Journal of Physical Sciences, Vol. 28, 2023, 17-22
ISSN: 2350-0352 (print), www.vidyasagar.ac.in/publication/journal
Published on 30 December 2023
DOI: http://dx.doi.org/10.62424/jps.2023.28.00.03

Almost Pseudo-Ricci Symmetric Mixed Generalized Quasi-Einstein Space-time

Mrityunjoy Kumar Pandit

Department of Mathematics, Jahangirnagar University
Savar, Dhaka-1342, Bangladesh
Email: mrityunjoyju@gmail.com

Received 1 September 2023; accepted 26 December 2023

ABSTRACT

In this paper, an almost pseudo-Ricci symmetric mixed generalized quasi-Einstein space-time has been revealed where the nature of associated vector fields is observed.

Keywords: Quasi-Einstein manifold, generalized quasi-Einstein manifold, mixed generalized quasi-Einstein space-time, almost pseudo-Ricci symmetric space.

AMS Mathematics Subject Classification (2010): 53C25

1. Introduction

Let (M^n, g) be an n dimensional Riemannian manifold with the metric g and let ∇ be the Levi-Civita connection of (M^n, g). According to Cartan an n dimensional Riemannian manifold M is called locally symmetric if $\nabla \mathrm{R} = 0$, where, R is the Reimannian curvature tensor of (M^n, g). The class of Riemannian symmetric manifolds is a very natural generalization of the class of manifolds of constant curvature.

During the last five decades, the notion of locally symmetric manifolds has been weakened by many authors [10, 11, 12, 13] in several ways to a different extent such as recurrent manifolds introduced by Walker [2], pseudo symmetric manifolds by Chaki [3], generalized pseudo symmetric manifold by Chaki [5]. The notion of a pseudo-Ricci symmetric manifold was introduced by Chaki [4] in 1988.

Definition 1. A Riemannian manifold (M^n, g) $(\mathrm{n} > 2)$ is called pseudo-Ricci symmetric if the Ricci tensor S of type $(0, 2)$ of the manifold is not identically zero and satisfies the condition

$$(\nabla_X \mathrm{S})(\mathrm{Y}, \mathrm{Z}) = 2\mathrm{A}(\mathrm{X})\mathrm{S}(\mathrm{Y}, \mathrm{Z}) + \mathrm{A}(\mathrm{Y})\mathrm{S}(\mathrm{X}, \mathrm{Z}) + \mathrm{A}(\mathrm{Z})\mathrm{S}(\mathrm{Y}, \mathrm{X}),$$

Where ∇ denotes the Levi-Civita connection and A is a non-zero 1-form such that $\mathrm{g}(\mathrm{X}, \mathrm{U}) = \mathrm{A}(\mathrm{X})$ for all vector fields X, U being the vector field corresponding to the associated 1-form A. An n dimensional manifold of this kind is denoted by $(\mathrm{PRS})_n$.

As an extended class of pseudo-Ricci symmetric manifolds introduced by Chaki, recently Chaki and Kawaguchi [6] introduced the notion of almost pseudo-Ricci symmetric manifolds.

Definition 2. A Riemannian manifold (M^n, g) is said to be an almost pseudo-Ricci symmetric manifold if its Ricci tensor S of type $(0,2)$ is not identically zero and satisfies the condition

$$(\nabla_X S)(Y, Z) = [A(X) + B(X)]S(Y, Z) + A(Y)S(X, Z) + A(Z)S(Y, X), \quad (1.1)$$

where ∇ denotes the operator of covariant differentiation with respect to the metric tensor g and A, B are nowhere vanishing 1-forms such that $g(X, U) = A(X)$ and $g(X, V) = B(X)$ for all X and U, V are called the basic vector fields of the manifold. In such a case, A and B are called the associated 1-forms and an n dimensional manifold of this kind is denoted by $A(PRS)_n$.

If $B = A$ then it reduces to a pseudo Ricci symmetric manifold. Thus pseudo Ricci symmetric manifold is a particular case of $A(PRS)_n$.

In general relativity, the matter content of the space-time is described by the energy-momentum tensor T which is to be determined from the physical considerations dealing with the distribution of matter and energy.

Definition 3. It is well known that a Riemannian manifold or a semi-Riemannian manifold (M^n, g) $(n > 2)$ is an Einstein manifold (Besse [1], 1987) if its non-zero Ricci tensor S of type $(0,2)$ is proportional to the metric tensor, i.e, S is of the form $S = kg$, where k is a constant, which reduces to $S = \frac{r}{n}g$, r being the scalar curvature (constant) of the manifold. Einstein's manifolds play an important role in Riemannian geometry as well as in the general theory of relativity.

A quasi-Einstein manifold was introduced by Chaki and Maity [7] as a generalization of Einstein manifolds.

Definition 4. A non-flat Riemannian manifold (M^n, g) $(n > 2)$ is defined to be a quasi-Einstein manifold if its Ricci tensor S of type $(0,2)$ is not identically zero and satisfies the following condition

$$S(X, Y) = \alpha g(X, Y) + \beta a(X)A(Y), \quad (1.2)$$

Where for all vector fields X,

$$g(X, U) = A(X), g(U, U) = 1.$$

We shall call A the associated 1-form and the unit vector field U is called the generator of the manifold. Such an n-dimensional quasi Einstein manifold is denoted by $(QE)_n$. The scalars α, β are known as the associated scalars of the manifold. From the above definition, it follows that every Einstein manifold is quasi-Einstein because if $\beta = 0$, clear the manifold reduces to an Einstein manifold. Quasi-Einstein Lorentzian manifolds are called perfect fluid space-times whenever A is time-like. Moreover, it was shown that the Robertson-Walker space-time is quasi-Einstein manifolds.

In 2001, Chaki [8] introduced the notion of generalized quasi Einstein manifolds.

Definition 5. A non-flat Riemannian manifold (M^n, g) $(n > 2)$ is called a generalized quasi-Einstein manifold if its Ricci tensor S of type (0,2) is not identically zero and satisfies the following condition

$$S(X,Y) = \alpha g(X,Y) + \beta A(X)A(Y) + \gamma B(X)B(Y), \quad (1.3)$$

where α, β, γ are certain scalar functions on (M^n, g) and $\beta \neq 0$, $\gamma \neq 0$. A, B are two non-zero 1-forms such that

$$g(X,U) = A(X),\ g(X,V) = B(X),\ g(U,V) = 0, g(U,U) = 1, g(V,V) = 1$$

U and V are two united vector fields which are orthogonal to each other.

A *and* B are said to be associated 1-forms and scalars α, β, γ are called associated scalars. The vector fields U and V are called the generators of the manifold. Such an n-dimensional manifold is denoted by $G(QE)_n$. Clearly, for γ = 0, the manifold reduces to a quasi-Einstein manifold and for β = γ = 0, the manifold reduces to an Einstein manifold. Generalized quasi-Einstein manifolds portray a generalization of Einstein manifolds and an extension of quasi-Einstein manifolds.

In 2007, Bhattacharya and De [9] introduced the notion of a mixed generalized quasi-Einstein manifold.

Definition 6. A non-flat Riemannian manifold is a mixed generalized quasi-Einstein manifold if its Ricci tensor S of type (0,2) is not identically zero and satisfies the following condition

$$S(X,Y) = \alpha g(X,Y) + \beta A(X)A(Y) + \gamma B(X)B(Y) + \delta[A(X)B(Y) + A(X)B(Y)], \quad (1.4)$$

where α, β, γ and δ are non-zero scalars and A, B are two non-zero 1-forms such that $g(X,U) = A(X)$, $g(X,V) = B(X)$ and $g(U,V) = 0$, where U, V are unit vector fields. The scalar functions α, β, γ, and δ are called associated scalars A *and* B are called associated 1-forms. U, V are called the generators of the manifold. Such n-dimension manifold is denoted by $MG(QE)_n$. If $\delta = 0$, the manifold reduces to a generalized quasi-Einstein manifold.

In cosmology and general relativity, mixed general quasi-Einstein manifolds have a significant role. After studying these papers, I am inspired to study almost pseudo-Ricci symmetry mixed with generalized quasi-Einstein space-time.

If we set $X = Y = e_i$ in the Ricci tensor of a $MG(QE)_n$ and take the summation over $i, 1 \leq i \leq n$, we obtain

$$r = n\alpha + \beta + \gamma, \quad (1.5)$$

where r is the scalar curvature of the manifold is expressed as a linear combination of the associated scalar functions α, β and γ. Aside from the fact that the scalar curvature of a manifold generalises its sectional curvature, in manifolds like $\mathrm{MG(QE)}_n$, it is expressed in terms of its associated scalar functions, it is a good relationship between scalar curvature and associated scalars that motivated the investigation of $\mathrm{MG(QE)}_n$ admitting some curvature tensor.

A Lorentzian four-dimensional manifold is said to be a mixed generalized quasi-Einstein space-time with the generator U as the unit timelike vector field if its nonzero

Ricci tensor of type (0,2) satisfies the equation (1.4). Here, A and B are non-zero 1-forms such that V is the heat flux vector field perpendicular to the velocity vector field U. Therefore, for any vector field X, we have

$$g(X,U) = A(X),\ g(X,V) = B(X),\ g(U,U) = A(U) = -1,\ g(V,V) = B(V) = 1,\ g(U,V) = 0.$$

2. Almost pseudo-Ricci symmetric mixed generalized Quasi-Einstein space-time

Here in this section, we consider an almost pseudo-Ricci symmetric mixed generalized quasi-Einstein space-time.

From (1.4), we have

$$S(X,Y) = \alpha g(X,Y) + \beta A(X)A(Y) + \gamma B(X)B(Y) + \delta[A(X)B(Y) + A(Y)B(X)].$$

From the above equation, we gain

$$(\nabla_Z S)(X,Y) = d\alpha(Z)g(X,Y) + d\beta(Z)A(X)A(Y) + \beta[(A)(X)A(Y) + A(X)(\nabla_Z A)(Y)] + d\gamma(Z)B(X)B(Y) + \gamma[(\nabla_Z B)(X)B(Y) + B(X)(\nabla_Z B)(Y)] + d\delta(Z)[A(X)B(Y) + A(Y)B(X)] + \delta[(\nabla_Z A)(X)B(Y) + A(X)(\nabla_Z B)(Y) + (\nabla_Z A)(Y)B(X) + A(Y)(\nabla_Z B)(X)] \quad (2.1)$$

Since this spacetime is almost pseudo Ricci symmetric therefore from equation (1.1), we get

$$(\nabla_Z S)(X,Y) = [A(Z) + B(Z)]S(X,Y) + A(X)S(Y,Z) + A(Y)S(X,Z). \quad (2.2)$$

The space-time is mixed generalized quasi-Einstein. Therefore, from equation (1.4) we obtain,

(i) $S(X,Y) = \alpha g(X,Y) + \beta A(X)A(Y) + \gamma B(X)B(Y) + \delta[A(X)B(Y) + A(Y)B(X)$.

(ii) $S(Y,Z) = \alpha g(Y,Z) + \beta A(Y)A(Z) + \gamma B(X)B(Y) + \delta[A(Y)B(Z) + A(Z)B(Y)]$.

(iii) $S(X,Z) = \alpha g(X,Z) + \beta A(X)A(Z) + \gamma B(X)B(Z) + \delta[A(X)B(Z) + A(Z)B(X)$.

There by using $(i), (ii), (iii)$ in equation (2.3), we get

$$(\nabla_Z S)(X,Y) = [A(Z) + B(Z)][\alpha g(X,Y) + \beta A(X)S(Y,Z) + \gamma A(Y)S(X,Z) + \delta[A(X)B(Y) + A(Y)B(X)] + A(X)[\alpha g(Y,Z) + \beta A(Y)A(Z) + \gamma B(X)B(Y) + \delta[A(Y)B(Z) + A(Z)B(Y)] + A(Y)[\alpha g(X,Z) + \beta A(X)A(Z) + \gamma B(X)B(Z) + \delta[A(X)B(Z) + A(Z)B(X)]. \quad (2.3)$$

Putting (2.3) in (2.1), we obtain

$$[A(Z) + B(Z)][\alpha g(X,Y) + \beta A(X)A(Y) + \gamma B(X)B(Y) + \delta\{A(X)B(Y) + A(Y)B(X)\}] + A(X)[\alpha g(Y,Z) + \beta A(Y)A(Z) + \gamma B(Y)B(Z) + \delta\{A(Y)B(Z) + A(Z)B(Y)\}] + A(Y)[\alpha g(X,Z) + \beta A(X)A(Z) + \gamma B(X)B(Z) + \delta\{A(X)B(Z) + A(Z)B(X)\}] = d\alpha(Z)g(X,Y) + d\beta(Z)A(X)A(Y) + \beta[(\nabla_Z A)(X)A(Y) + A(X)(\nabla_Z A)(Y)] + d\gamma(Z)B(X)B(Y) + \gamma[(\nabla_Z B)(X)A(Y) + B(X)(\nabla_Z B)(Y)] + d\delta(Z)[A(X)B(Y) + A(Y)B(X)] + \delta[(\nabla_Z A)(X)B(Y) + A(X)(\nabla_Z B)(Y) + (\nabla_Z A)(Y)B(X) + A(Y)(\nabla_Z B)(X)] \quad (2.4)$$

Contracting the equation (2.4) over X and Y, we get

$$[A(Z) + B(Z)](4\alpha - \beta + \gamma) + [\alpha A(Z) - \beta A(Z) - \delta B(Z)] + [\alpha A(Z) - \beta A(Z) - \delta B(Z)] = 4d\alpha(Z) - d\beta(Z) + d\gamma(Z)$$

$$\Rightarrow (6\alpha - 3\beta + \gamma)A(Z) + (4\alpha - \beta + \gamma - 2\delta)B(Z) = d(4\alpha - \beta + \gamma)(Z) \quad (2.5)$$

If we take $4\alpha-\beta+\gamma=constant=c$(say), (2.6)

Then $d(4\alpha-\beta+\gamma)=0$. (2.7)

Then by using (2.6) and (2.7) in (2.5), we get

$$(c+2\alpha-2\beta)A(Z)+(c-2\delta)B(Z)=0,$$

$$\text{i.e. } A(Z)=-\frac{(c-2\delta)}{(c+2\alpha-2\beta)}B(Z).$$

Therefore, we can conclude the following theorem:

Theorem 2.1. On almost pseudo Ricci symmetric mixed generalized quasi Einstein space-time the associated 1-forms A and B are related in the way such that
$A(Z)=-\frac{(c-2\delta)}{(c+2\alpha-2\beta)}B(Z)$ if we consider $4\alpha-\beta+\gamma=constant$.

From the above theorem we have

$A(Z)=-\frac{(c-2\delta)}{(c+2\alpha-2\beta)}B(Z)$,

i.e. $g(Z,U)=-\frac{(c-2\delta)}{(c+2\alpha-2\beta)}g(Z,V)$.

Therefore, we can assert the following corollary:

Corollary 2.2. On almost pseudo Ricci symmetric mixed generalized quasi Einstein space-time the generators U and V are in opposite directions.

Replacing $X=Y=U$ in (2.4), we obtain

$$[A(Z)+B(Z)](-\alpha+\beta)-[\alpha A(Z)-\beta A(Z)-\delta B(Z)]-[\alpha A(Z)-\beta A(Z)-\delta B(Z)]$$
$$=-d\alpha(Z)+d\beta(Z)-2\delta(\nabla_z B)(U),$$

i.e. $(-3\alpha+3\beta)A(Z)+(-\alpha+2\delta)B(Z)=-d\alpha(Z)+d\beta(Z)-2\delta(\nabla_z B)(U)$. (2.8)

Replacing $X=Y=V$ in (2.4), we obtain

$$[A(Z)+B(Z)](\alpha+\gamma)=d\alpha(Z)+d\gamma(Z)+2\delta(\nabla_z B)(U). \quad (2.9)$$

As two generators U and V are mutually perpendicular, $g(U,V)=0$. Therefore,
$Z\big(g(U,V)\big)=g(\nabla_Z U,V)+g(U,\nabla_Z V)=0$.

Then it becomes

$$(\nabla_Z B)(U)=-(\nabla_Z A)(V). \quad (2.10)$$

Subtracting (2.8) from (2.9) and using (2.10), we have

$(4\alpha-3\beta+\gamma)A(Z)+(2\alpha-\beta+\gamma-2\delta)B(Z)=d(2\alpha-\beta+\gamma)(Z)$. (2.11)

If we take $2\alpha-\beta+\gamma=constant=m$(say), (2.12)

Then $d(2\alpha-\beta+\gamma)=0$. (2.13)

Then by using (2.12) and (2.13) in (2.11), we get

$$(m+2\alpha-2\beta)A(Z)+(m-2\delta)B(Z)=0,$$

$$\text{i.e. } A(Z)=-\frac{(m-2\delta)}{(m+2\alpha-2\beta)}B(Z),$$

$$\text{i.e. } g(Z,U)=-\frac{(m-2\delta)}{(m+2\alpha-2\beta)}g(Z,V).$$

Therefore, we can conclude the following theorem:

Theorem 2.3. On an almost pseudo-Ricci symmetric mixed generalized quasi Einstein manifold if the vector fields are the associated vector fields then the generators U and V are in opposite directions.

Acknowledgements.
The author gratefully acknowledges the reviewer's valuable comments and suggestions for the improvement of the paper.

REFERENCES

1. A.L.Besse, *Einstein manifolds*. Springer Science & Business Media, 2007.
2. A.G. Walker, On ruses space of recurrent curvature, *Prof. London Math. Soc.*, 52 ((1950) 36-64.
3. M. C. Chaki, On pseudo symmetric manifolds, *An. Stiint. Univ. Al. I. Cuza. Din lasi Romanian*, 33 (1987) 53-58.
4. M.C. Chaki, On pseudo-Ricci symmetric manifolds, *Buig. J. Phys.*, 15 (1988) 525-531.
5. M.C. Chaki, On generalized pseudo symmetric manifolds, *Publ. Math. Debrecen*, 45 (1994) 305-312.
6. M.C. Chaki and T. Kawaguchi, On almost pseudo-Ricci symmetric manifolds, *Tensor N. S*, 68 (2007) 10-14.
7. M.C. Chaki and R.K. Maity, On pseudo-Einstein manifolds, *Math. Debrecen*, 57 (2000) 297.
8. M.C. Chaki, On generalized quasi Einstein manifolds, *Publ. Math. Debrecen*, 58 (2001) 638.
9. M.Tarafdar and A. Mayra, Semi pseudo-Ricci symmetric manifold, *J. Indian Inst. Sci.*, 73(6) (1993) 591-596.
10. A. Bhattacharyya and T. De, On mixed generalized quasi-Einstein manifolds, *Differential Geometry-Dynamical Systems*, 9 (2007) 40-46.
11. M. Tarafdar, On pseudo-symmetric and pseudo-Ricci-symmetric P-Sasakian manifold, *An. Stiint. Univ. Al. I. Cuza lasi sect. 1 a Mat.*, 37(2) (1991) 211-214.
12. B. Pal and A. Bhattacharyya, Almost pseudo-Ricci symmetric viscous fluid space-time, *Mathematical Combinatorics,* 4 (2015) 41-52.
13. K. Chattopadhyay, N. Bhunia and A. Bhattacharyya, On Ricci-symmetric mixed generalized quasi-Einstein spacetime, *Bull. Cal. Math. Soc*, *110*(6) (2018) 513-524.
14. N. Bhunia, S. Pahan and A. Bhattacharyya, Application of hyper-generalized quasi-Einstein spacetimes in general relativity, *Proceedings of the National Academy of Sciences, India Section A: Physical Sciences*, 91 (2021) 297-307.