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General Solution of the Diophantine Equation $M_p^x + (M_q + 1)^y = (lz)^2$

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ABSTRACT

In this article, I study and solve the exponential Diophantine equation $M_p^x + (M_q + 1)^y = (lz)^2$ where M_p and M_q are Mersenne primes, l is a prime number, and x,y and z are non-negative integers. Several illustrations are presented as well as cases where no solution of the given Diophantine equation is present.

Keywords: Diophantine equation, Mersenne Primes, Integer solution

AMS Mathematics Subject Classification (2010): 11D61, 11D72, 11A41

1. Introduction

The Diophantine equation is one of the most attractive and exciting categories of problems in number theory. over the years, several researchers have been studying the Diophantine equation of the form $a^x + b^y = z^2$. This includes Aggarwal, Burshtein, Sroysang, Rabago among others([1-8], [11-13]). Some have studied these equations about Mersenne primes(see definition 1). Their work primarily focuses on the case where one of the bases a and b is a Mersenne prime. Sroysang [14] proved that the solutions of $3x + 2y = z^2$ are $(0,1,2)$; $(3,0,3)$ and $(2,4,5)$. Asthana and Singh [6] proved that $3x + 13y = z^2$ has exactly four non-negative integer solutions, and these are $(1,0,2)$, $(1,1,4)$; $(3,2,14)$ and $(5,1,6)$. Rabago [13] proved that the triples $(4,1,10)$ and $(1,0,2)$ are the only solutions to the Diophantine equation $3^x + 19^y = z^2$, and that $(2,1,10)$ and $(1,0,2)$ are the only two solutions to $3^x + 91^y = z^2$. Sroysang [14] also showed that the $7^x + 8^y = z^2$ has the only solution $(x, y, z) = (0,1,3)$. Chotchaisthit [9] aimed to study $px + (p + 1)y = z^2$ in the set of non-negative integers and where p is a Mersenne prime.

In this article, I have found a general solution of the exponential Diophantine equation $M_p^x + (M_q + 1)^y = (lz)^2$ where M_p and M_q are Mersenne Primes, x, y and z are non-negative integers and l is a prime number. The motivation behind this article is to find a solution involving the general Mersenne prime instead of taking any particular value of it. Methods of modular arithmetic and factorization of polynomials are used in proving the results of this article.

2. Main results

The following definitions and lemmas are needed for this article.

Definition 2.1. *A Mersenne prime is a prime number of the form* $2^p - 1$ *where* p *is a prime number and is denoted by* M_p.

Lemma 2.1. *All Mersenne Primes are congruent to* $3(mod4)$
Proof. As any Mersenne Prime is of the form $2^p - 1$, we can clearly say that $p \geq 2$. Now as $p \geq 2$, then $2^p \equiv 0(\text{mod}4)$ and hence, $2^p - 1 \equiv 3(\text{mod}4)$.

Lemma 2.2. *(Mihailescu's theorem)(see [10]) The only solution to the Diophantine equation* $a^x - b^y = 1$ *is* $a = 3,\ b = 2,\ x = 2$ *and* $y = 2$ *where* $min\{a, b, x, y\} > 1$.

At first, we consider the case when $l = 2$. Hence, the following is the first main theorem of this paper.

Theorem 2.1. *Every non-negative integer solution to the equation* $M_p^x + (M_q + 1)^y = (2z)^2$ *is the following tuple* $(M_p, M_q, x, y, z) = (3, M_q, 1{,}0{,}1)$.

Proof: Let us first consider the case when one of the exponent x and y is zero.

Case-I At first, if we assume $x = 0$, then we get the following equation

$$2^{qy} = 4z^2 - 1. \tag{2.1}$$

Subcase-(a) If $y = 0$, then from equation (2.1), we get $4z^2 = 2$ which is a contradiction.

Subcase-(b) If $y = 1$, then from the equation (2.1), we get that $(2z)^2 - 2^q = 1$. From the lemma (2), the solution of this equation is only possible if $z = \frac{3}{2}$ which is a contradiction.

Subcase-(c) If $y > 1$, then the equation (2.1) can be written as $(2z)^2 - 2^{qy} = 1$. By the lemma (2), we must have $qy = 3$. Now as q is a prime number, we get $q = 3$ and $y = 1$ which is a contradiction to our assumption.

Case-II Now we assume $y = 0$. Then we get the following equation

$$M_p^x + 1 = (2z)^2. \tag{2.2}$$

Subcase-(a) If $x = 0$, then from equation (2.2), we get $4z^2 = 2$ which is a contradiction.

Subcase-(b) If $x = 1$, then the equation (2.2) can be written as $(2z)^2 = 2^p$. Now let $Z = 2z$ and $Z = 2^a$. Then we get $2^{2a} = 2^p$ which in turn gives $p = 2a$. As p is a prime number, we get $a = 1$ and $p = 2$. Hence $Z = 2$ and finally, $z = 1$. Hence $(M_p, M_q, x, y, z) = (3, M_q, 1.0{,}1)$.

Subcase-(c) if $x > 1$, then from the equation (2.2), we get that $(2z)^2 - (2^p - 1)^x = 1$. By the lemma (2), we get that $2^p = 3$ which is a contradiction.

Case-III. Now we consider the case when $\{x, y\} \geq 1$. From the lemma (1), we know that $M_p \equiv 3(\text{mod}4)$ and $(M_q + 1) \equiv 0(\text{mod}4)$. Hence,

$$(M_p^x + (M_q + 1)^y) \equiv \begin{cases} 3(\text{mod}4), & x \ is \ odd \\ 1(\text{mod}4), & x \ is \ even \end{cases}$$

Now as $4z^2 \equiv 0(\text{mod}4)$, the equation has no solution when $\{x, y\} \geq 1$.

Now we will consider the case when l is an odd prime. Then we get the following theorem

Theorem 2.2. *Every non-negative integer solution to the equation* $M_p^x + (M_q + 1)^y = (lz)^2$ *takes one of the following form:*

(a)$(M_p, M_q, x, y, z)=(M_p, 7,0,1,1)$

(b) $(M_p, M_q, x, y, z) = (M_p, M_q, 2, \frac{p+2}{q}, \frac{2^p+1}{l})$.

Proof: Let us first assume that one of the exponents x and y is zero.

Case-I At first, we assume that $x = 0$.Then we get the equation

$$1 + (M_q + 1)^y = (lz)^2. \tag{2.3}$$

Subcase-(a) If $y = 0$, then from the equation (2.3), we get $2 = (lz)^2$ which is a contradiction.

Subcase-(b) If $y = 1$, then from the equation (2.3), we get that $(lz)^2 - 2^q = 1$. Hence from the lemma (2), we can conclude that $l = 3$, $z = 1$, and $q = 3$. Hence, $(M_p, M_q, x, y, z) = (M_p, 7,0,1,1)$ is the only solution when $l = 3$.

Subcase-(c) If $y > 1$, then from the equation (2.3), we get $(lz)^2 - 2^{qy} = 1$. Again, from the lemma (2), we get that $qy = 3$. By using the primality of q, we get that $q = 3$ and $y = 1$ which is a contradiction.

Case-II Now we assume $y = 0$. Then we get the equation

$$(M_p)^x + 1 = (lz)^2. \tag{2.4}$$

Subcase-(a) If $x = 0$, then from the equation (2.4), we get $2 = (lz)^2$ which is a contradiction.

Subcase-(b) If $x = 1$, then from the equation (2.4), we get $2^p = (lz)^2$. Now as $2^p \not\equiv 0(\text{mod} l)$ and $(lz)^2 \equiv 0(\text{mod} l)$, so the above equation has no solution.

Subcase-(c) If $x > 1$, then from the equation (2.4), we get that $(lz)^2 - (M_p)^x = 1$. By the lemma (2), we get that $2^p = 3$ which is a contradiction.

Case-III Now we consider the case when $\{x, y\} \geq 1$. Now as we know $(lz)^2 \equiv 1(\text{mod}4)$ when z is odd and

$$(M_p^x + (M_q + 1)^y) \equiv \begin{cases} 3(\text{mod}4), & x \textit{ is odd} \\ 1(\text{mod}4), & x \textit{ is even} \end{cases}$$

. From the above, we can conclude that the equation $M_p^x + (M_q + 1)^y = (lz)^2$ has a solution only if x is even and z is odd. Thus there exists a positive integer k such that $x = 2k$ and we get the equation $M_p^{2k} + 2^{qy} = (lz)^2$. This equation can be written as

$$(lz + M_p^k)(lz - M_p^k) = 2^{qy}. \tag{2.5}$$

There exist two non-negative integers α and β with $\alpha > \beta$ such that $\alpha + \beta = qy$. Then the equation ?? can be written as

$$(lz + M_p^k)(lz - M_p^k) = 2^{\alpha+\beta}. \tag{2.6}$$

we claim that $\gcd(lz + M_p^k, lz - M_p^k) \neq 1$. Suppose our assumption is wrong. Then $(lz + M_p^k, lz - M_p^k) = 1$. Now from the equation (2.6), we can say that $lz - M_p^k = 1$. We know from the lemma (1) that $M_p \equiv 3(\text{mod}4)$ and hence $M_p^k \equiv 1,3(\text{mod}4)$. As l is an odd prime and z is also odd, then $lz \equiv 1,3(\text{mod}4)$. So, $lz - M_p^k \equiv 0,2(\text{mod}4)$ which is contradiction to the fact $lz - M_p^k \equiv 1(\text{mod}4)$. Now as $\gcd(lz + M_p^k, lz - M_p^k) \neq 1$, we take $lz + M_p^k = 2^\alpha$ and $lz - M_p^k = 2^\beta$. it implies that $2M_p^k = 2^\beta(2^{\alpha-\beta} - 1)$ and by comparing odd and even parts, we get the system of equations:

$$\begin{cases} 2^\beta = 2. \\ 2^{\alpha-\beta} - 1 = M_p^k. \end{cases}$$

From the above equation, we know that $\beta = 1$ and hence,

$$1 = 2^{\alpha-1} - M_p^k. \tag{2.7}$$

By using the lemma (2), we can say that the equation (2.7) has no solution if $\alpha > 2$ and $k > 1$. If $\alpha = 2$, then from the equation (2.7), we get that $M_p^k = 1$. This gives $k = 0$ which is a contradiction to that k being a positive integer. So the only possibility is $k = 1$ and hence $x = 2$. Now putting the value $k = 1$ in the equation (2.7), we get $M_p + 1 = 2^{\alpha-1}$ or equivalently, $\alpha = p + 1$. Putting the values $\alpha = p + 1$ and $\beta = 1$ in the relation $\alpha\beta = qy$, we get $y = \frac{p+2}{q}$. Now putting the values x and y in the equation $M_p^x + (M_q + 1)^y = (lz)^2$, we get that $z = \frac{2^p+1}{l}$.

Remark 2.1. *From the theorem (2), we can say that the equation $M_p^x + (M_q + 1)^y = (lz)^2$ has a positive integer solution $(M_p, M_q, 2, \frac{p+2}{q}, \frac{2^p+1}{l})$. It means that given a Mersenne prime M_p, the solution can only be found if q divides $(p + 2)$ and also l divides $(2^p + 1)$ where l is an odd prime. We need to be careful about q and pick only those q for which M_q is a Mersenne prime.*

Let us now look at a couple of examples of the Theorem 2.2.

Example 2.1. *Find all possible positive integer solutions of the equation $8191^x + (M_q + 1)^y = (3z)^2$ where M_q is a Mersenne prime and q is a prime number.*

Solution: Here $M_p = 8191$ implies $p = 13$. From the theorem (2), we get the condition that q divides $p + 2 = 15$ which in turn gives $q = 3$ or $q = 5$. If $q = 3$, we have $M_q = 7$, $y = 5$ and when $q = 5$, we have $M_5 = 2^5 - 1 = 31$ and $y = 3$. Observe that both M_3 and M_5 are Mersenne primes. Also for both the cases, $l = 3$ and hence $z = \frac{2^{13}+1}{3} = 2731$. Hence the solution set are $(M_p, M_q, x, y, z) = (8191,7,2,5,2731)$ and $(M_p, M_q, x, y, z) = (8191,31,2,3,2731)$.

Example 2.2. *Find the positive integer solutions of the equation $7^x + 4^y = (7z)^2$*
Solution: Here $M_p = 7$ where $p = 3$ and $M_q = 3$ where $q = 2$. Hence, from the theorem 2, the solution set is $(x, y, z) = (2, \frac{p+2}{q}, \frac{2^p+1}{l})$ if q divides $(p + 2)$ and l divides $2^p + 1$. Now as $l = 7$ does not divide $2^p + 1 = 9$, the equation has no solution in positive integers.

Example 2.3. *Find the positive integer solutions of the equation $3^x + 8^y = (5z)^2$.*
Solution: Here $M_p = 3$ where $p = 2$ and $M_q = 7$ where $q = 3$. Hence, from the

theorem 2, the solution set is $(x, y, z) = (2, \frac{p+2}{q}, \frac{2^p+1}{l})$ if q divides $(p+2)$ and l divides $2^p + 1$. As $q = 3$ does not divide $p + 2 = 4$, the equation has no solution in positive integers.

3. Conclusion and future work

In this article, using the modular arithmetic method, with the help of Mihailescu's theorem 2 and using the fact that every Mersenne prime is of the form $4k + 3$, we have been able to show the complete list of positive integer solutions of the Diophantine equation $M_p^x + (M_q + 1)^y = (lz)^2$ where l is a prime.

The following table represents the solution of the Diophantine equation $M_p^x + (M_q + 1)^y = (lz)^2$ for the first couple of Mersenee Primes:

M_p	p	$p+2$	q	M_q	2^p+1	l	(x, y, z)
3	2	4	2	3	5	5	(2,2,1)
7	3	5	5	7	9	3	(2,1,3)
31	5	7	7	31	33	3	(2,1,11)
31	5	7	7	31	33	11	(2,1,3)
127	7	9	3	7	129	3	(2,1,43)
127	7	9	3	7	129	43	(2,1,3)

Table 1: Some possible integer solution of the equation $M_p^x + (M_q + 1)^y = (lz)^2$

The next table represents some particular cases of the Diophantine equation $M_p^x + (M_q + 1)^y = (lz)^2$ where no solutions can be obtained. The un-solvability of these equations is due to two main reasons namely q does not divide $(p+2)$ or l does not divide $(2^p + 1)$.

M_p	p	$p+2$	q	M_q	2^p+1	l	$M_p^x + (M_q + 1)^y = (lz)^2$
3	2	4	5	31	5	3	$3^x + 32^y = (3z)^2$
7	3	5	7	127	9	5	$7^x + 128^y = (5z)^2$
31	5	7	3	7	33	7	$31^x + 8^y = (7z)^2$
127	7	9	5	31	129	13	$127^x + 32^y = (13z)^2$

Table 2: Some of the unsolvable cases of the equation $M_p^x + (M_q + 1)^y = (lz)^2$

Now for possible extensions, the reader may try to solve the following Diophantine equations:

(i) $M_p^x + (M_q + k)^y = z^2$, where $k \geq 1$, and M_p and M_q are Mersenne primes.
(ii) $M_p^x + (M_q + 1)^y = z^n$, where $n \geq 1$, and M_p and M_q are Mersenne primes and,
(iii) $M_p^x + (M_q + k)^y = z^n$, where $k, n \geq 1$, and M_p and M_q are Mersenne primes.

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