, the Catalan conjecture was proved by Mihailescu [3]. The Diophantine equation [9-12] of the form $a^x - b^y = z^2$ play an important role for the study of no-negative integer solutions and this work has motivated me to investigate for the non-negative integer solutions of the exponential Diophantine equation $27^x - 11^y = z^2$. Just a while ago, Chikkavarapu [13] have generalized the exponential Diophantine equation $23^x - 19^y = z^2$. Buosi et al. [10] also have investigated the exponential Diophantine equation $p^x - 2^y = z^2$ with $p = k^2 + 2$ a prime number of the integer solutions. I refer the reader to [9, 11, 12] for the non-negative integer solutions.

In 2007, Acu [5] have studied about the exponential Diophantine $2^x + 5^y = z^2$ with two non-negative integer solutions such as $(x, y, z) = (3,0,3)$ and $(x, y, z) = (2,1,3)$. In this article, the work of Chikkavarapu [13] play an important role to proof the individual result of exponential Diophantine equation.

2. Primary results

It is very difficult to decide whether a Diophantine equation has any solutions, no solutions or how many solutions due to no general method exists.

In this paper, we have studied the exponential Diophantine equation with the favor of the following Lemma 2.1 which is Catalan's Conjecture [2] and is already proved by Mihailescu [3].

Lemma 2.1. (Catalan's Conjecture [2] or Mihailescu's Theorem [3])
The quadruple $(a, x, b, y) = (3,2,2,3)$ is the only integer solution for the Diophantine equation $a^x - b^y = 1$, where a, x, b, y are integers with $\min\{a, x, b, y\} > 1$.

Lemma 2.2. The exponential Diophantine equation $1 - 11^y = z^2$ has non-negative integer solution $(y, z) = (0, 0)$.

Proof: Let y and z be non-negative integers. Now let us consider the exponential Diophantine equation: $1 - 11^y = z^2$ (1)

To prove the given Lemma, we take two cases:

Case-1: When $y = 0$, we get from equation (1) $z = 0$ and this implies $(y, z) = (0, 0)$ is a solution of equation (1).

Case-2: When $y > 0$, then $z^2 = 1 - 11^y$ is obviously a negative integer which is a contradiction to the fact that z^2 is a non-negative integer.
Hence it is clear that, there is only possible non-negative integer solution is $y = 0$ and $z = 0$.

Lemma 2.3. The exponential Diophantine equation $27^x - 1 = z^2$ has only one non-negative integer solution $(x, z) = (0, 0)$.

Proof: Let x and z be non-negative integers and consider the exponential Diophantine equation $27^x - 1 = z^2$ (2)
Case-1: For $x = 0$
In the case of $x = 0$, from the equation (2), it is clear that $z = 0$ and in such a way that $(x, z) = (0, 0)$ is a solution of (2).
For $z = 1$, the equation (2) implies $27^x = 2$

Case-2: For $x > 0$
If we take $x = 1$, we have $z^2 = 26$, this having no solution.

3. Main result

Theorem 3.1. Let x, y, z be non-negative integers. The exponential Diophantine equation $27^x - 11^y = z^2$ has two non-negative integer solution $(x, y, z) = (0,0,0)$ *and* $(1,1,4)$.

Proof: Let us consider that x, y *and* z be non-negative integers such that

$$27^x - 11^y = z^2 \tag{3}$$

To prove the above equation we consider the four cases and two subcases:

Case 1: If we put $x = 0$ in (3), we have $1 - 11^y = z^2$. Then by lemma 2.1, it is clear that $(y, z) = (0,0)$. Thus $(x, y, z) = (0,0,0)$ is a solution of (3).

Case 2: If we substitute $y = 0$ in (3), we have $27^x - 1 = z^2$. Then by lemma 2.2 a solution $(x, z) = (0,0)$ is obtained. Thus we get a solution $(x, y, z) = (0,0,0)$.

Case 3: Putting $x = 1$ and $y = 1$ in (3), it gives a solution $(x, y, z) = (1,1,4)$ of the equation (3)

Case 4: If we take $x > 1$ and $y > 1$. Now from the equation (3) we get $27^x \equiv 3^x \equiv \begin{cases} 1 \bmod 4, \; if \;\; x \; is \; even \\ 3 \bmod 4, \; if \;\; x \; is \; odd \end{cases}$ and $11^y \equiv 3^y \equiv \begin{cases} 1 \bmod 4, \; if \;\; y \; is \; even \\ 3 \bmod 4, \; if \;\; y \; is \; odd \end{cases}$

Now

$$z^2 = 27^x - 11^y \equiv 3^x - 3^y \equiv \begin{cases} 0 \bmod 4 \; if \;\; both \; x \; and \;\; y \; are \; even & (4) \\ 0 \bmod 4 \; if \;\; both \; x \; and \;\; y \; are \; odd & (5) \\ -2 \bmod 4 \; if \;\; x \; is \; even \; and \;\; y \; is \; odd & (6) \\ 2 \bmod 4 \; if \;\; x \; is \; odd \; and \;\; y \; is \; even & (7) \end{cases}$$

Neglecting equation (6) and (7). Hence both x and y are even or both x and y are odd.

Now we can explain two possible subcases which are as follows:

Subcase 1: Let both x and y are even non-negative integers. Suppose that $x = 2m$ and $y = 2n$. Here m and n are non-negative integers. Then the equation (3) becomes

$$27^{2m} - 11^{2n} = z^2 \tag{8}$$

(8) becomes $11^{2n} = 27^{2m} - z^2 = (27^m - z)(27^m + z)$ (9)

(9) becomes $11^{2n} = 27^{2m} - z^2 = 11^u \left(27^m + z\right)$ (10)

where $\left(27^m - z\right) = 11^u$ and u is a non-negative integers.

(10) becomes $\left(27^m + z\right) = 11^{2n-u}$ (11)

Adding (9) and (10), we get

$$2\left(27^m\right) = 11^{2n-u} + 11^u = 11^u\left(1 + 11^{2n-2u}\right).$$

It is clear that $11^u = 1$ and $2\left(27^m\right) = \left(1 + 11^{2n-2u}\right)$.

From the above relation we must have the solution $(u, m, n) = (0, 0, 0)$.

Subcase 2:

Let both x and y are odd non-negative integers. Let $x = 2m+1$ and $y = 2n+1$, for some non-negative integers m, n.

Then from equation (3), we have

$$27^{2m+1} - 11^{2n+1} = z^2 \tag{12}$$

This implies,

$11^{2n+1} = 27^{2m+1} - z^2 = 27^{2m}.27 - z^2 = 27^{2m}(25+2) - z^2$ (13)

From equation (13), we get

$11^{2n+1} - 27^{2m}(2) = 27^{2m}(25) - z^2 = \left(27^m(5)\right)^2 - z^2 = \left(27^m(5) - z\right)\left(27^m(5) + z\right)$ (14)

Let $\left(27^m(5) - z\right) = 11^u$, u is a non-negative integer (15)

Then from (14), we have

$\left(27^m(5) + z\right) = \left(11^{2n+1} - 27^{2m}(2)\right)11^{-u}$ (16)

Adding (15) and (16) we have

$27^m(10) = 11^u + \left(11^{2n+1} - 27^{2m}(2)\right)11^{-u}$. This implies

$27^m(10) = 11^u\left[1 + \left(11^{2n+1} - 27^{2m}(2)\right)11^{-2u}\right]$ (17)

From the equation (17), we have the following relations:

$11^u = 1$ and $\left[1 + \left(11^{2n+1} - 27^{2m}(2)\right)11^{-2u}\right] = 27^m(10)$

From the above relations, it is obvious that $u = 0$ only and so that:

$1 + \left(11^{2n+1} - 27^{2m}(2)\right) = 27^m(10)$ (18)

This implies $1 \equiv 0 \bmod 4$ is a contradiction.

Hence $27^x - 11^y = z^2$ has exactly two non-negative integer solutions $(x, y, z) = (0, 0, 0), (1, 1, 4)$.

3. Conclusion

In this s5tudy, we have presented the basic concepts in Number theory with the help of the factoring method and modular method, to solve the exponential Diophantine equation

$27^x - 11^y = z^2$ and finally, we have proved that the exponential Diophantine equation $27^x - 11^y = z^2$ in which x, y and z are non-negative integers which has exactly two solutions $(x, y, z) = (0, 0, 0)$ and $(x, y, z) = (1, 1, 4)$.

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REFERENCES

1. L.E.Dickson, *First course in the theory of equations*, J. Wiley & sons, Incorporated, 1922.
2. E.Catalan, *Note extradite d'une letter addressee a l'editreur par Mr. E. Catalan*, Repetiteur a l'ecole polytechnique de Paris, (1844) 192-192.
3. P.Mihailescu, Primary cyclotomic units and a proof of Catalans conjecture, (2004) 167-195.
4. M.Le, S.Reese and S.Robert, A survey on the ternary purely exponential Diophantine equation $a^x + b^y = c^z$, arXiv preprint, arXiv: 1808.06557 (2018).
5. D.Acu, On a Diophantine equation, *General Mathematics*, 15(4) (2007) 145-148.
6. N.Burshtein, On the Diophantine Equation $p^x + q^y = z^2$, *Annals of Pure and Applied Mathematics*, 13(2) (2017) 229-233.
7. A.Suvarnamani, Solutions of the Diophantine equation $2^x + p^y = z^2$, *Int. J. Math. Sci. Appl.*, 1(3) (2011) 1415-1419.
8. S.Asthana and M.M.Singh, On the Diophantine equation $8^x + 113^y = z^2$, *International Journal of Algebra*, 11(15) (2017) 225-230.
9. S.Thongnak, W. Chuayjan and T. Kaewong, On the exponential Diophantine equation $2^x - 3^y = z^2$, *Southeast Asian Journal of Science*, 7(1) (2019) 1-4.
10. M.Buosi, L.P.Anderson and D.F.G. Santiago, On the exponential Diophantine Equation $p^x - 2^y = z^2$ with $p = k^2 + 2$, a prime number, *Southeast Asian Journal of Sciences*, 8(2) (2020) 103-109.
11. W.C.Thongnak and T.Kaewong, The solution of the exponential diophantine equation $7^x - 5^y = z^2$, *Mathematical Journal*, 66(703) (2021) 62-67.
12. S.W.C.Thongnak and T.Kaewong, On the exponential Diophantine equation $2^x - 3^y = z^2$, *Southeast Asian Journal of Science*, 7(1) (2019) 1-4.
13. Chikkavarapu, On the exponential diophantine equation $23^x - 19^y = z^2$, *Journal of Physical Sciences*, 27 (2022), 1-4.